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Ergänzungen und Übungen zur Vorlesung 'Mikrolokale Analysis' (SS 2005)

Blatt 2

Problem 1 (*Some Fourier distributions on* $\mathbb{R}$)

(*a*) Let $a \in \mathbb{R}$. Show that the limits

$$(x + i0)^a := \lim_{\epsilon \searrow 0} (x + i\epsilon)^a,$$

$$\log(x + i0) := \lim_{\epsilon \searrow 0} \log(x + i\epsilon).$$

exist in $\mathcal{S}'(\mathbb{R})$ Here, z^a resp. $\log z$ is defined as analytic function of z in the upper half plane $\{\operatorname{Im} z > 0\}$, as the analytic continuation from the usual (real and positive valued) branch for $z > 0$.

(**Hint:** For log and for $a > -1$ use dominated convergence. The other cases can be obtained from these by repeated differentiation.)

(*b*) (*Fourier transform of homogeneous distributions*)

A distribution u on $\mathbb{R}^n$ is *a-homogeneous* ($a \in \mathbb{R}$) if $u(tx) = t^a u(x)$ for all $t > 0$, where this is interpreted in the distribution sense (make this precise!). Show that $(x + i0)^a$ is a-homogeneous. Also show that, if u is a tempered distribution which is a-homogeneous then $\hat{u}$ is $(-a - n)$-homogeneous.

(*c*) Let $m \in \mathbb{R}$ and let $\chi \in C^\infty(\mathbb{R})$ equal zero on $(-\infty, 1)$ and one on $(2, \infty)$. Then $\chi(\xi)\xi^m$ is a symbol of order m. Show that, as oscillatory integral,

$$\int_{\mathbb{R}} e^{ix\xi} \chi(\xi) \xi^m \, d\xi = r_m(x) + \begin{cases} c_m (x + i0)^{-m-1} & \text{if } m \neq -1 \\ c_{-1} \log(x + i0) & \text{if } m = -1 \end{cases}$$

for a smooth function r_m and some constant c_m.

(**Hint:** For $m > -1$ show first that the function

$$\xi_+^m := \begin{cases} \xi^m & \text{if } \xi > 0 \\ 0 & \text{if } \xi \leq 0 \end{cases}$$

is a tempered distribution on $\mathbb{R}$ whose Fourier transform is $c_m(x + i0)^{-m-1}$, and deduce the claim from this. The other cases can be reduced to this case by repeated differentiations (and subsequent integrations using problem 1).)

Remark: A good reference for this and the following problem is volume I of Hörmander's book 'The Analysis of Partial Differential Operators'.

Problem 2 (*Topology on C_0^∞*)

Convergence in $C_0^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, is defined as follows: $\phi_i \to \phi$ if and only if

- there is a compact $K \subset \Omega$ containing the support of ϕ and of *all* the ϕ_i, and
- $D^\alpha \phi_i \to D^\alpha \phi$ uniformly on K, for all multi-indices α (that is, $\phi_i \to \phi$ in the C^∞ sense).

Use this definition to show:

(a) A linear functional on $C_0^\infty(\Omega)$ is a distribution if and only if it is continuous in the sense of sequences.

(b) Let $\Omega_1 \subset \mathbb{R}^{n_1}$, $\Omega_2 \subset \mathbb{R}^{n_2}$. Then the map

$$\begin{aligned} \mathcal{D}'(\Omega_2) \times C_0^\infty(\Omega_1 \times \Omega_2) &\to C_0^\infty(\Omega_1) \\ (f, \phi) &\mapsto (x \mapsto < f, \phi(x, \cdot) >) \end{aligned}$$

is continuous in the sense of sequences.