D. Grieser
Mathematisches Institut
Universität Bonn
Zi. 12, Beringstr. 4
email: grieser@math.uni-bonn.de

Übungen zur Vorlesung 'Mikrolokale Analysis' (SS 2005)

Blatt 1

Problem 1 Let $a_k \in \mathbb{C}$ for $k \in \mathbb{Z}$. Show that

$$f = \sum_{k=-\infty}^{\infty} a_k \delta_k \in \mathcal{D}'(\mathbb{R}),$$

that is, the sum converges in $\mathcal{D}'(\mathbb{R})$.

Show that $f \in \mathcal{S}'(\mathbb{R})$ if and only if there are $C, N > 0$ such that

$$|a_k| \leq C(1 + |k|)^N$$

for all k.

Problem 2 (*'Integration' of distributions*)

Let u be a distribution on $\mathbb{R}$ satisfying $u' = 0$. Show that u is a constant. That is, there is $c \in \mathbb{R}$ such that $\langle u, \phi \rangle = \int c\phi$ for all test functions ϕ.

Deduce that if a distribution on $\mathbb{R}$ has a smooth derivative then it is smooth itself.

Problem 3 Show that $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$.

(Hint: Fix a function $\psi \in C_0^\infty(\mathbb{R}^n)$ which equals one near the origin $0 \in \mathbb{R}^n$. Let $\psi_N(x) = \psi(x/N)$. Show that $\psi_N g \to g$ as $N \to \infty$, for $g \in \mathcal{S}'$. Using properties of the Fourier transform, conclude that $\psi_M(\psi_N \hat{f})\check{}$ gives an approximation of $f \in \mathcal{S}'$ by C_0^∞-functions, for suitable large M, N. Convergence here is in the sense of $\mathcal{S}'$ always.)

Problem 4 Prove that any distribution on $\mathbb{R}^n$ with support equal to $\{0\}$ is a finite sum of the form

$$\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha D^\alpha \delta, \quad a_\alpha \in \mathbb{C}.$$

(Hint: If the order of the distribution is N, use Taylor's theorem to order $N+1$ on the test functions.)