Notes on Fluid Flow and Complex Analysis

Consider a fluid flowing in a domain $G \subset \mathbb{C}$. Let $V(z)$ be the velocity of the particle at $z \in G$.

Assumptions: 1. The flow $V(z)$ is independent of time.
2. locally irrotational: $\operatorname{curl} V = 0$ $\quad$ ($\operatorname{curl} V := \operatorname{rot} V$)
3. incompressible and no sources: $\operatorname{div} V = 0$
4. $V(z)$ is tangential to ∂G if $z \in \partial G$, the boundary.

Notes: 2. $\Leftrightarrow \int_\gamma V \cdot T = 0$ $\quad \forall$ closed Jordan curves γ whose interior $\subset G$.

3. $\Leftrightarrow \int_\gamma V \cdot N = 0$ $\quad$ - - - - - - - - - -

$T =$ tangent to γ, $N =$ (outward) normal to γ.

4.: Since $V(z)$ may not be defined for $z \in \partial G$ (it could be ∞), this is to be understood in some limiting sense.

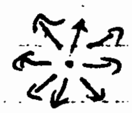
2. and 3. imply that $f(z) = \overline{V(z)} = V_1(z) - iV_2(z)$ is analytic,
where $V = (V_1, V_2) = V_1 + iV_2$

Complex potential: Any (potentially multivalued) function F with $F' = f$.

Fact: $\operatorname{Im} F$ is constant along each flowline (easy calculation)

Flowline: A curve $z(t)$ with $V(z(t)) = \dot{z}(t) := \frac{d}{dt} z(t)$ for all t.
(= path of a fluid particle)

Circulation along a closed path γ is $\int_\gamma f(z)\,dz$.

Assumption 4 implies that the circulation is always real.
(There are no sources "outside G", as there would be for $f(z) = \frac{1}{z}$ on $G = \mathbb{C} - \{0\}$, for example: )

A <u>single valued potential</u> F exists if and only if the <u>circulation equals zero</u> along all closed paths in G.

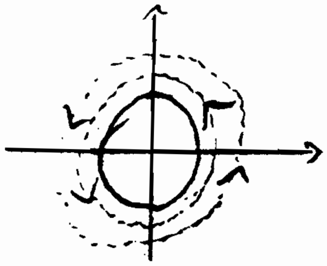
This is the case, for example, if G is <u>simply connected</u>.

Note that <u>$\operatorname{Im} F$</u> is <u>always singlevalued</u> since the circulation is real and different F-values differ by multiples $n \cdot C$, $n \in \mathbb{Z}$, of the circulation C along some path.

For example, for $f(z) = \frac{i}{z}$ on $\mathbb{C} - \{|z| < 1\}$,

$$F(z) = i \log z = i \ln r - \arg z$$

and $\operatorname{Im} F = \ln r$ is single valued.



<u>Main problem</u>: Find all possible flows in a given domain G, maybe with additional conditions, e.g. at ∞.

<u>ex</u>: The only flows on $G = \mathbb{C}$ which are bounded, $|V(z)| \le C \ \forall z$, are the uniform flows $V(z) = v \ \forall z$. (some $v \in \mathbb{C}$)
(Liouville's theorem restated)

<u>Conformal mapping</u>: If G_1 and G are two domains and a flow $f(w), F(w)$ is known in G, and if $g: G_1 \to G$ is conformal and bijective

then $F_1(z) = F(g(z))$ is a flow in G_1, and

$$f_1(z) = f(g(z)) \cdot g'(z).$$

This implies that circulation along $\gamma \subset G_1$ (w.r.t. f_1) equals circulation along $g(\gamma) \subset G$.

(We assume that g maps ∂G_1 "nicely" to ∂G, which is usually the case.)

(Easy) <u>Fact</u>: g maps flow lines of f_1 to flow lines of f.

Flow around an obstacle $\mathcal{O}$:
let $\mathcal{O} \subset \mathbb{C}$ be closed and bounded (compact)*, $G = \mathbb{C} - \mathcal{O}$
The circulation around $\mathcal{O}$ is the circulation along any closed curve encircling $\mathcal{O}$ once. By Cauchy's Theorem, it does not depend on which curve is chosen.

Examples 1) $G_1 = \mathbb{C} - \{|z| \le 1\}$, $G_2 = \mathbb{C} - [-1,1]$

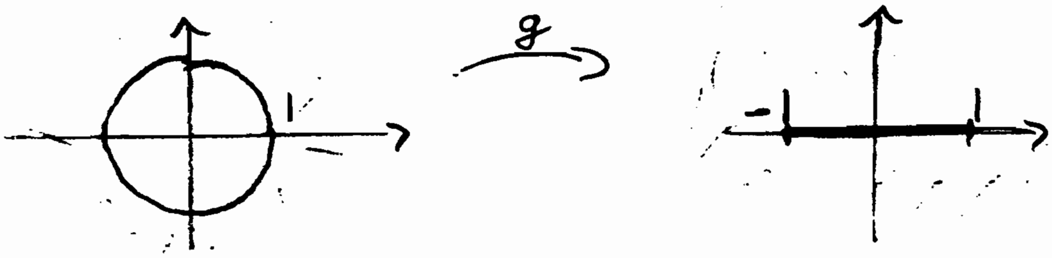
$$g(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

One class of (obvious) flows on G is $f(w) = a$, any $a \in \mathbb{R}$.
So $F(w) = aw$ and

$$F_1(z) = \frac{a}{2}\left(z + \frac{1}{z}\right)$$

$$f_1(z) = \frac{a}{2}\left(1 - \frac{1}{z^2}\right)$$

As $z \to \infty$, the velocity tends to $v = \frac{a}{2}$.



There is another (easy to see) class of flows for G_1: $\tilde{f}_1 = \frac{C}{2\pi i} \cdot \frac{1}{z}$.

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Superposition principle: If $f, \tilde{f}$ are possible flows then so is $f + \tilde{f}$.
(clear from assumptions)

General flow around circular obstacle: $F(z) = v\left(z + \frac{1}{z}\right) + \frac{C}{2\pi i} \log z$
where $v \in \mathbb{R}$, $C \in \mathbb{R}$.
($\bar{v}$ = velocity at ∞, C = circulation around obstacle).

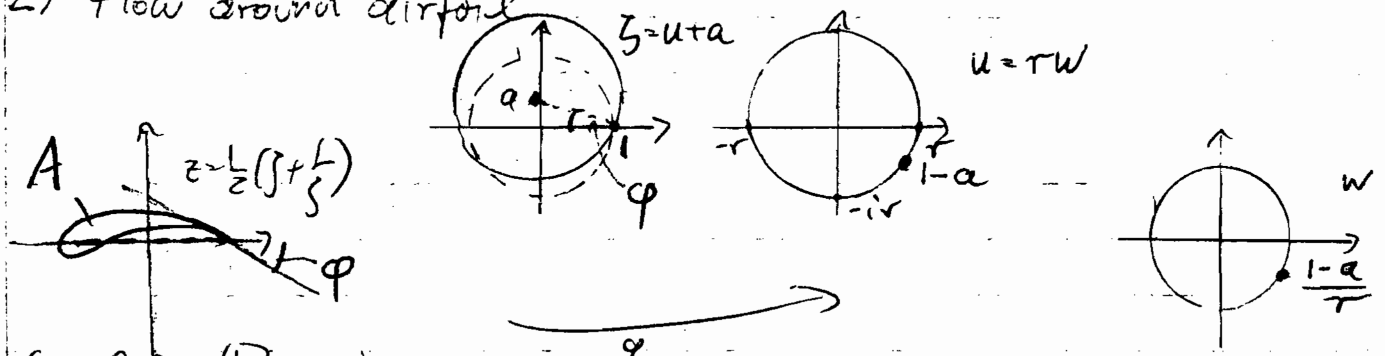
We proved, using Schwarz reflection principle, that these are the only possible flows!
(assuming $V(z) \to v$ as $z \to \infty$)

* and simply connected!

Note: We assumed that the velocity at ∞ is real ("horizontal"). Replace z by $e^{i\varphi}z$ to get velocity $e^{i\varphi}v$ at ∞.

Then, "translating back" to the w-plane $\mathbb{C}-[-1,1]$, one gets some interesting non obvious flows.

2) Flow around airfoil



($\varphi>0$ small)

$g(z)=\frac{1}{r}\left(z+\sqrt{z^2-1}-a\right)$ $\qquad$ $a=1-re^{-i\varphi}$. Condition: $\operatorname{Re} a \neq 0$.

Each flow $F(w)=v\left(w+\frac{1}{w}\right)+\frac{C}{2\pi i}\log w$, $f(w)=v\left(1-\frac{1}{w^2}\right)+\frac{C}{2\pi i}\frac{1}{w}$

induces a flow in the z-plane around A with velocity $\frac{2}{r}v$ at infinity since $g'(z)\to\frac{2}{r}$ as $z\to\infty$. (Plug in $g(z)$ for w).

Stagnation points: Points where the velocity is zero.
In the w-plane, $f(w)=0 \iff w_s=i\frac{C}{4\pi v}\pm\sqrt{1-\left(\frac{C}{4\pi v}\right)^2}$

If $\left|\frac{C}{4\pi v}\right|\le 1$, there is a unique $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$ with $\sin\theta=\frac{C}{4\pi v}$, and then

$$w_s=e^{i\theta} \text{ or } e^{i(\pi-\theta)}.$$

For $\left|\frac{C}{4\pi v}\right|=1$ there is only one stagnation point, for $\left|\frac{C}{4\pi v}\right|>1$ there is one outside the obstacle and one inside; only the one outside matters.

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The velocity around the airfoil A is determined by
$$f_1(z) = f(g(z))\, g'(z).$$

Since $g'(1) = \infty$, this is finite at $z=1$ only if $f(g(z)) = 0$, i.e.
$$f\left(\frac{1-a}{r}\right) = 0, \quad \text{i.e.} \; -\theta = \varphi \quad \text{or} \quad \boxed{\sin\varphi = -\frac{C}{4\pi v}}$$

<u>Joukowski hypothesis</u>: The flow around A adjusts itself such that the velocity at the "trailing edge" $z=1$ is finite.

This is an experimental fact, does not follow from the theory, although it is made plausible by the theory.

We assume $v > 0$, therefore "trailing edge".

<u>Conclusion</u>. The circulation C around A is uniquely determined by the Joukowski hypothesis. It is negative.

<u>Force exerted by a flow on the obstacle $\mathcal{O}$</u>:

<u>Blasius's theorem</u>: Force $= \mathcal{F} = -i\frac{\rho}{2} \overline{\int_{\partial\mathcal{O}} f^2(z)\,dz}$

where ρ = density of the fluid.

Obtained as follows: The force on a piece ds of the boundary is $p\,ds$ in absolute value, where p = pressure, and the direction is $-N$, normal towards the obstacle, so
$$\mathcal{F} = -\int_{\partial B} p N\,ds.$$

Bernoulli's law gives a quadratic dependence of p on the velocity $|f|$:
$$p = p_0 - \frac{\rho}{2}|f|^2, \quad p_0 \text{ constant},$$
and a short calculation yields the formula.

If the circulation is known, this can be simplified:

<u>Kutta-Joukowski theorem:</u> If $V(z) \to v \in \mathbb{C}$ as $z \to \infty$ then

$$\mathcal{F} = -i\rho v \cdot C.$$

(proof: write out Laurent series for $f(z)$ and use residue theorem)

<u>Note</u>: a) $\mathcal{F}$ is perpendicular to v ! (no "drag")
b) $\mathcal{F} = 0$ if $C = 0$.
c) $\mathcal{F}$ points upward for the airfoil (so planes can fly).

<u>Remark on uniqueness of the flow around A:</u> For every obstacle $\mathcal{O}$ (compact, simply connected) there is a unique number $\rho \geq 0$ and conformal map

$$g: \mathbb{C} - \mathcal{O} \longrightarrow \mathbb{C} - \{|z| \leq \rho\}$$

with Laurent series $g(z) = z + a_0 + \frac{a_1}{z} + \dots$, i.e. $g'(z) \to 1$ as $z \to \infty$.

This follows easily from the Riemann mapping theorem. The uniqueness part is an easy consequence of problem 4b) on PS 7.

Using the uniqueness of flows around a circular obstacle, a simple consideration shows that this implies uniqueness of the flow around $\mathcal{O}$ for given C and v

<u>Note on physical interpretation:</u> The model used here becomes very inaccurate for higher speeds v. For example, friction will then play a larger role and create turbulence. This phenomenon is more difficult to study.

<u>Some literature on Fluid Dynamics</u>

D.J. Acheson: Elementary Fluid Dynamics (nice and short, mathematically a little impre… Oxford 1990

Chorin, Marsden: A mathematical introduction to fluid mechanics, Springer 79

Darrozes, François: Mécanique des Fluides Incompressibles, Springer 82

For the whole course: P. Henrici: Applied and Computational Complex Analysis, Vol. I, Wiley 74