The Free Boundary Problem in the Optimization of Composite Membranes

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Abstract

In this paper, continuing our earlier article [CGIKO], we study qualitative properties of solutions of a certain eigenvalue optimization problem. Especially we focus on the study of the free boundary of our optimal solutions on general domains.

1 Introduction and Summary of results

In this note, we study some qualitative properties of solutions of a certain eigenvalue optimization problem. Our note is a continuation and also a summarization of the key results of our earlier article [CGIKO]. Our problem can be stated in physical terms as :

Problem(P) Build a body of a prescribed shape out of given materials of varying densities, in such a way that the body has a prescribed mass and with the property that the fundamental frequency of the resulting membrane (with fixed boundary) is lowest possible.

The physical problem can be re-formulated as a more general mathematical problem. More precisely, we are given $\Omega \subset \mathbf{R}^n$, a bounded domain with Lipschitz boundaries and numbers $\alpha > 0, A \in [0, |\Omega|]$ (with $|\cdot|$ denoting volume). For any measurable set $D \subset \Omega$, let χ_D be the characteristic function and $\lambda_\Omega(\alpha, D)$ the lowest eigenvalue λ of the problem,

$$\begin{aligned} -\Delta u + \alpha\chi_D u &= \lambda u \quad \text{on} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1}$$

Define,

$$\Lambda_\Omega(\alpha, A) = \inf_{D\subset\Omega, |D|=A} \lambda_\Omega(\alpha, D). \tag{2}$$

Any minimizer for (2) will be called an optimal configuration for the data (Ω, α, A). If D is an optimal configuration and $u = u_{\alpha,D}$ satisfies (1), then $(u_{\alpha,D}, D)$ will be called an optimal pair (or solution). The mathematical problem then reads,

Problem (M) Study existence, uniqueness and qualitative properties of optimal pairs.

We will hereon always work under the nomalization

$$\int_\Omega u^2 = 1, \;\; u \geq 0. \tag{3}$$

Furthermore, changing D by sets of measure zero does not affect $\lambda_\Omega(\alpha, D)$ or u, thus sets D that differ by sets of measure zero will be said to be equal.

A basic tool that we use to analyse our problem is a variational characterization of eigenvalues, precisely,

$$\lambda_\Omega(\alpha, D) = \inf_{u\in H_0^1(\Omega)} R_\Omega(u, D), \quad R_\Omega(u, D) = \frac{\int_\Omega |\nabla u|^2 + \alpha\int_\Omega \chi_D u^2}{\int_\Omega u^2}.$$

The minimizer u is well known to exist and is an eigenfunction. Thus, for $\Lambda_\Omega(\alpha, A)$ we have

$$\Lambda_\Omega(\alpha, A) = \inf_{u\in H_0^1(\Omega), |D|=A} R_\Omega(u, D).$$

The theorem that follows is basic to the questions we hope to treat in this paper. The proof of this theorem is to be found in [CGIKO]. To state our theorem we will need to introduce some notation. First, we will consistently use the notation $\{u = t\}$ for $\{x \in \Omega; u(x) = t\}$, and $\{u \leq t\}$ for $\{x \in \Omega; u(x) \leq t\}$.

Theorem 1 ([CGIKO]) *For any $\alpha > 0$ and $A \in [0, |\Omega|]$, there exists an optimal pair. Moreover any optimal pair $(u_{\alpha,D}, D)$ has the following properties:*

(a) $u_{\alpha,D} \in C^{1,\delta}(\Omega) \cap W^{2,2}(\Omega) \cap C^\gamma(\overline{\Omega})$ for every $\delta < 1$ and some $\gamma > 0$.

(b) D is a sub-level set of $u_{\alpha,D}$, i.e. there exists $t > 0$ such that

$$D = \{x \in \Omega; u_{\alpha,D}(x) \leq t\}.$$

(c) The value of α for which $\Lambda_\Omega(\alpha, A) = \alpha$, is unique.
(d) Every level set $\{u_{\alpha,D} = s\}, s \neq t$ has measure zero. If in addtion $\Lambda_\Omega(\alpha, A) \neq \alpha$, the free boundary, $\mathcal{F} = \{u_{\alpha,D} = t\}$ has measure zero.

We use the notation $\overline{\alpha}_\Omega(A)$ for the unique value of α in part (c) above, that is,

$$\overline{\alpha}_\Omega(A) = \Lambda_\Omega(\overline{\alpha}_\Omega(A), A) \tag{4}$$

We also see right away that our problem to determine an optimal pair $(u_{\alpha,D}, D)$ which seemed linear is now a non-linear problem,

$$\begin{aligned} -\Delta u_{\alpha,D} + \alpha\chi_{\{u_{\alpha,D}\leq t\}} u_{\alpha,D} &= \Lambda_\Omega(\alpha, A) u_{\alpha,D} \quad \text{on} \quad \Omega \\ u_{\alpha,D} &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{5}$$

Another important remark is that because any optimal configuration D is a sub-level set and $u_{\alpha,D} = 0$ on $\partial\Omega$, the set D will always contain a tubular neighborhood of $\partial\Omega$, i.e. D always contains a boundary layer.

For notational convenience, from now on we will drop the subscript Ω and write $\Lambda(\alpha, A)$ for $\Lambda_\Omega(\alpha, A)$, and use the simpler notation u for $u_{\alpha,D}$, in situations where no confusion arises. $\|\cdot\|_\infty$ will denote the supremum norm in $L^\infty(\Omega)$ and $\|\cdot\|_2$ the norm in $L^2(\Omega)$.

The main focus in this paper will be on the free boundary $\{u = t\}$ on a general domain Ω.

Before we state the theorems that we prove in this paper, we continue summarizing some of the salient results of [CGIKO]. It is proved there that problem (M) generalizes problem (P). In fact for $\alpha \leq \overline{\alpha}_\Omega(A)$, the solutions of problem (M) and (P) are in one to one correspondence. Another natural questions that arises is if D inherits natural symmetries that Ω possesses, and if given Ω, does there exist a unique optimal configuration D. The answer is negative on general domains unless Ω has very strong topological restrictions. Symmetrization and rearrangement invariant integral methods allow one to prove the next theorem.

Theorem 2 ([CGIKO]) *Assume Ω is symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$. That is for each fixed $x' = (x_2, \cdots, x_n)$ the set*

$$\{x_1 : (x_1, x') \in \Omega\}$$

is either empty or an interval of the form $(-c, c)$. Then any optimal solution (u, D) is symmetric with respect to the hyperplane $\{x_1 = 0\}$ and u is decreasing in x_1 for $x_1 \geq 0$.

Theorem 2 implies the next corollary, which is the only uniqueness result proved in [CGIKO].

Corollary 1 *Let $\Omega = \{|x| < 1\}$ be the ball. Then the optimal configuration is unique for any α, A and furthermore D is an annular region,*

$$D = \{x; r(A) < |x| < 1\}.$$

On other domains we encounter the phenomenon of symmetry breaking. Specifically in [CGIKO] we show symmetry breaking phenomena on annular domains in $\mathbf{R}^2$ and on dumbbell shaped domains. We have

Theorem 3 ([CGIKO]) *Fix any $\alpha > 0$ and $\delta \in (0, 1)$. Let $\Omega_a = \{x \in \mathbf{R}^2; a < |x| < a + 1\}$. Then there exists $a_0 = a_0(\alpha, \delta)$, such that whenever $a > a_0$ and D is an optimal configuration for Ω_a with parameters α and $A = \delta|\Omega_a|$, then D is not rotationally symmetric.*

Because Ω_a is rotationally invariant, Theorem 3 implies that there are infinitely many choices for the optimal configuration D on annuli. On dumbbell shaped domains we have, in addition to symmetry breaking, some extra information on D. We define the dumbbell shaped domain Ω_h by

$$\Omega_h = B_1((-2, 0)) \cup B_1((2, 0)) \cup ((-2, 2) \times (-h, h)), \tag{6}$$

where $B_r(p) = \{x \in \mathbf{R}^2; |x - p| < r\}$. We call the disks $B_r(p)$, the lobes of the dumbbell and the strip $(-2, 2) \times (-h, h)$ the handle. We have

Theorem 4 ([CGIKO]) *For any given $\alpha > 0$ and $A \in (0, 2\pi)$, there exists $h_0 > 0$ such that for domains Ω_h of (6) with $h < h_0$,*
(a) Any optimal pair (u, D) is not symmetric with respect to the x_2-axis.
(b) If $A > \pi$, then for any optimal pair (u, D), D^c is totally contained in one of the lobes $B_1((\pm 2, 0))$.

We end our summary of results from [CGIKO] with a theorem on convex domains.

Theorem 5 ([CGIKO]) *Suppose Ω is convex and has a smooth boundary. Then there exists $\alpha_0(\Omega, A) > 0$, such that for any $\alpha < \alpha_0$ and any optimal configuration D, one has*
(a) $\partial D \cap \Omega$ is real-analytic.
(b) D^c is convex.

Theorem 5 should be compared with the basic theorem of Brascamp-Lieb [BL] that establishes the convexity of level lines of the first eigenfunction on convex domains, when $\alpha = 0$. Theorem 5 extends the result of [BL] to some values of $\alpha > 0$, but it is completely open if Theorem 5 extends to the case of all $\alpha > 0$. The proof of Theorem 5 uses Theorems 6 and 8 below, which are proved in this paper. But Theorem 5 is not used in this paper, so there is no circular reasoning.

We now turn to the results that we prove in this paper. Our focus will primarily be on general domains and in particular on the free boundary for the optimal pair (u, D). In a general domain Ω the first eigenfunction ψ (with standard L^2 normalization) that is,

$$\begin{aligned} -\Delta\psi &= \mu_1\psi \quad \text{on} \quad \Omega \\ \psi &= 0 \quad \text{on} \quad \partial\Omega, \quad \int_\Omega \psi^2 = 1, \end{aligned} \tag{7}$$

is real-analytic in the interior and this places very strong restrictions on the exceptional sets, i.e. places on the level sets of ψ, where $\nabla\psi = 0$. If we were in $\mathbf{R}^2$, the exceptional set would consist of points. In this analysis unique continuation plays a role, since we easily see that if $w = \psi_{x_i}$ then from (7), $-\Delta w = \mu_1 w$, and thus unique continuation yields some information on the zero set of w. Thus in our problem it is clear the free boundary $\{u = t\}$ on a general domain will possess an exceptional set, since ψ in general has one, but any attempt to understand the fine structure of the exceptional set, Hausdorff measure, rectifiability etc., through a unique continuation approach is difficult. The reason being, unique continuation will not apply, since u is only weakly regular. This prevents us from obtaining an equation for $w_1 = u_{x_i}$. In addition a further difficulty is that (5) is an equation of the type $-\Delta u + V(x)u = 0$, and this prevents us from obtaining a homogeneous equation that is satisfied by $w_2 = u - t$ to which we may apply unique continuation to study the level surface $\{u = t\}, t \neq 0$.

Another approach is to view our problem (5) as a perturbation in α from the problem (7). This approach suffers from the fact that we do not get additional information for large α. One may view this again as a difficulty arising from lack of continuation properties in α. Two results in this direction are proved here.

Theorem 6 *For $s \geq 0$, let $[\Omega]^s = \{\psi \leq s\}$, where ψ is the normalized first eigenfunction of problem (7). Fix $A \in [0, |\Omega|]$ and choose t_Ω such that*

$|[\Omega]^{t_\Omega}| = A$. *Then for any* $\delta > 0$*, there is* $\alpha_0 = \alpha_0(\delta, \Omega)$ *such that if* $\alpha < \alpha_0$ *and* D *is an optimal configuration for* (α, A)*, then* $|t - t_\Omega| < \delta$ *and*

$$[\Omega]^{t_\Omega - \delta} \subset D \subset [\Omega]^{t_\Omega + \delta}.$$

The basic lemma that is used to prove Theorem 6 can be used to analyse the limiting behavior as $A \to |\Omega|$. We have

Theorem 7 *Let* Ω *be a smooth bounded domain. Let* $\alpha > 0$ *be fixed. Let* ψ *be the function of (7) and let*

$$M = \max_{\Omega} \psi.$$

Then for any $\delta > 0$*, there is* $A_0 = A_0(\delta, \alpha, \Omega) < |\Omega|$*, such that whenever* $A > A_0$ *and* D *is an optimal configuration for* (α, A)*, then*

$$D^c \subset \{\psi > M - \delta\}.$$

The meaning of Theorem 6 is that the free boundary for our optimization problem, that is the set $\{u = t\}$, is "trapped" between the levels $t_\Omega - \delta$ and $t_\Omega + \delta$ of the first eigenfunction ψ for the domain Ω. However, this information is too weak to conclude anything fine about the free boundary even for small $\alpha > 0$. Theorem 7 on the other hand indicates that as $A \to |\Omega|$, D^c coalesces onto the set where ψ achieves its maximum, ψ being the first eigenfunction on Ω, see (7). Now, keeping in mind part (b) of Theorem 4, it is likely that D^c may coalesce onto a strict subset of the set where ψ achieves its maximum.

Even though we have been unable to apply unique continuation to study the free boundary for large α, it is still possible to apply the Hopf lemma [GT, Lemma 3.4] and get some information on the free boundary. The most basic result on the regularity of the free boundary is the following.

Theorem 8 *If* D *is an optimal configuration,* $x \in \partial D$ *and* $\nabla u(x) \neq 0$ *then* ∂D *is a real analytic hypersurface near* x.

Using this and a refined analysis of the set of points where the gradient vanishes, we obtain:

Theorem 9 *Let* $\alpha \geq \Lambda(\alpha, A)$*. Let* $\mathcal{F} = \{u = t\}$ *denote the free boundary set. Then there is a subset* $\mathcal{E}$ *of* $\mathcal{F}$ *such that*

(a) $\mathcal{E}$ is a G_δ set.
(b) $\mathcal{F} \setminus \mathcal{E}$ is a real-analytic, $n-1$ dimensional sub-manifold of $\mathbf{R}^n$.
(c) If moreover $\alpha > \Lambda(\alpha, A)$, then for every $x_0 \in \mathcal{F}$ and every $\epsilon > 0$, the ball $B_\epsilon(x_0)$ contains points of both $\{u > t\}$ and $\{u < t\}$.

We refer to the set $\mathcal{E}$ as the exceptional set. From the construction of the exceptional set, we will deduce further geometric information regarding the free boundary. This is the content of Proposition 2, which we do not state here (see section 2).

We now give a sufficient condition that ensures that the hypothesis of Theorem 9, $\alpha \geq \Lambda(\alpha, A)$ is fulfilled.

Proposition 1 *Let $\alpha > \mu_1(\Omega)$, where $\mu_1(\Omega)$ is the first Dirichlet eigenvalue for $-\Delta$ on Ω. Then there exists $A_0 = A_0(\alpha)$, such that $\alpha \geq \Lambda(\alpha, A)$ for all $A < A_0$. Furthermore, for $C_1 = ||\psi||_\infty^{-2}$ and for fixed $A \in (0, C_1)$, there exists α_0 such that $\alpha \geq \Lambda(\alpha, A)$ for all $\alpha \geq \alpha_0$.*

We lastly investigate the effect of the curvature of $\partial\Omega$ on the free boundary and the "thickness" of the optimal configuration. As observed earlier in the remarks after Theorem 1, D always contains a tubular neighborhood of the boundary. The theorem that follows demonstrates in the model case of an annulus in $\mathbf{R}^2$, that at places where $\partial\Omega$ has large "negative" curvature, one finds that D is "thin". To state our result we will set up some notation. Let

$$\Omega_\epsilon = \{x \in \mathbf{R}^2; \epsilon < |x| < 1\}, \quad B = \{x \in \mathbf{R}^2; |x| < 1\}.$$

For fixed ϵ_0, let $A < \pi(1 - \epsilon_0^2)$. For $\epsilon < \epsilon_0$ and any fixed $\alpha > 0$, let (u_ϵ, D_ϵ) denote the optimal pair for Ω_ϵ, with lowest eigenvalue $\Lambda_{\Omega_\epsilon}(\alpha, A)$, with constraint $|D_\epsilon| = A$. We now claim that for $\mu_1(B)$, the first Dirichlet eigenvalue of the unit disk, we have

$$\Lambda_{\Omega_\epsilon}(\alpha, A) > \mu_1(B). \tag{8}$$

Since $\Omega_\epsilon \subset B$, using (u_ϵ, D_ϵ) as a trial pair in the variational characterization for $\Lambda_{\Omega_\epsilon}(\alpha, A)$, we have

$$\Lambda_{\Omega_\epsilon}(\alpha, A) = \int_{\Omega_\epsilon} |\nabla u_\epsilon|^2 + \alpha \int_{\Omega_\epsilon} \chi_{D_\epsilon} u_\epsilon^2 > \int_{\Omega_\epsilon} |\nabla u_\epsilon|^2 \geq \mu_1(B).$$

This establishes, (8). As a consequence of (8), imposing the hypothesis $\alpha \leq \mu_1(B)$, ensures that for every $\epsilon \geq 0$, $\alpha < \overline{\alpha}_{\Omega_\epsilon}(A)$ (see the definition (4)).

Next, if $\alpha < \overline{\alpha}_{\Omega_\epsilon}(A)$, and if D_ϵ is radially distributed, Theorem 2 of [CGIKO], yields that D_ϵ has the form,

$$D_\epsilon = \{x \in \mathbf{R}^2; \epsilon < |x| < r_\epsilon \quad \text{or} \qquad R_\epsilon < |x| < 1\} \tag{9}$$

for some r_ϵ, R_ϵ, $\epsilon < r_\epsilon < R_\epsilon < 1$. Thus if $\alpha \leq \mu_1(B)$ we may assume that if D_ϵ is radially distributed, then the set D_ϵ has the form described by (9) for every $\epsilon > 0$. We have

Theorem 10 *Assume $\alpha \leq \mu_1(B)$, and $A > 0$ is prescribed. Given this choice of α and A, let (u_ϵ, D_ϵ) be an optimal pair with $|D_\epsilon| = A$. Assume D_ϵ is radially distributed and hence of the form (9). Then,*

$$\limsup_{\epsilon \to 0} r_\epsilon = 0.$$

Thus the implication is that D_ϵ thins out on the boundary layer in contact with $\{x; |x| = \epsilon\}$, the inner boundary of $\partial\Omega_\epsilon$. As $\epsilon \to 0$, the curvature of $\{x; |x| = \epsilon\}$ is increasing and negative as seen from Ω_ϵ. Thus in this model case one may conclude that diam (D) is small on parts of D which are in contact with pieces of $\partial\Omega$, where the curvature of $\partial\Omega$ is large and where $\partial\Omega$ is concave as seen from Ω.

The paper [CGIKO] discusses the historical antecedents of this problem and the interested reader is referred to this paper for a discussion. Furthermore the optimization problem discussed here, is amenable to being modelled on a computer. Details of the numerical simulation are available in [CGIKO] and the interested reader may find the shape of the optimal configuration in many types of domains obtained by these numerical studies.

Finally, we thank the referee for bringing to our attention the results in the work of Alt and Caffarelli [AC] and also suggesting Problem 7 in the section on open problems.

2 Proofs of the Theorems

In this section we prove Theorems 6-10 and Proposition 1. We begin with the proof of Theorem 6. We need a preparatory Lemma, that is well-known in perturbation theory and in the Physics literature [B, Appendix 39, p. 469]. We want a slightly more precise form, though the technique of proof is standard and the basic idea follows from [B].

Lemma 1 *Fix $D \subset \Omega$. Let $u_{\alpha,D}$ be the (positive, L^2 normalized) first eigenfunction of $-\Delta+\alpha\chi_D$ with eigenvalue $\lambda(\alpha, D)$. Then there is a constant $C = C_\Omega$ such that for $0 \leq \alpha \leq 1$ (ψ, μ_1 refers to (7)),*
(a) $0 \leq \lambda(\alpha, D) - \mu_1 \leq \alpha$,
(b) $\|u_{\alpha,D} - \psi\|_{H^2(\Omega)} \leq C\alpha$,
(c) $\|u_{\alpha,D} - \psi\|_{L^\infty(\Omega)} \leq C\alpha$.

Proof of Lemma 1: Recall we have set $u_{\alpha,D} = u$. Note

$$\begin{aligned} \lambda(\alpha, D) \leq \int_\Omega (|\nabla\psi|^2 + \alpha\chi_D\psi^2) &\leq \mu_1 + \alpha\int_\Omega \psi^2 \\ &\leq \mu_1 + \alpha. \end{aligned}$$

Thus, $\lambda(\alpha, D) - \mu_1 \leq \alpha$. Next,

$$\lambda(\alpha, D) = \int_\Omega (|\nabla u|^2 + \alpha\chi_D u^2) \geq \mu_1 + \alpha\int_\Omega \chi_D u^2.$$

Thus, $\lambda(\alpha, D) - \mu_1 \geq 0$, and we have (a). To prove (b), let $\{\psi_k\}_{k=1}^\infty$ be an orthogonal basis of eigenfunction of $-\Delta$ with Dirichlet boundary conditions (Note $\psi_1 = \psi$ of problem (6)). The corresponding eigenvalues will be denoted by $\{\mu_k\}_{k=1}^\infty$, where it is well-known that μ_1 is simple. Expanding u, we have $u = \sum_{j=1}^\infty \beta_j\psi_j$, and thus $(-\Delta - \mu_1)u = \sum_{j=2}^\infty \beta_j(\mu_j - \mu_1)\psi_j$ and

$$\|(\Delta + \mu_1)u\|_2^2 = \sum_{j=2}^\infty \beta_j^2(\mu_j - \mu_1)^2, \tag{10}$$

where $\|\cdot\|_2$ denotes the $L^2(\Omega)$ norm. From $-\Delta u + \alpha\chi_D u = \lambda(\alpha, D)u$, we get

$$(-\Delta - \mu_1)u = (\lambda(\alpha, D) - \mu_1)u - \alpha\chi_D u.$$

Therefore applying (a), $\|(\Delta + \mu_1)u\|_2 \leq C\alpha$. Since μ_1 is simple, there exists $\delta > 0, \delta = \delta(\Omega)$, such that $\mu_j - \mu_1 \geq \delta > 0$ for $j \geq 2$. Then from (10) we get

$$\sum_{j=2}^\infty \beta_j^2 \leq \frac{C\alpha^2}{\delta^2}. \tag{11}$$

We re-write u as $u = \beta_1\psi_1 + \Psi$, and (11) gives $\|\Psi\|_2 \leq C\alpha\delta^{-1}$. Now $1 = \|u\|_2 = \beta_1^2 + \|\Psi\|_2^2$, thus

$$|\beta_1 - 1| \leq \frac{\|\Psi\|_2^2}{1 + \beta_1} \leq \frac{C\alpha^2}{\delta^2}.$$

Here we used the fact that $\beta_1 = (u, \psi_1) > 0$, because both u and ψ_1 are positive. Therefore,

$$\|u - \psi_1\|_2^2 = (\beta_1 - 1)^2 + \|\Psi\|_2^2 \leq \frac{C\alpha^2}{\delta^2} \leq C\alpha^2. \tag{12}$$

All the remaining consequences follow from (12). ¿From (6),

$$-\Delta(u - \psi_1) = (\lambda(\alpha, D) - \mu_1)u + \mu_1(u - \psi_1) - \alpha\chi_D u. \tag{13}$$

We re-write (13) as

$$-\Delta(u - \psi_1) - \mu_1(u - \psi_1) = (\lambda(\alpha, D) - \mu_1)u - \alpha\chi_D u = g.$$

Now from [GT, Theorem 8.15] again, it follows that

$$\|u - \psi_1\|_\infty \leq C\|u - \psi_1\|_2 + C\alpha.$$

Using (12) on the right side,

$$\|u - \psi_1\|_\infty \leq C\alpha,$$

which is part (c). Using $\|g\|_\infty \leq C\alpha$ and part (c), we conclude $\|\Delta(u - \psi_1)\|_\infty \leq C\alpha$, which is (b). $\square$

Theorem 6 and 7 are now consequences of Lemma 1.

Proof of Theorem 6: Apply Lemma 1 (c) to the optimal pair (u, D). Choose $\alpha_0 = \delta/(2C)$, so that $\|u - \psi_1\|_\infty \leq \delta/2$ for $\alpha \leq \alpha_0$. From Theorem 1, if $x \in D, u(x) \leq t$, and so $\psi(x) \leq t + \delta/2$ and hence $D \subset [\Omega]^{t+\delta/2}$. In a similar way we establish $[\Omega]^{t-\delta/2} \subset D$. Thus we have

$$[\Omega]^{t-\delta/2} \subset D \subset [\Omega]^{t+\delta/2}.$$

From the statement above, we get $|[\Omega]^{t-\delta/2}| \leq A \leq |[\Omega]^{t+\delta/2}|$, and thus by continuity, there exists t_Ω such that $A = |[\Omega]^{t_\Omega}|$, and $|t_\Omega - t| < \delta/2$. From this assertion the assertions of Theorem 6 follow. $\square$

Proof of Theorem 7: We begin by showing that a slight modification of the proof of Lemma 1 yields,

$$\|u - \psi_1\|_\infty \leq C_{\alpha,\Omega}(|\Omega| - A). \tag{14}$$

We show first,

$$|\mu_1 - (\Lambda(\alpha, A) - \alpha)| \leq C_{\alpha,\Omega}(|\Omega| - A). \tag{15}$$

We re-write our equation for u as

$$-\Delta u - \alpha\chi_{D^c}u = (\Lambda - \alpha)u, \quad \Lambda = \Lambda(\alpha, A). \tag{16}$$

From (16) we have

$$\mu_1 - \alpha\int_\Omega \chi_{D^c}u^2 \leq \int_\Omega |\nabla u|^2 - \alpha\int_\Omega \chi_{D^c}u^2 = \Lambda - \alpha.$$

Thus,

$$\mu_1 - (\Lambda - \alpha) \leq \alpha\int_\Omega \chi_{D^c}u^2 \leq C|D^c| = C(|\Omega| - A).$$

Next, we have

$$\Lambda - \alpha \leq \int_\Omega (|\nabla\psi|^2 - \alpha\chi_{D^c}\psi^2) \leq \mu_1 - \alpha\int_\Omega \chi_{D^c}\psi^2$$

which yields $0 \leq \mu_1 - (\Lambda - \alpha)$. The assertion (15) follows. Using (15) and the equation (16) we can proceed as in Lemma 1 to obtain (14). If $|\Omega| - A < \delta/(2C_{\alpha,\Omega})$, from (14) we see $[\Omega]^{t-\delta/2} \subset D = \{u \leq t\} \subset [\Omega]^{t+\delta/2}$. Since $|D| = A$, we have $A \leq |[\Omega]^{t+\delta/2}| = |\{\psi \leq t + \delta/2\}|$. Thus if in addition $A > A_0$, we can arrange the situation so as to have $M - t \leq \delta/2$. So $[\Omega]^{M-\delta} \subset [\Omega]^{t-\delta/2} \subset D$. The conclusion $[\Omega]^{M-\delta} \subset D$ is readily seen to be equivalent to the assertion made in Theorem 7. $\square$

We need some preparatory lemmas before we prove Theorem 9. As usual $u = u_{\alpha,D}$ will denote the solution to our optimization problem and consequently u will satisfy (5).

Lemma 2 *(a) Fix any $\alpha > 0$. Let the free boundary set be $\mathcal{F}$, $\mathcal{F} = \{u = t\}$. We let $D^+ = \{x; u(x) > t\}$. Assume D^+ satisfies an interior sphere condition with respect to $x_0 \in \mathcal{F}$, that is there exists a ball B, $B \subset D^+$ and $\partial B \cap \mathcal{F} = \{x_0\}$. Then $|\nabla u(x_0)| \neq 0$.*
(b) Let $D^- = \{x; u(x) < t\}$ (in the situation $\alpha \neq \overline{\alpha}_\Omega(A)$, $D^- = D$). Assume that $\alpha \geq \Lambda_\Omega(\alpha, A)$. Let D^- satisfy an interior sphere condition with respect to $x_0 \in \mathcal{F}$, that is there exists a ball B, $B \subset D^-$ and $\partial B \cap \mathcal{F} = \{x_0\}$. Then $|\nabla u(x_0)| \neq 0$.

Proof of Lemma 2: The proof of both parts of our lemma rely on Hopf's lemma [GT, Lemma 3.4]. We prove (a). Set $\phi = t - u$. We observe that in the ball $B \subset D^+$, u satisfies from (5)

$$-\Delta u = \Lambda(\alpha, A)u.$$

Thus $\Delta\phi = \Lambda(\alpha, A)u \geq 0$ in B, and $\phi < 0$ on B with $\phi(x_0) = 0$. Hopf's lemma then yields $|\nabla\phi(x_0)| = |\nabla u(x_0)| \neq 0$.

(b). The proof of this part is similar to part (a). Since $B \subset D^-$, from (5) we see on B we have

$$-\Delta u + \alpha u = \Lambda u.$$

Since $\alpha \geq \Lambda$, we easily see $\Delta u \geq 0$ on B. Thus on $B \subset D^-$ we have $\Delta\phi \leq 0, \phi > 0$ on B and $\phi(x_0) = 0$. Thus Hopf's lemma again yields $|\nabla\phi(x_0)| = |\nabla u(x_0)| \neq 0$. □

Lemma 3 *Let $h(\eta, p), \eta \in \mathbf{R}, p = (p_1, \cdots, p_n) \in \mathbf{R}^n$ be a locally bounded function. Let $w \in C^1(\Omega)$ satisfy*

$$\Delta w = h(w, \nabla w). \tag{17}$$

Assume furthermore h is smooth in the variable p. Assume at the point $x_0 \in \Omega$, $\nabla w(x_0) \neq 0$. Then there exists a ball B, $x_0 \in B$, such that the set,

$$\{x \in B; w(x) = w(x_0)\} = \mathcal{S}$$

is a smooth hypersurface of $\mathbf{R}^n$. If in addition h is real-analytic in the variable p, the set $\mathcal{S}$ is also real-analytic.

Proof of Lemma 3: Since $h(w, \nabla w)$ is locally bounded, it follows by elliptic estimates that $w \in C^{1,\gamma} \cap W^{2,s}$, $s < \infty$. Thus by the implicit function theorem, since $\nabla w(x_0) \neq 0$, we conclude that $\mathcal{S}$ is a $C^{1,\gamma}$ hypersurface for all $\gamma < 1$. Now we shall improve the regularity of the hypersurface $\mathcal{S}$. By a rotation of coordinates we may assume $w_{x_i}(x_0) = 0, i = 1, \cdots, n-1$ and $w_{x_n}(x_0) \neq 0$. Let $x' = (x_1, \cdots, x_{n-1})$ and consider the map,

$$\Psi : B \to \mathbf{R}^n, \quad \Psi(x', x_n) = (x', w(x', x_n)).$$

We denote points in the image of Ψ, by $y = (y', y_n)$ where $y' = (y_1, \cdots, y_{n-1})$ and $y_n = w(x', x_n)$. Let Ψ^{-1} denote the inverse map to Ψ, which will exist if B is picked to be small. We have,

$$\Psi^{-1}(y', y_n) = (y', F(y', y_n)).$$

Now,

$$F(x', w(x', x_n)) = x_n.$$

Differentiating the equation above we get the equations,

$$F_{y_i} + w_{x_i} F_{y_n} = 0, i = 1, \cdots, n-1, \quad \text{and} \quad F_{y_n} w_{x_n} = 1. \tag{18}$$

By the chain rule,

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i} + w_{x_i} \frac{\partial}{\partial y_n}, \qquad \frac{\partial}{\partial x_n} = w_{x_n} \frac{\partial}{\partial y_n}.$$

From (18), $w_{x_i} = -F_{y_i}/F_{y_n}, i = 1, \cdots, n-1$ and $w_{x_n} = 1/F_{y_n}$. Thus,

$$\begin{aligned}
\Delta w &= \sum_{i=1}^{n-1} \Big(\frac{\partial}{\partial y_i} + w_{x_i} \frac{\partial}{\partial y_n} \Big) \Big(\frac{-F_{y_i}}{F_{y_n}} \Big) + w_{x_n} \frac{\partial}{\partial y_n} \Big(\frac{1}{F_{y_n}} \Big) \\
&= \sum_{i=1}^{n-1} \Big(\frac{\partial}{\partial y_i} - \frac{F_{y_i}}{F_{y_n}} \frac{\partial}{\partial y_n} \Big) \Big(\frac{-F_{y_i}}{F_{y_n}} \Big) + \frac{1}{F_{y_n}} \frac{\partial}{\partial y_n} \Big(\frac{1}{F_{y_n}} \Big) \\
&= LF.
\end{aligned}$$

Next we freeze the coefficients of L at x_0. Since $w_{x_i}(x_0) = 0, i = 1, \cdots, n-1$ and $w_{x_n}(x_0) = a \neq 0$, we see from (18), $F_{y_i} = 0$ and $F_{y_n} = a^{-1}$ at $\Psi(x_0)$. At $y_0 = \Psi(x_0)$, we have

$$\begin{aligned}
LF &= -\frac{1}{F_{y_n}} \sum_{i=1}^{n-1} F_{y_i y_i} - \frac{1}{F_{y_n}^3} F_{y_n y_n} \\
&= -\frac{1}{F_{y_n}} \Big[\sum_{i=1}^{n-1} F_{y_i y_i} + \frac{1}{F_{y_n}^2} F_{y_n y_n} \Big].
\end{aligned}$$

Thus if B is picked suitably small, LF is an elliptic, quasi-linear operator. Since w satisfies (17), F satisfies

$$LF = h\Big(y_n, \mathcal{A}(\frac{-F_{y_i}}{F_{y_n}}, \frac{1}{F_{y_n}}) \Big) \tag{19}$$

where $\mathcal{A}$ is a fixed matrix in $O(n)$, the rotation group, associated with our rotation of coordinates, and $\mathcal{A}(z_i, z_n)$, denotes the product of $\mathcal{A}$ and the vector $(z_1, \cdots, z_n), (z_i) = (z_1, \cdots, z_{n-1})$. Since $w \in C^{1,\gamma} \cap W^{2,s}$ for all $\gamma < 1, s < \infty$, we see the coefficients of L are in $C^{0,\gamma} \cap W^{1,s}$, $s < \infty$. Differentiating equation (19) in any of the variables $y_i, i \neq n$, we may apply a standard bootstrap argument using well-known elliptic estimates for example [GT, Theorem 9.11] to conclude from the fact that h is smooth in the variable p, that F is smooth in any of the variables $y_i, i = 1, \cdots, n-1$. If one has in addition that h is real-analytic in the variables p, then applying the results of Morrey [M, Theorem C] or Friedman [Fr, Theorems 1, 4], we can conclude that F is real-analytic in the variables $y_i, i = 1, \cdots, n-1$. Now the defining equation for $\mathcal{S}$ is given by $x_n = F(x_1, \cdots, x_{n-1}, w(x_0))$. Since F is smooth (real-analytic) depending on the regularity of h in the p variables, it follows that $\mathcal{S}$ is a smooth (real-analytic) manifold, depending on the fact that h is smooth (real-analytic) in the p variables. □

We are now ready to prove Theorems 8 and 9.

Proof of Theorem 8: The PDE satisfied by u, that is (5), can be written as

$$\Delta u = h(u),$$

where $h(\eta) = -\Lambda(\alpha, A)\eta + \alpha\chi_G(\eta)\eta, G = \{\eta; \eta \leq t\}$. Thus $h \in L^\infty_{loc}(\mathbf{R})$ and the hypotheses of Lemma 3 apply to u. The theorem follows. □

Proof of Theorem 9: We construct the exceptional set $\mathcal{E}$. Let

$$K_n = \{x \in \Omega;\ \text{distance } (x, \mathcal{F}) = \frac{1}{n}\}.$$

Now define

$$F_n = \{x \in \mathcal{F};\ \text{distance } (x, K_n) = \frac{1}{n}\}.$$

The sets K_n and F_n are closed sets for all $n \in \mathbf{N}$. We define the exceptional set by

$$\mathcal{E} = \mathcal{F} \setminus (\cup_{n=1}^\infty F_n).$$

Since each set $\mathcal{F} \cap F_n^c$ is open in $\mathcal{F}$, $\mathcal{E}$ is a G_δ set. This proves Theorem 9 (a).

We shall now prove that for each point $x_0 \in \mathcal{F} \setminus \mathcal{E}$, we can construct either an interior ball $B \subset D^+$, such that $\partial B \cap \mathcal{F} = \{x_0\}$ or an interior ball $B \subset D^-$, such that $\partial B \cap \mathcal{F} = \{x_0\}$. Since we are assuming $\alpha \geq \Lambda(\alpha, A)$, Lemma 2 ensures that for each $x_0 \in \mathcal{F} \setminus \mathcal{E}$, $\nabla u(x_0) \neq 0$. By Theorem 8, there exists a ball B_0, centered at x_0, such that, $\mathcal{F} \cap B_0$ is a real analytic manifold. So we now verify our claims regarding the interior spheres. Fix a point $x_0 \in \mathcal{F} \setminus \mathcal{E}$. Then $x_0 \in F_n$ for some n. Let $z_0 \in K_n$, such that $|x_0 - z_0| = 1/n$. We claim the ball $B_{1/n}(z_0)$ is totally contained in D^+ or D^-. Suppose there are points $z_1 \in D^+$ and $z_2 \in D^-$ in $\overline{B_{1/n}(z_0)}$. Then by the continuity of u, the line segment joining z_1 to z_2 which also lies inside $B_{1/n}(z_0)$ will contain a point of $\mathcal{F}$. Thus distance $(z_0, \mathcal{F}) < 1/n$ and hence $z_0 \notin K_n$. Now pick a ball $B', B' \subset B_{1/n}(z_0)$, and B' centered along the radius joining x_0 to z_0 and with $\partial B' \cap \partial B_{1/n}(z_0) = \{x_0\}$. Clearly $\overline{B' \setminus \{x_0\}}$ is contained either in D^+ or D^- and $\overline{\partial B'} \cap \mathcal{F} = \{x_0\}$. The hypotheses of Lemma 2 are fulfilled. Theorem 9 (b) now follows.

To prove Theorem 9 (c), we can argue via the strong maximum principle. Since $\alpha > \Lambda(\alpha, A)$, applying Theorem 1 (d), we see that $D = \{u < t\}$. If for some $\epsilon > 0$, the ball $B_\epsilon(x_0)$, $x_0 \in \mathcal{F}$ contains no points of D^- and only points $\{u \geq t\}$, then on $B_\epsilon(x_0)$, u satisfies

$$-\Delta u = \Lambda(\alpha, A) u \geq 0.$$

So $\Delta u \leq 0$ on $B_\epsilon(x_0)$ and $u \geq t$ on $B_\epsilon(x_0)$ with $u(x_0) = t$ which is a contradiction. We may argue as in Lemma 2, part (b) and show on $B_\epsilon(x_0)$ there are also points of D^+ for every $\epsilon > 0$. $\square$

To discuss further geometric properties of $\mathcal{E}$, we introduce,

$$K_\epsilon = \{x \in \Omega;\ \text{distance } (x, \mathcal{F}) = \epsilon\},$$

$$F_\epsilon = \{x \in \mathcal{F};\ \text{distance } (x, K_\epsilon) = \epsilon\}.$$

We have

Proposition 2 *For every $\alpha > 0$,*
(a) $F_{\epsilon_1} \subset F_{\epsilon_2}$ for $\epsilon_1 > \epsilon_2$.

As a consequence of (a), we have
(b) $\cup_{n \geq 1} F_n = \cup_{\epsilon > 0} F_\epsilon$, $F_n \subset F_m$ for $m > n$, and $\mathcal{E} = \mathcal{F} \setminus \cup_{\epsilon > 0} F_\epsilon$.
(c) If $\alpha > \Lambda(\alpha, A)$, then for every point $x_0 \in \mathcal{E}$, any ball $B_\epsilon(x_0)$ contains points $y_+ \in D^+$ and $y_- \in D^-$ such that distance $(y_\pm, \mathcal{F}) < |y_\pm - x_0|$.
(d) Furthermore, if $z_0 \in \mathcal{F}$, such that $|y_+ - z_0| =$ distance $(y_+, \mathcal{F})$, then D^+ satisfies an interior sphere condition in the sense of Lemma 2 with respect to z_0. Thus by the proof of Theorem 9, there is a neighborhood B of z_0, such that $\{x \in B : u(x) = u(z_0)\}$ is a real-analytic hypersurface. A similar statement holds for y_-.

The meaning of (b) is that since $\mathcal{E} = \mathcal{F} \cap (\cap_{n=1}^{\infty} F_n^c)$ and $F_n^c \supset F_m^c$ for $m > n$, the exceptional set is really a G_δ set formed by the intersection of the nested open sets F_n^c. The meaning of (c), (d) is that the behaviour of the free boundary in the neighborhood of the exceptional set at least in $\mathbf{R}^2$ is that of isolated singularities with a conical structure, with the cone having its vertex at $x_0 \in \mathcal{E}$. The cone locally divides $\mathbf{R}^2$ into at least two components, one component is contained in D^+ and the other is contained in D^-. This geometric picture is only heuristic since it still needs to be rigorously established that the component of $\mathcal{F}$ that contains z_0 also contains $x_0 \in \mathcal{E}$. Only then can we conclude there is a true conical singularity at x_0.

Proof of Proposition 2: Fix $x \in F_{\epsilon_1}$. Then by definition, one can find $z \in K_{\epsilon_1}$, such that $|z - x| = \epsilon_1$. Since $z \in K_{\epsilon_1}$, the ball $B_{\epsilon_1}(z)$ will contain no points of $\mathcal{F}$ in the interior. Now consider the radial line connecting z and x, and locate on this line a point y such that $|y - x| = \epsilon_2$. The ball $B_{\epsilon_2}(y) \subset B_{\epsilon_1}(z)$, and x is the sole point in $\mathcal{F}$ on $\partial B_{\epsilon_2}(y)$. Now by definition $y \in K_{\epsilon_2}$, and since $x \in \mathcal{F}$ and $|x - y| = \epsilon_2$, $x \in F_{\epsilon_2}$. This proves (a), and (b) is then an elementary consequence of set theory.

Next, by Theorem 9 (c) we know that the ball $B_\epsilon(x_0)$ with $x_0 \in \mathcal{E}$ (in general for any point in $\mathcal{F}$ actually), contains points $y_\pm$ in $D^\pm$ respectively. Let $|y_+ - x_0| = \delta$. We claim distance $(y_+, \mathcal{F}) < \delta$. If distance $(y_+, \mathcal{F}) \geq \delta$, then $y_+ \in K_\tau, \tau =$ distance $(y_+, \mathcal{F}), \tau \geq \delta$. Since $|y_+ - x_0| = \delta \leq \tau$, it follows that $x_o \in F_\tau$. Thus $x_0 \notin \mathcal{E}$. This proves part (c), since an argument similar to the one above takes care of the point $y_- \in D^-$.

Lastly we prove part (d). Now, $|y_+ - z_0| = \tau$, $z_0 \in \mathcal{F}$. Thus by the argument employed in Theorem 9(b) it is easily seen, that the open ball $B_\tau(y_+)$ contains only points of D^+. Again employing the argument of Theorem 9 (b), we can find a ball $B' \subset B_\tau(y_+)$ such that $\partial B' \cap \mathcal{F} = \{z_0\}$ and hence B' is the desired interior ball. $\square$

We now prove Proposition 1.

Proof of Proposition 1: Arguing as in Lemma 1(a), but now using the fact that $\|\psi\|_\infty \leq C$, [GT, Theorem 8.15], we have

$$\Lambda(\alpha, A) \leq \int_\Omega |\nabla\psi|^2 + \alpha \int_\Omega \chi_D \psi^2 \leq \mu_1 + \alpha\|\psi\|_\infty^2 A.$$

Thus,

$$\Lambda(\alpha, A) \leq \mu_1 + \alpha C_0 A, \quad C_0 = \|\psi\|_\infty^2. \tag{20}$$

Since $\alpha > \mu_1$, we can find $A_0 > 0$ such that $\alpha C_0 A_0 \leq \alpha - \mu_1$. It follows from (20) that for $A < A_0$, $\alpha \geq \Lambda(\alpha, A)$. The second part of Proposition 1 also follows from (20). Select $C_1 = C_0^{-1}$. If $A < C_1$, $C_0 A = 1 - \epsilon, \epsilon > 0$. Thus for $\alpha > \alpha_0$, $\mu_1 \leq \epsilon\alpha$, and hence by (20), $\Lambda(\alpha, A) < \alpha$. □

Proof of Theorem 10: We set $\Lambda_\epsilon = \Lambda_{\Omega_\epsilon}(\alpha, A)$, $\Lambda = \Lambda_B(\alpha, A)$. D will denote the optimal configuration for B, that is $D = \{x; r_0 < |x| < 1\}, \pi(1 - r_0^2) = A$. We also need to consider the first Dirichlet eigenvalue λ_ϵ, of the problem,

$$\begin{aligned} -\Delta g + \alpha\chi_D g &= \lambda_\epsilon g \quad \text{in} \quad \Omega_\epsilon \\ g|_{\partial\Omega_\epsilon} = 0. & \end{aligned}$$

We claim

$$0 \leq \Lambda_\epsilon - \Lambda \leq C|\log\epsilon|^{-1}. \tag{21}$$

Extending u_ϵ to $\{|x| < \epsilon\}$ by setting $u_\epsilon = 0$ on $\{|x| < \epsilon\}$ and using this extended function as a trial function with D_ϵ as a trial configuration on $B = \{|x| < 1\}$, we see $\Lambda \leq \Lambda_\epsilon$ and so we have the left side in (21). Next by Theorem 2 in Swanson [S],

$$0 \leq \lambda_\epsilon - \Lambda \leq c|\log\epsilon|^{-1}. \tag{22}$$

In fact $0 \leq \lambda_\epsilon - \Lambda$ is simply a consequence of domain monotonicity. By the variational characterization $\Lambda_\epsilon \leq \lambda_\epsilon$, and thus from (22) we easily have (21).

We now establish $\limsup_{\epsilon\to 0} r_\epsilon = 0$, by contradiction. Assume there is a sequence $\epsilon_j \searrow 0$, $\lim_{j\to\infty} r_{\epsilon_j} = \delta > 0$. Then $R_{\epsilon_j} \to b_\delta$ and since $|D_{\epsilon_j}| = A$, the limit set $\mathcal{D}_\delta$ is:

$$\mathcal{D}_\delta = \{x; |x| < \delta \quad \text{or} \quad b_\delta < |x| < 1\}, \quad |\mathcal{D}_\delta| = A.$$

We set,

$$D_{\delta,\epsilon} = \{x; \epsilon < |x| < \delta \quad \text{or} \quad b_\delta < |x| < 1\}.$$

We use the notation $\lambda_\epsilon(D_{\delta,\epsilon})$ for the first Dirichlet eigenvalue on Ω_ϵ for the problem,

$$\begin{aligned} -\Delta w \quad + \quad \alpha\chi_{D_{\delta,\epsilon}} w = \lambda_\epsilon(D_{\delta,\epsilon}) w \quad \text{on} \quad \Omega_\epsilon \\ w \quad = \quad 0 \quad \text{on} \quad \partial\Omega_\epsilon, \quad \int_{\Omega_\epsilon} w^2 = 1. \end{aligned} \tag{23}$$

$\lambda(\mathcal{D}_\delta)$ will denote the first Dirichlet eigenvalue for $-\Delta + \alpha\chi_{\mathcal{D}_\delta}$ on B. We claim,

$$|\Lambda_{\epsilon_j} - \lambda(\mathcal{D}_\delta)| \to 0 \quad \text{as} \quad \epsilon_j \to 0. \tag{24}$$

We have

$$\begin{aligned} |\Lambda_{\epsilon_j} - \lambda(\mathcal{D}_\delta)| &\leq |\Lambda_{\epsilon_j} - \lambda_{\epsilon_j}(D_{\delta,\epsilon_j})| + |\lambda_{\epsilon_j}(D_{\delta,\epsilon_j}) - \lambda(\mathcal{D}_\delta)| \\ &= J_1 + J_2. \end{aligned}$$

By using Theorem 2 in [S], one can conclude $J_2 \leq C|\log \epsilon_j|^{-1} \to 0$ as $\epsilon_j \to 0$. We now show $J_1 \to 0$. Let (u_ϵ, D_ϵ) be the optimal pair for Ω_ϵ. Then, using the eigenfunction w from (23), and the uniform bounds $\|w\|_{L^\infty(\Omega_\epsilon)} \leq C$, C independent of $\epsilon > 0$, which follows from [GT, Theorem 8.15], we have

$$\begin{aligned} \Lambda_\epsilon &\leq \int_{\Omega_\epsilon} (|\nabla w|^2 + \alpha\chi_{D_\epsilon} w^2) \\ &= \int_{\Omega_\epsilon} (|\nabla w|^2 + \alpha\chi_{D_{\delta,\epsilon}} w^2) + \alpha \int_{\Omega_\epsilon} (\chi_{D_\epsilon} - \chi_{D_{\delta,\epsilon}}) w^2 \\ &\leq \lambda_\epsilon(D_{\delta,\epsilon}) + C\alpha |D_\epsilon \bigtriangleup D_{\delta,\epsilon}|, \end{aligned}$$

where $D_\epsilon \bigtriangleup D_{\delta,\epsilon}$ is the symmetric difference of the sets $D_\epsilon, D_{\delta,\epsilon}$. Thus for the sequence ϵ_j, we easily have $|D_\epsilon \bigtriangleup D_{\delta,\epsilon}| \to 0$ as $j \to \infty$. We conclude

$$\Lambda_{\epsilon_j} \leq \lambda_{\epsilon_j}(D_{\delta,\epsilon_j}) + o(1), \quad j \to \infty.$$

Likewise using the function u_ϵ in the argument above, we have

$$\begin{aligned} \lambda_{\epsilon_j}(D_{\delta,\epsilon_j}) &\leq \Lambda_{\epsilon_j} + C\alpha |D_{\epsilon_j} \bigtriangleup D_{\delta,\epsilon_j}| \\ &\leq \Lambda_{\epsilon_j} + o(1), \quad j \to \infty. \end{aligned}$$

Thus,

$$|\lambda_{\epsilon_j}(D_{\delta,\epsilon_j}) - \Lambda_{\epsilon_j}| \to 0 \quad \text{as} \quad j \to \infty.$$

Thus $J_1 \to 0$ as $\epsilon_j \to 0$. This establishes (24). We infer from (21) and (24) that as $j \to \infty$,

$$|\Lambda - \lambda(\mathcal{D}_\delta)| \leq |\Lambda - \Lambda_{\epsilon_j}| + |\Lambda_{\epsilon_j} - \lambda(\mathcal{D}_\delta)| \to 0.$$

Hence $\Lambda = \lambda(\mathcal{D}_\delta)$. However, this contradicts Corollary 1 of our introduction, a proof of which is supplied in [CGIKO]. $\square$

3 Open Problems and Conjectures

A number of open problems and conjectures can be stated based on the numerical data in [CGIKO] and the rigorous results there and also on results proved here. We will outline some.

Problem 1 *: (Uniqueness of the optimal configuration) The only domain for which we have established the uniqueness of the optimal configuration is the ball, see Corollary 1. Is D unique if Ω is convex?*

Problem 2 *:(Continuation Problem) Theorem 5 states that on a convex domain Ω, D^c is convex for small $\alpha > 0$. Is it possible to continue along the values $\alpha > 0$, to obtain convexity of D^c for all $\alpha > 0$?*

Problem 3 *:(The free boundary on general domains) The free boundary $\mathcal{F}$ on general domains will contain an exceptional set $\mathcal{E}$ as constructed in the proof of Theorem 8. What is the Hausdorff dimension of $\mathcal{E}$? Is at least $\mathcal{F}$ a rectifiable set? One suspects $\mathcal{E}$ consists of points, where real-analytic arcs intersect if $\Omega \subset \mathbf{R}^2$.*

Problem 4 *:(Monotonicity of D) Suppose $A < A'$, then does this imply $D_{\alpha,A} \subset D_{\alpha,A'}$? If symmetry breaking occurs this statement needs to be modified. Nevertheless on domains where the optimal configuration is unique, does the above monotonicity hold?*

Problem 5 *:(Symmetry breaking on annuli) When Ω is an annulus, what is the shape of D precisely? The proof of Theorem 3 in our introduction as presented in [CGIKO] and the numerical computations presented in [CGIKO] suggest that D^c lies between two rays $\theta = 0$ and $\theta = \beta$. In fact the results in [CGIKO] suggest that, $\beta = \pi/N, N = N(\alpha, |D|/|\Omega|)$, and $N \to \infty$ as $|D| \to |\Omega|$.*

Problem 6 *:(Influence of the boundary curvature) In Theorem 10 we saw in a model case that the diameter of D is affected by the curvature of $\partial\Omega$. Investigate this phenomena on general domains Ω.*

Problem 7 *:(The reduced Free Boundary) Federer has introduced the notion of the reduced boundary of a set, [F], 4.5.5. Alt and Caffarelli in [AC] show* *that for their minimization problem, the free boundary of the minimizer is regular on the reduced boundary of the set, where the minimizer is positive. In the same token is it true in our case, that the set $\mathcal{F} \setminus \mathcal{E}$ coincides with the reduced boundary of the sub-level set:*

$$D = \{x \in \Omega; u_{\alpha,D} \leq t\}.$$

The difficulty in establishing this equivalence in our situation, seems to be that unlike [AC], here $t > 0$, and this is again related in another way with failure of unique continuation.

We refer the interested reader to [CGIKO] for further problems and conjectures.

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