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Ergänzungen und Übungen zur Vorlesung 'Mikrolokale Analysis' (SS 2005)

Blatt 3

Problem 1 (*A classical problem on divergent series*)

An old question is to make sense of the divergent sum

$$\sum_{j=0}^{\infty}(-1)^j = 1 - 1 + 1 - 1 + \dots .$$

Here are some simple methods:

$$(1-1) + (1-1) + \ldots = 0,$$
$$1 - (1-1) - (1-1) - \ldots = 1,$$
$$\lim_{x\to -1, |x|<1} \sum_{j=0}^{\infty} x^j = \lim_{x\to -1} 1/(1-x) = 1/2.$$

The last method is called Abel summation. If you want to know more about these things, look at Hardy's book 'Divergent Series'.

The analogue of the 'regularisation' used for oscillatory integrals is

$$\lim_{T\to\infty} \sum_{j=0}^{\infty} (-1)^j \chi(\frac{j}{T})$$

where $\chi \in C_0^\infty(\mathbb{R})$ and $\chi(x) = 1$ for x near zero.

(a) Show that the result is 1/2.

(b) Which special choice of χ yields Abel summation? (This shows that weaker assumptions on χ should suffice.)

(c) Calculate the oscillatory integral

$$\int_0^\infty \sin x \, dx.$$

What would be the results of various 'bracketings' (as used for the series above)?

(This shows that it's not obvious how to make sense of oscillatory integrals – you may get different answers.)

Problem 2 (*Decay of the Fourier transform of measures on surfaces*)

(a) Let $f(x) = a(x)\delta(x_1)$ in $\mathbb{R}^n$, where $a \in C_0^\infty(\mathbb{R}^n)$. Let ω be a fixed unit vector. Show that, for $\lambda \to \infty$,

$$\hat{f}(\lambda\omega) = \begin{cases} O(\lambda^{-N}) & \text{if } \omega \neq (1, 0, \ldots, 0), \\ \text{constant} & \text{if } \omega = (1, 0, \ldots, 0) \end{cases}$$

(The constant is non-zero if $\int a(0, x_2, \ldots, x_n)\, dx_2 \ldots dx_n \neq 0$.)

(b) Show that the Fourier transform of $f(x) = a(x)\delta(|x| - 1)$, $a \in C^\infty(\mathbb{R}^n)$, is smooth and satisfies

$$|\hat{f}(\lambda\omega)| \leq C\lambda^{-(n-1)/2}, \quad \lambda \geq 1,$$

for all unit vectors ω, where C is an absolute constant.

(**Hint:** Suppose first that a is supported near $x = (1, 0, \ldots, 0)$. Introduce new coordinates $y_1 = |x| - 1, y_2 = x_2, \ldots, y_n = x_n$ and use stationary phase. Reduce the general case to this by a partition of unity and rotation.)

(**Note:** Note the difference in the behavior of $\hat{f}$ in (a) and (b). The reason behind this is that the singular support of f in (a) is flat, while it is curved in (b).)

Problem 3 (*Lattice points in circles*)

(a) Using Problem 3(b), or otherwise, show that the Fourier transform of the characteristic function of the unit ball, $\chi(x) = 1$ if $|x| \leq 1$ and zero otherwise, satisfies

$$|\hat{\chi}(\xi)| \leq C(1 + |\xi|)^{-(n+1)/2}.$$

(b) Show that the number of lattice points inside a circle, $N(r) := \{(k, l) \in \mathbb{Z}^2 : k^2 + l^2 \leq r^2\}$, satisfies

$$N(r) = \pi r^2 + O(r^{2/3}), \quad r \to \infty.$$

(**Hint:** Let $\chi_r(x) = \chi(x/r)$ be the characteristic function of the ball $B_r(0)$. Then $N(r) = \sum_{p\in\mathbb{Z}^2} \chi_r(p)$. Approximate χ_r by a smooth function $\chi_r * \phi_r$ where $\phi_r(x) = r^{2\alpha}\phi(xr^\alpha)$, for a fixed $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\int \phi = 1$, and $\alpha > 0$ to be determined. Here, $(f * g)(x) = \int f(y)g(x-y)dy$ is the convolution, which satisfies $(f * g)\hat{} = \hat{f} \cdot \hat{g}$. Use the Poisson summation formula and optimize over α.)

Note: It is a classical, still unsolved problem (the 'circle problem'), to determine for which exponents s one has $N(r) = \pi r^2 + O(r^s)$ as $r \to \infty$. It is easy to see that $s = 1$ works. One can show that necessarily $s > 1/2$. The smallest value to date, found with the help of sophisticated number theory, is $s = 46/73 = 0.630$... See for example http://mathworld.wolfram.com/GausssCircleProblem.html for history and references.