

Basics of the b -Calculus

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ABSTRACT. R. B. Melrose's b -calculus provides a framework for dealing with problems of partial differential equations that arise in singular or degenerate geometric situations. This article is a somewhat informal short course introducing many of the basic ideas of this world, assuming little more than a basic analysis and manifold background. As examples, classical pseudodifferential operators on manifolds and b -pseudodifferential (also known as totally characteristic) operators on manifolds with boundary are discussed.

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1. Introduction

This article gives a leisurely introduction to the b -calculus of R. B. Melrose. Here, we use the term ' b -calculus' in a broad sense: A geometrically inspired way of viewing and solving problems about smooth functions and distributions (especially their asymptotic and singular behavior) and differential equations (especially as they arise in singular geometric situations); a set of concepts introduced to realize this view mathematically; and a set of basic and general theorems about these concepts. The b -calculus in the narrower, technical sense (as a set of operators) will also be discussed.

The style of this article is rather informal. We emphasize examples, motivations and intuition and often refer to the literature for full proofs and the most general definitions. While the ultimate goal is to extend the classical pseudodifferential operator (Ψ DO) calculus, large parts (Sections 2 and 3) are interesting in other contexts as well. Knowledge of the classical Ψ DO calculus is not a prerequisite. Rather, it will be introduced, if sketchily, as the simplest instance of the more general theory to be developed.

We begin with some general considerations on solving linear partial differential equations (PDE), to show in which direction we aim. Since it is usually impossible to get an explicit solution, one wants to study existence and uniqueness, and qualitative properties of solutions. The PDE may contain parameters, then one wants to study how these things depend on the parameters. (For example, spectral problems are of this kind.)

To fix ideas, let us look at the case of an *elliptic* partial differential operator P with smooth coefficients on some manifold X_0 (for example, the Laplacian on a Riemannian manifold) and at the equation

$$(1.1) \quad Pu = f.$$

Solving for u in terms of f means finding an inverse Q of P (which we assume to exist for the moment, between suitably chosen function spaces). 'Knowing' Q would mean knowing its Schwartz kernel, that is the distribution, also denoted Q , on $X_0 \times X_0$ satisfying $(Qf)(x) = \int_{X_0} Q(x, x') f(x') dx'$ (also known as Green's function). Many important properties of equation (1.1) may be read off from certain partial information on Q :

- A. The location and nature of the singularities of Q (i.e. places where the distribution Q is not a C^∞ function).
- B. The asymptotic behavior of Q when approaching the 'boundary' of $X_0 \times X_0$ (i.e. when leaving any compact subset).

We will refer to this information as the *singularity structure* of Q .

Then we reformulate problem (1.1) as:

Main Problem: *Given the singularity structure of P , determine the singularity structure of $Q = P^{-1}$.*

(Here P is also identified with its Schwartz kernel.) If P depends smoothly on parameters in a space T , then one wants to find the singularity structure of Q on the space $X_0 \times X_0 \times T$.^{1 2} Since the coefficients of P are smooth functions on $X_0 \times T$, the singularity structure of P only depends on their asymptotic behavior 'near the

¹In applications, if we start with an operator that has 'singular' coefficients (at some place or parameter value) then we take X_0 and T to be the set where the coefficients are smooth. Similarly, if we start with an operator on a 'singular space' (e.g. a manifold with boundary) then X_0 is the smooth part of that space (the interior of the manifold with boundary).

²Note that many problems of linear analysis are specializations of the Main Problem, for example: asymptotics of eigenvalues and eigenfunctions under singular perturbations, mapping and Fredholm properties of elliptic operators, heat kernel asymptotics (the latter in the analogous parabolic setup).

boundary' of $X_0 \times T$ (i.e. when leaving any compact subset); for example, they may blow up or degenerate ('non-uniform ellipticity') in various ways. If P is not invertible then one asks the same questions for approximate inverses (parametrix) of P .

For example, if X_0 is compact (and P elliptic as before) and there are no parameters then the singularity structure is given by point A above only, and the classical pseudodifferential calculus tells us that Q is smooth outside the diagonal and has 'conormal' singularities on the diagonal, and gives a recipe for calculating these modulo smooth functions (i.e. the complete symbol of Q). As an example with parameter, consider $P_z = -\Delta + z$ on a compact Riemannian manifold X_0 , for $z \in [1, \infty)$. Then the singularity structure of P_z^{-1} describes not only the conormal singularity at the diagonal, but also the asymptotic behavior of the resolvent kernel as $z \rightarrow \infty$.

The goal of the b -calculus is to solve the Main Problem for a fairly broad class of singularity structures of P , the so-called boundary fibration structures. This general goal still seems out of reach, but a growing list of instances shows the versatility of the b -calculus in treating problems arising in geometric analysis (see the references given below).

Figure 1 shows a rough outline of the b -calculus approach to the Main Problem. An arrow means 'is used for'. A *calculus* is a set of operators with a fixed

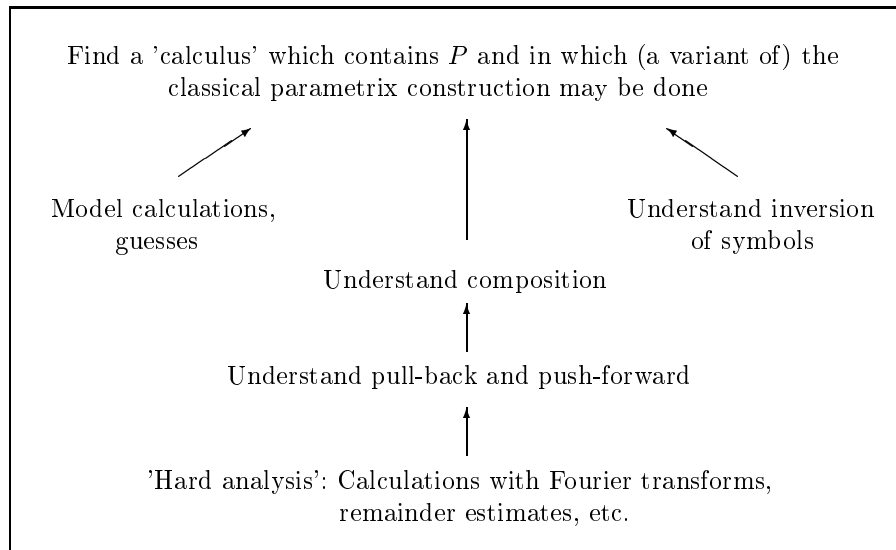


FIGURE 1. The b -calculus hierarchy for inverting P

singularity structure, which is closed (at least conditionally) under composition, together with a collection of *symbols*, i.e. rules that assign to each operator certain

'simplified' operators, usually by some sort of (partial) freezing of coefficients. 'Understanding' composition etc. means determining the singularity structure of the composition from the singularity structures of the factors. In particular, understanding inversion of symbols is another instance of the Main Problem itself, but for a simpler class of operators, and this shows the iterative nature of the problem. The lowest level of the iteration is inversion of constant coefficient operators, which may be done directly using the Fourier transform.

Therefore, in the construction of a calculus one expects the Fourier transform to play a central role, and this is reflected in the definition and occurrence of conormal distributions. However, calculations involving the Fourier transform tend to be messy and to obscure essential structures; this is why it is banished to the bottom level in Figure 1: The Fourier transform is only used explicitly in analyzing pull-back and push-forward of conormal distributions; composition is then reduced to a combination of pull-back and push-forward operations.³ The 'essential structures' are added on the higher levels and encoded geometrically. This parallels the b -calculus way of describing singularity structure (see below) and is one of its fundamental characteristics:

Fundamental principles of the b -calculus

- (1) Many complications may be understood geometrically, the analysis may be reduced to a few fundamentals.
- (2) All concepts which are introduced should be defined in a coordinate-invariant way. If they depend on choices, the exact freedom in these choices should be determined. This helps in understanding the concepts themselves.
- (3) Operators are always described by their Schwartz kernels.
- (4) All differential objects (e.g. densities, differential operators) should be written as b -objects (i.e. using $dx/x, x\partial/\partial x$ etc. instead of $dx, \partial/\partial x$ near a boundary $\{x = 0\}$). (This is specific to the b -calculus in the narrow sense.)

The aim of this article is to explain Figure 1 and to illustrate the use and power of these principles. Our first task is to elucidate what we mean by 'singularity structure'. The main point will be that complicated behavior of a function (or distribution) may often be described economically by 'blowing up' the underlying space and then looking at a rather 'simple' function on the new space. The resulting spaces are *manifolds with corners*, and this is the reason for the central role they play in the b -calculus: They are simultaneously simple enough to allow for simple analysis, and general enough to describe many phenomena.

³This should be taken with a grain of salt, but gives a general guideline. In the 'full b -calculus' in Section 4 the Fourier transform, in the guise of the Mellin transform, is also used for inverting the 'indicial operator'; this belongs to the 'Understand inversion of symbols' part in Figure 1.

In Section 2 we introduce manifolds with corners and discuss the singularity structure of smooth functions.

In Section 3 we discuss the lower three lines of Figure 1 and conormal distributions. We spend some time to explain the central role played by the Push-Forward Theorem. As an illustration of the second arrow from below in Figure 1, we define classical pseudodifferential operators (Ψ DOs) and show that they are closed under composition.

Finally, the top levels of Figure 1 are addressed in Section 4. We first recall the essential ingredients of the classical Ψ DO calculus that permit the construction of a parametrix for elliptic operators. We then show how this may be generalized to operators P with the simplest non-trivial singularity structure, the so-called b -differential operators on (the interior of) a manifold with boundary. In applications, these occur in the context of manifolds with infinite cylindrical ends or with conical singularities. Starting from a model calculation we construct the small and full b -calculus and sketch the parametrix construction. Note that, up to now, there does not seem to be a systematic way to construct a calculus in general (i.e. finding the right 'Ansatz'). This is the hardest part, and it usually involves a lot of trial and error.

In the Appendix we collect definitions and basic properties of some objects which are characteristic for the geometric view of the b -calculus.

Prerequisites for Sections 2 and 3 are minimal (basic analysis and manifold theory), except for 3.3 where an acquaintance with distributions is assumed. In addition, in Section 4 some basic functional analysis (e.g. compact operators) is needed, and some vague ideas about elliptic operators are useful, though not strictly necessary. The many footnotes mostly give additional details and can be skipped at first reading.

The reader who wants only a quick impression of the b -calculus should at least skim the following definitions, remarks, and examples: 2.2, 2.3, 2.5, 2.6, 2.7, 2.8.3, 3.1, 3.2, 3.4, 3.5, 3.8, 3.9, 3.10, 3.12, 3.13.2, 4.1, 4.2; and Subsections 2.3, 3.3 (if unfamiliar with conormal distributions), 4.1 (if unfamiliar with classical Ψ DOs), and 4.2, 4.3.

Literature: R. Melrose's 'green book' [26] gives a detailed exposition of the b -calculus (in the narrow sense) on manifolds with boundary; the first papers on this were [23], [31], and its extension to manifolds with corners is discussed in [14], [33]. Other 'calculi' (i.e. singularity structures of P , alias boundary fibration structures) are analyzed in [4], [7], [8], [16], [17], [18], [19], [20], [21], [26], [27], [28], [30], [32], [34], usually with applications to problems of geometric analysis. [25] gives a condensed presentation of the basic theorems (Pull-Back and Push-Forward Theorem) on manifolds with corners. In [24] (an ICM-talk) boundary fibration structures are introduced and a general strategy for constructing associated pseudodifferential calculi is outlined. Of an expository nature are also [28], [29]. The unfinished and long-awaited book [22] will be the ultimate source for all the details; currently you can get it on the www, so be quick before it disappears

again! Comparisons with other approaches to singular analysis are made in [6], [12] in this book. See [12] for many references to other approaches.

Why an article about the b -calculus, given all of these beautiful writings? I was told by some that they would like to learn about the b -calculus, but find it hard to get into the style in which it is usually presented: Often, things are expressed in ways that many analysts are not used to. My aim was to bridge this gap by explaining some of the basics and highlighting some of the ideas which are usually hidden between the lines. I hope to make this beautiful world accessible to a larger audience.

Some specific points in which this presentation differs from others are:

- I propose a notion of 'asymptotic type' of a function as a blow-up under which it becomes polyhomogeneous conormal (Definition 2.7). While this is clearly implicit in existing treatments, the explicit notion suggests naturally the problem of determining a type of a push-forward (or pull-back) of f from a type for f . Melrose's Push-Forward and Pull-Back Theorems answer this only partially (see Remarks 3.11.2 and 3.13).
- I give an alternative definition of the central notion of b -fibration, which I believe to be more intuitive (see Definition 3.9).
- I discuss the relation between the notions of 'type' and 'regimes'; the former originates in pure mathematics (algebraic geometry) while the latter is widely used in applied mathematics.
- For reasons of space I do not discuss the general notion of boundary fibration structures, nor any other instances besides the b - Ψ DOs. Also missing are the b -vector fields and associated b -bundles (but they are implicit in Principle 4) and the b -Sobolev spaces.

ACKNOWLEDGEMENT

I am deeply grateful to R. B. Melrose for introducing me into this world. Clearly, all the important ideas are due to him. I hope he will not disagree with the particular slant given to some things here, which reflect my own understanding and interests.

2. Geometry

We begin with the b -calculus way of describing the asymptotic behavior of smooth functions. Thus, we are given a non-compact manifold Z_0 and want to find a 'good' way to describe how the value $u(z)$ of a smooth function $u : Z_0 \rightarrow \mathbb{C}$ (or \mathbb{R}) behaves when z approaches the 'boundary' of Z_0 , i.e. leaves any compact subset of Z_0 . Here are a few examples, along with naive attempts to describe their asymptotic behavior:

EXAMPLES 2.1.

- (1) $Z_0 = (0, \infty)$, $u(x) = 1/x$: has a 'first order pole' at zero and vanishes to first order at infinity.

- (2) $Z_0 = (0, 1)^2$, $u(x, y) = xy$: extends smoothly to a neighborhood of $[0, 1]^2$ (i.e. the asymptotics is given by Taylor expansion around any boundary point of $[0, 1]^2$).
- (3) $Z_0 = (0, 1)^2$, $u(x, y) = 1/xy$: similar to 2., except that negative powers are allowed.
- (4) $Z_0 = \mathbb{R}^2 \setminus \{(0, 0)\}$, $u(x, y) = \sqrt{x^2 + y^2}$: 'decays linearly to $(0, 0)$ from all directions'.
- (5) $Z_0 = (0, \infty)^2$, $u(x, y) = \sqrt{x^2 + xy + y^3}$: smooth at the coordinate axes except at $(0, 0)$; near $(0, 0)$: complicated (x^2 dominates for $x > y$, xy for $y > x > y^2$, and y^3 for $y^2 > x$).
- (6) $Z_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2, x_3 > 0\}$, $u(x_1, x_2, x_3) = x_3$: similar to 4.

(In Examples 4.-6. we did not consider the behavior at infinity.)

Observation 1: It is useful to add certain 'boundary' points to Z_0 , so that one can talk for example about the behavior of a function 'at $(0, 0)$ ' when actually referring to its behavior in $Z_0 \cap U$ for U an arbitrarily small neighborhood of $(0, 0)$. (In Example 1 this means adding a 'point at infinity' also.)

Observation 2: Descriptions of asymptotic behavior must refer to certain coordinates (e.g. in Example 1 the standard coordinate on \mathbb{R} , both near zero and near infinity; in Example 4 it is natural to use polar coordinates, then u is just a smooth function of $r \geq 0$, vanishing at zero; for Example 5 it is less clear what 'good' coordinates would be).

To explain the b -calculus description of asymptotics, we proceed in two steps:

First step: We introduce what is considered model behavior: Z_0 is the interior of a manifold with corners Z , and the functions have joint asymptotic expansions in all variables in the corners. Melrose calls these functions 'polyhomogeneous conormal'. We prefer short words here, so we call them 'nice'.

Second step: We show how more general asymptotics may (often) be reduced to this model case by specifying an identification (diffeomorphism) of Z_0 with the interior of some manifold with corners W . The most common way to define such a diffeomorphism is by blow-up, which we also discuss.

We will see that this gives a very geometric way to describe the 'asymptotic type' of a function. Good references for this section are [25] and [26] (besides the all-encompassing unpublished [22]).

2.1. Manifolds with corners and nice functions.

DEFINITION 2.2.

- (1) A *manifold with corners* (mwc) is a topological space locally modelled on pieces of the form $[0, \infty)^k \times \mathbb{R}^{n-k}$, for various $k \in \{0, \dots, n\}$ (in the

same sense as a manifold is modelled on pieces \mathbb{R}^n and a manifold with boundary (mwb) on pieces \mathbb{R}^n and $[0, \infty) \times \mathbb{R}^{n-1}$.

- (2) A mwc Z is the union of its interior Z° and its boundary ∂Z . The boundary is the union of the *boundary hypersurfaces* (bhs's) of Z which are themselves mwc's.⁴
- (3) A *boundary defining function* (bdf) of a boundary hypersurface H of Z is a function $\rho : Z \rightarrow [0, \infty)$ such that $\rho^{-1}(0) = H$, ρ is smooth up to the boundary, and $d\rho \neq 0$ on H . (See below for the definition of smoothness.)⁵

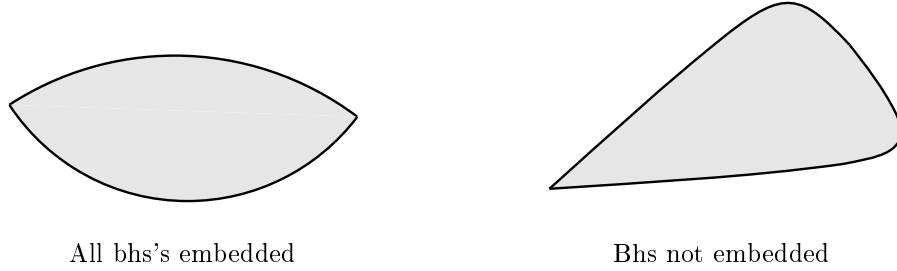


FIGURE 2

The simplest examples are \mathbb{R} , $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_+ \times \mathbb{R}$ (all mwb) and \mathbb{R}_+^2 (the simplest corner). Also, $[0, 1]^n$ is a mwc, but $Z = \overline{Z}_0$ in Example 2.1.6 is not. See Figure 5 for a more complicated mwc. On \mathbb{R}_+ , a bdf is given by $\rho(x) = x$. But note that many others are possible. The cartesian product of two mwc's is again a mwc (and the product of two mwb's is a mwc, but not a mwb, which is one reason for introducing the notion of mwc; for another reason see 2.5.1).

Every mwc can be embedded in a manifold: For example $\mathbb{R}_+^k \times \mathbb{R}^{n-k} \hookrightarrow \mathbb{R}^n$ in the obvious way; by definition, an embedding is a map which looks locally like that.

If Z is a mwc then we will speak of a 'function u on Z ' even if u is only defined in the interior of Z .

The role of bdf's is that they are the coordinates in terms of which the asymptotic behavior of functions will be described.

We now define nice functions. These should be thought of as slightly more general than functions smooth up to the boundary, so we discuss these shortly.

2.1.1. Functions smooth up to the boundary. These are, per definition, restrictions to Z of smooth functions on M , where $Z \hookrightarrow M$ is some embedding into a manifold. However, it is desirable to characterize this intrinsically, just using the

⁴When mwc's are defined this way, a bhs may happen to be only immersed rather than embedded, see Figure 2. In the b -calculus it is also always assumed that the boundary hypersurfaces are embedded (and connected); this detail won't matter for a while.

⁵The assumption on H to be embedded implies the existence of a bdf for H .

values of the function on the interior Z° . *Seeley's extension theorem* (see [22]) says that

u is smooth up to the boundary iff all derivatives of all orders of u are bounded on bounded subsets of Z° .

(Of course, a bounded subset is one whose closure in Z is compact.) For a characterization in terms of asymptotics see Remark 2.4.3 below.

2.1.2. *Nice (polyhomogeneous conormal) functions.* These are functions that behave like sums of products of terms like $x_i^\alpha \log^p x_i$, $i \in I$, near a corner defined by $\{x_i \geq 0, i \in I\}$. We call them nice since their behavior under integration can be analyzed fairly easily (see Section 3), and a discrete set of numbers is sufficient to describe their asymptotic behavior completely.

To set the stage, we consider a manifold with boundary first:

DEFINITION 2.3. Let Z be a manifold with boundary $H = \partial Z$.

- (1) An *index set* is a discrete subset $F \subset \mathbb{C} \times \mathbb{N}_0$ such that every 'left segment' $F \cap \{(z, p) : \operatorname{Re} z < N\}$, $N \in \mathbb{R}$ is a finite set. Also, it is assumed that $(z, p) \in F, p \geq q \implies (z, q) \in F$.
- (2) Given an index set F , a smooth function u on Z° is called polyhomogeneous conormal (in short, *nice*) with respect to F if, on a tubular neighborhood $[0, 1) \times H$ of H , one has

$$(2.1) \quad u(x, y) \sim \sum_{(z, p) \in F} a_{z, p}(y) x^z \log^p x \quad \text{as } x \rightarrow 0$$

with $a_{z, p}$ smooth on H .⁶⁷⁸

Thus, an index set tells us which terms $x^z \log^p x$ may occur in the asymptotics of u at the boundary. The finiteness condition on F ensures that (2.1) makes sense. \log 's are allowed since they often appear naturally, e.g. when integrating smooth functions (see Example 3.2.2). Note that x is a bdf for H (at least near H , which is all that matters).

REMARKS 2.4 (Contents and limitations of Definition 2.3).

⁶The meaning of \sim is this: Let $u_N(x, y)$ be the sum of (2.1) restricted to $\operatorname{Re} z \leq N$. Then

$$(2.2) \quad |u(x, y) - u_N(x, y)| \leq C_N x^N$$

for all N , uniformly on compact subsets of H , plus analogous estimates when taking any number of $x\partial_x$ and ∂_y derivatives.

⁷It is easy to check that this definition is independent of the choice of identification of a neighborhood of H with $[0, 1) \times H$, if one assumes that F satisfies

$$(2.3) \quad (z, p) \in F \implies (z + 1, p) \in F.$$

In any case, any index set can be 'completed', i.e. enlarged to a smallest index set having this property.

⁸Instead, one could consider finite asymptotics, but it messes up the notation. We prefer complete asymptotics (and C^∞ functions) so we can focus on more important things.

- (1) In the simplest case of $Z = \mathbb{R}_+$, we allow functions like $x^{-3}, \log x$, but no 'fast oscillation' like $\sin 1/x$. But $e^{-1/x}$ is nice (for any index set F , e.g. $F = \emptyset$).
- (2) The exponents α are not allowed to depend on y ; thus the 'variable asymptotics' of Schulze ([36], Section 2.3) is excluded.
- (3) Exercise: u is nice with respect to the index set $0 := \{(n, 0) : n \in \mathbb{N}_0\}$ iff u is smooth up to the boundary.

We now turn to manifolds with corners, and this is where the story gets really interesting.

For simplicity, we will only consider the mwc \mathbb{R}_+^2 . The extension to the general case is not difficult, see for example [4], [19], [22], [25] (for corners of higher codimension use induction, for additional \mathbb{R} -variables assume smooth dependence).

An *index family* \mathcal{E} for a mwc Z is an assignment of an index set to each bhs. For $Z = \mathbb{R}_+^2$ we simply write $\mathcal{E} = (E, F)$ if E is associated with the x -axis (i.e. $\{y = 0\}$) and F is associated with y -axis (i.e. $\{x = 0\}$).

DEFINITION 2.5. Let (E, F) be an index family for \mathbb{R}_+^2 . A function u on \mathbb{R}_+^2 is polyhomogeneous conormal (in short, *nice*) with respect to (E, F) if it has an asymptotic expansion in x as in (2.1), where the coefficients $a_{z,p}$ are functions on \mathbb{R}_+ that are nice with index set E (in the sense defined above for the mwb \mathbb{R}_+).⁹

Again, it is easy to see that u is smooth on \mathbb{R}_+^2 iff u is nice with respect to the index family $(0, 0)$, with 0 from Remark 2.4.3. Examples 2.1.1-3 are nice. (However, in Example 1 this describes only the behavior near zero; see below for the behavior near infinity.)

REMARK 2.6. The MAIN POINT is that all coefficients in the expansion (2.1) lie in the same singularity class. As a non-example, let us consider the function $u(x, y) = \sqrt{x^2 + y^2}$ on \mathbb{R}_+^2 . Clearly, u extends smoothly to the boundary except at $(0, 0)$. Therefore, for each fixed $y > 0$, one has an asymptotic expansion

$$\sqrt{x^2 + y^2} \sim \sum_{i=0}^{\infty} a_i(y) x^i \quad \text{as } x \rightarrow 0,$$

⁹Here, a good definition of \sim is harder to come by since the remainder in (2.2) should be allowed to be singular in y (at $y = 0$), but not too singular. One way around this is to require

$$|u(x, y) - u_N(x, y)| \leq C_N y^{-M} x^N$$

for some fixed M and all N (plus similar remainder estimates for the derivatives), *plus* an analogous expansion and estimate with x and y interchanged. Again, one has coordinate invariance. See also 2.5.1.

and similarly with x and y interchanged. But the coefficient functions $a_i(y)$ become more and more singular as $i \rightarrow \infty$. This is seen easily by writing

$$(2.4) \quad \sqrt{x^2 + y^2} = y \sqrt{1 + (x/y)^2} = y \sum_0^{\infty} c_i \left(\frac{x}{y}\right)^{2i}$$

$$(2.5) \quad = y + \frac{1}{2} \frac{x^2}{y} - \frac{1}{8} \frac{x^4}{y^3} + \dots$$

with the Taylor series $\sqrt{1+t} = \sum_0^{\infty} c_i t^i = 1 + t/2 - t^2/8 + \dots$ (for $|t| < 1$). It is easy to see from this that u is not nice with respect to any index family.

Therefore, niceness means having a 'joint' (or uniform) asymptotic expansion, simultaneously as all variables tend to zero, in the corner.

2.2. More general asymptotic behavior. Asymptotic type. Though $u(x, y) = \sqrt{x^2 + y^2}$ is not nice on \mathbb{R}_+^2 , we saw in (2.4) (and its analogue with x and y interchanged) that:

$$(2.6) \quad \begin{array}{l} u \text{ is nice as a function of } y \text{ and } x/y \text{ (for } x/y \text{ bounded), and} \\ u \text{ is nice as a function of } x \text{ and } y/x \text{ (for } y/x \text{ bounded).} \end{array}$$

We want to take this as a characterization of the 'asymptotic type' of u . A beautiful way to do this is through the following construction:

DEFINITION 2.7. Let Z_0 be any manifold. Let W be a compact mwc and $\beta : W^o \rightarrow Z_0$ a diffeomorphism. Call a function u on Z_0 of (asymptotic) type β if β^*u is a nice function on W .¹⁰

We then also say that u is resolved by β . Some people would call β a 'singular coordinate change'. Of course, we may also specify an index family \mathcal{E} on W and then speak of 'type β with index family \mathcal{E} '. The compactness of W means that the asymptotics of u is controlled 'in all directions'. We will freely consider non-compact W as well, when we are only interested in the behavior of β^*u on a compact part of W .

EXAMPLES 2.8.

- (1) To describe the behavior of Example 2.1.1 at infinity, one should say in which sense $[0, \infty]$ is a mwc (actually, a mwb). This may be done by choosing any diffeomorphism $\beta : (0, \infty) \rightarrow (0, 1)$ which is equal to the identity near zero and to the map $x \mapsto 1 - 1/x$ near infinity¹¹. Then u from Example 2.1.1 has type β with index sets $\{(-1, 0)\}$ at zero and $\{(1, 0)\}$ at one.
- (2) Let $\beta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2, (\xi, \eta) \mapsto (\xi\eta, \eta)$. Then u has type β iff $u(\xi\eta, \eta) = v(\xi, \eta)$ with v nice. Writing $\xi\eta = x, \eta = y$ we see that this means exactly that u is nice as a function of y and x/y (for bounded y and x/y).

¹⁰Here $\beta^*u = u \circ \beta$, the pull-back. One can think of β as a distortion lens, then β^*u is simply u , looked at through this lens. Of course, a function has many types.

¹¹This corresponds to the common usage in complex analysis, where behavior of $u(z)$ 'at infinity' is described by behavior of $u(1/z)$ at zero.

- (3) (Polar coordinates) Let $W = \mathbb{R}_+ \times [0, \pi/2]$, $Z = \mathbb{R}_+^2$ and
- $$(2.7) \quad \beta(r, \theta) = (r \cos \theta, r \sin \theta).$$

If $u(x, y) = \sqrt{x^2 + y^2}$ on \mathbb{R}_+^2 then $\beta^*u(r, \theta) = r$, so u has type β . We will see below that (2.6) is equivalent to u having this type β (see Remarks 2.9). Of course, the same formula (2.7) works for $Z_0 = \mathbb{R}^2 \setminus \{(0, 0)\}$ and $W = \mathbb{R}_+ \times S^1$ (with $S^1 = [0, 2\pi]/0 \sim 2\pi$ the circle), which makes u in Example 2.1.4 of type β .

Note that by considering β^*u we 'spread out' the values of u near 0 over a whole strip (a neighborhood of $\{0\} \times S^1$).

- (4) Let $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\xi \mapsto e^{-1/\xi}$. Then $\beta(\xi) = x$ iff $\xi = (\log 1/x)^{-1}$, so u has type β iff u is nice as a function of $(\log 1/x)^{-1}$. This is used in [7], for example.
- (5) Exercise: Find a type for Examples 2.1.5 and 6 (near $(0, 0)$).

Often, Z_0 is given as dense subset of a mwc Z , but the functions u of interest are not nice (see Example 2.1.4 when considered on \mathbb{R}_+^2 , or Example 5). In this case, a space W and map β can often be obtained by a procedure called 'blow-up'. We discuss this next.

2.3. Blow-up. Blow-up is a way to obtain new mwc's from old. It is used to resolve functions on a mwc and to desingularize (algebraic) subsets of a mwc (see Subsection 2.4).

The simplest non-trivial case of a blow-up is given by polar coordinates (Example 2.8.3). We will discuss this case in some detail and then sketch the general construction.

2.3.1. *Blowing up $(0, 0)$ in \mathbb{R}_+^2 .* Consider the 'polar coordinates map' (2.7). Note that, for $p \in \mathbb{R}_+^2$, $\beta^{-1}(p)$ is a point unless $p = (0, 0)$ when it is the interval $\{0\} \times [0, \pi/2]$. Therefore, we say that W is obtained from Z by 'blowing up $(0, 0)$ '. We write $W = [Z, (0, 0)]$ and call β the blow-down map. The bhs's of W are called lb = $\{\theta = \pi/2\}$, rb = $\{\theta = 0\}$ (the left and right boundary) and ff = $\{r = 0\}$ (the front face).

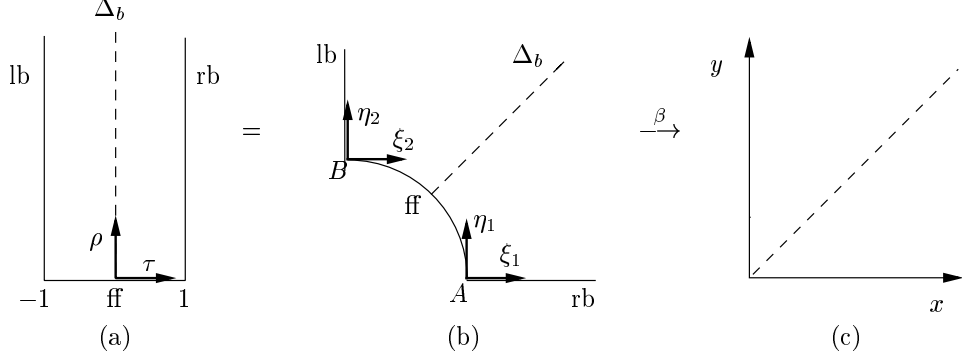
When drawing pictures, some people prefer to draw W as in Figure 3(b) while others prefer 3(a). In spirit these correspond roughly to using two different sets of coordinate systems on W , which are often more convenient to use than (r, θ) :¹²

Coordinate systems on $[\mathbb{R}_+^2, (0, 0)]$

- (1) (Projective coordinates) For $y \ll x$, we have $r = \sqrt{x^2 + y^2} \approx x$ and $\theta = \arctan y/x \approx y/x$. This suggests considering

$$(2.8a) \quad \xi_1 = x, \eta_1 = \frac{y}{x}$$

¹²The occurrence of the transcendental functions \sin and \cos in (2.7), with all their special properties (e.g. $\sin' = \cos$) is rather accidental and usually distracts from what really matters, e.g. the asymptotic behavior when approaching the boundary. There is no way to completely erase such accidents, but the following two options come close to it.

FIGURE 3. The blow up $[\mathbb{R}_+^2, (0, 0)]$ of \mathbb{R}_+^2

as coordinates on W . Indeed, from $\xi_1 = r \cos \theta, \eta_1 = r \sin \theta$ we see that they define local coordinates for $\theta \neq \pi/2$, i.e. on $W \setminus \text{lb}$, with ξ_1 a bdf for ff and η_1 a bdf for rb. Similarly,

$$(2.8b) \quad \xi_2 = \frac{x}{y}, \quad \eta_2 = y$$

define coordinates on $W \setminus \text{rb}$, with ξ_2 a bdf for lb and η_2 a bdf for ff. In these two coordinates systems β takes the simple form

$$(2.8c) \quad \begin{aligned} \beta_1(\xi_1, \eta_1) &= (\xi_1, \xi_1 \eta_1) \\ \beta_2(\xi_2, \eta_2) &= (\xi_2 \eta_2, \eta_2). \end{aligned}$$

(2) ('Rational polar coordinates') Define (for $x, y > 0$)

$$(2.9) \quad \rho = x + y, \quad \tau = \frac{x - y}{x + y}.$$

Writing $x = r \cos \theta, y = r \sin \theta$ one easily sees that $\rho = r a(\theta), \tau = b(\theta)$ with $a > \text{const} > 0$ and $b : [0, \pi/2] \rightarrow [-1, 1]$ a diffeomorphism, thus $(\rho, \tau) \in \mathbb{R}^+ \times [-1, 1]$ may be regarded as new coordinates on W .

Solving (2.9) for x, y one obtains the form of the blow-down map as

$$\beta(\rho, \tau) = \left(\frac{1}{2}\rho(1 + \tau), \frac{1}{2}\rho(1 - \tau) \right).$$

Bdf's are given by ρ for ff, $1 + \tau$ for lb, and $1 - \tau$ for rb.

REMARKS 2.9.

(1) Rather than beginning with polar coordinates one may define W and β directly by glueing two coordinate patches, i.e.

$$(2.10) \quad W = \mathbb{R}_+^2 \sqcup \mathbb{R}_+^2 / \sim$$

where $(\xi_1, \eta_1) \sim (\xi_2, \eta_2) :\Leftrightarrow \beta_1(\xi_1, \eta_1) = \beta_2(\xi_2, \eta_2)$

with $\beta_{1/2}$ from (2.8c). This identification is done precisely in order for β to be injective on W° . Note that injectivity is essential for the whole idea of defining asymptotic types of functions on Z_0 using the map β .

- (2) The first remark together with (2.8a) and (2.8b) shows that (2.6) holds iff β^*u is nice on $[\mathbb{R}_+^2, (0, 0)]$.
- (3) (2.10) is the way that blow-up is usually defined in algebraic geometry, except that \mathbb{R}_+ is replaced by \mathbb{C} , so that all spaces involved are smooth complex varieties without boundary.
- (4) An advantage of the (ρ, τ) coordinates is that they are global on W . While the projective coordinates may feel cumbersome at first since they are not global, they have several advantages: The bdf's are simply the ξ_i and η_i , calculations tend to be very simple, and in many problems they occur naturally (see [6], for example).

2.3.2. More general blow-ups. In general, given mwc's Z and $Y \subset Z$ (satisfying certain conditions), one constructs $W = [Z, Y]$, the 'blow-up of Z along Y ', together with a smooth map $\beta : W \rightarrow Z$, the 'blow-down map', which is a diffeomorphism $W^\circ \rightarrow Z^\circ \setminus Y$. This is done as follows:

- (1) Blow-up of an interior point of a 2-dimensional mwc replaces it by a circle, see Example 2.8.3.
- (2) In higher dimensions, blow-up of an interior point replaces it by a sphere. For example,

$$[\mathbb{R}^n, 0] = \mathbb{R}_+ \times S^{n-1}, \quad \beta(r, \omega) = r\omega$$

for $r \in \mathbb{R}_+, \omega \in S^{n-1}$. Similarly, $[\mathbb{R}_+^n, 0] = \mathbb{R}_+ \times S_+^{n-1}$ where $S_+^{n-1} = S^{n-1} \cap \mathbb{R}_+^n$. (r, ω) provide polar coordinates on $[\mathbb{R}^n, 0]$. Projective coordinates are $\xi_1 = x_1, \xi_2 = x_2/x_1, \dots, \xi_n = x_n/x_1$ on $\{\omega_1 \neq 0\}$, and similarly on all other $\{\omega_i \neq 0\}$.

- (3) More generally, one can blow up closed submanifolds $Y \subset Z$: Assume first that Y lies in the interior of Z . Locally, the pair (Y, Z) is just $(\mathbb{R}^k \times \{0\}^{n-k}, \mathbb{R}^n)$, and we simply set

$$[\mathbb{R}^n, \mathbb{R}^k \times \{0\}^{n-k}] = \mathbb{R}^k \times [\mathbb{R}^{n-k}, 0],$$

with β as in 2. above. One can check that this is independent of the coordinates chosen (up to diffeomorphism that intertwines the β 's) and can therefore be glued together to a global blow-up $\beta : [Z, Y] \rightarrow Z$. A more intuitive model for the space $[Z, Y]$ is $Z \setminus U_r(Y)$, where $U_r(Y) = \{z \in Z : \text{dist}(z, Y) < r\}$ with respect to some Riemannian metric on Z , for r sufficiently small (at least when Y is compact). (But then β is more complicated to write down.)

- (4) The construction from 3. can be extended directly to mwc's Y hitting the boundary of Z , if this 'hitting' is transversal in a suitable sense. (As a non-example, consider the parabola $Y = \{x, x^2\} \subset Z = \mathbb{R} \times \mathbb{R}_+$ and try to define a blow-up!) The exact condition is that near any $p \in Y$

coordinates can be chosen such that $p = 0$ and, locally, $Z = \mathbb{R}_+^k \times \mathbb{R}^{n-k}$ and $Y = Z \cap S$ for some coordinate subspace S . Such Y are called *p-submanifolds*. $[Z, Y]$ is a mwc.

REMARKS 2.10.

- (1) As already indicated, all these blow-ups are defined invariantly (i.e. no choices are made, beyond Z and Y , to define $[Z, Y]$ up to diffeomorphism that preserves β).
- (2) The construction above yields an *elementary blow-up*. Sometimes, one needs an *iterated blow-up*. That is, one chooses a p-submanifold Y' in $[Z, Y]$ and considers $[[Z, Y], Y']$, or iterates even further. This is needed when describing more complicated asymptotics of functions. For example, blowing up first $(0, 0)$ in \mathbb{R}_+^2 and then the point B in Figure 3(b) resolves the function $u(x, y) = \sqrt{x^2 + xy + y^3}$ from Example 2.1.5 (exercise!). Also, the 'triple b -space' in the b -pseudodifferential calculus is an iterated blow-up (see Figure 5).
- (3) When using an (iterated) blow-up $\beta : W \rightarrow Z$ to describe asymptotic behavior of functions we have:
 - (a) If u is nice then u is of type β .
 - (b) The behavior of u on a compact part of Z is reflected by the behavior of β^*u on a compact part of W . (Cf. the remark on compactness after Definition 2.7.)
 (a) follows from the fact that β is a b -map (see below), and (b) is just the properness of β (i.e. $\beta^{-1}(\text{compact}) = \text{compact}$).

2.3.3. *b-maps*. An important property of blow-down maps is that they are b -maps. We define these now. Recall that if w is any point in a mwc W then a neighborhood of w can be identified with $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$, with w corresponding to 0 (k depends on w).

DEFINITION 2.11. A map $f : W \rightarrow Z$ between mwc's is a *b-map* at $w \in W$ if for some (and therefore any) identification of neighborhoods of w and $z = f(w)$ with $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$ and $\mathbb{R}_+^{k'} \times \mathbb{R}^{n'-k'}$, respectively, sending w, z to zero, the map has 'product type', i.e.

$$f = (f_1, \dots, f_{n'}),$$

$$(2.11) \quad f_i(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = a_i(x) \prod_{j=1}^k x_j^{\alpha_{ij}}, \quad \text{for } i = 1, \dots, k',$$

with a_i smooth and non-vanishing near zero, and non-negative integers α_{ij} .

f is a b -map if it is a b -map at every point.¹³

¹³Melrose calls such maps *interior b-maps*. For a general b -map he allows that instead of (2.11) one has $f_i \equiv 0$ for some i , i.e. that $f(W) \subset \partial Z$ (assuming W connected). In this article we never use these general b -maps.

In particular, b -maps are smooth up to the boundary. Examples 2.8.2 and 3 are b -maps, while 4 is not.

REMARKS 2.12. (Intuition and properties of b -maps)

- (1) Condition (2.11) is an algebraic counterpart to the weaker geometric condition that, near z , the zero set $f_i^{-1}(0)$ is a union of bhs's through z (more globally: The preimage of any bhs of Z is a union of bhs's of W). See 3.1 for a more detailed discussion of the boundary geometry of b -maps.
- (2) The composition of b -maps is a b -map. Projective coordinates show that the blow-down map for an elementary blow-up (and therefore for any blow-up) is a b -map.
- (3) If β is a b -map then $\beta^*(\text{nice}) = \text{nice}$ since $\log xy = \log x + \log y$ and $\log a$ is smooth for $a > 0$. For a more precise statement see the 'pull-back theorem', Theorem 3.12.
- (4) If β is just smooth then $\beta^*(\text{smooth}) = \text{smooth}$, but in general $\beta^*(\text{nice})$ is not nice; for example, for $\beta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $(x, y) \mapsto x + y$ and $u(t) = \log t$ we get $\beta^*u(x, y) = \log(x + y)$ which is not nice on \mathbb{R}_+^2 . Similarly, $\sqrt{x + y}$ is not nice.

2.4. Embedded blow-up. So far, we have not addressed Example 2.1.6. $Z = \overline{Z_0}$ is not a manifold with corners, so the blow-up construction above does not apply directly to the construction of an appropriate 'blow-up space' W . However, Z_0 is embedded in \mathbb{R}^3 , which *is* a manifold. So one may blow up 0 (the singular point of Z) in \mathbb{R}^3 and then take W to be the closure of the preimage of Z_0 :

$$W := \overline{\beta^{-1}(Z_0)}, \quad \text{with } \beta : [\mathbb{R}^3, 0] \rightarrow \mathbb{R}^3 \text{ the blow-down map.}$$

Using polar coordinates on $[\mathbb{R}^3, 0] = \mathbb{R}_+ \times S^2$, i.e. $\beta(r, \omega) = r\omega$, $\beta^{-1}(x) = (|x|, x/|x|)$, we get

$$W = \mathbb{R}_+ \times C, \quad C = \{\omega_1^2 + \omega_2^2 = \omega_3^2, \omega_3 > 0\} \subset S^2.$$

C is a smooth curve – a circle – on S^2 , so W is a mwb and $\beta : W^\circ \rightarrow Z_0$ a diffeomorphism, and

$$\beta^*x_3 = r\omega_3.$$

Since ω_3 is a smooth function on S^2 , this is a nice function by Definition 2.3.

This procedure is called 'embedded blow-up' (or embedded desingularization). Hironaka showed in his famous 'resolution of singularities' work that such an embedded blow-up exists for any (semi-)algebraic set (and (semi-)algebraic function on it) in \mathbb{R}^n , and can be obtained by an iterated blow-up. (These authors use the 'projective' blow-up, but it should be easy to transfer the result to our situation.) See [9], [1].

2.5. Invariance, regimes, etc. Here we collect some more remarks on the idea of 'asymptotic type'.

2.5.1. *On invariance.* Definition 2.5 (and its generalization to any mwc) is coordinate invariant if all index sets in \mathcal{E} satisfy the condition (2.3). This means: Let \tilde{x}, \tilde{y} be any other bdf's for the y - and x -axis in \mathbb{R}_+^2 , respectively. (In particular \tilde{x}, \tilde{y} define coordinates near $(0, 0)$.) Then u is nice with index family \mathcal{E} when expressed in terms of \tilde{x}, \tilde{y} iff it is in terms of x, y . The reason is that both x/\tilde{x} and y/\tilde{y} are smooth and non-zero up to the boundary.

Thus, although coordinates (i.e. bdf's) are needed to write down the particular asymptotics of u ,

*the class of nice functions on a mwc with a given index family
 \mathcal{E} is defined independent of coordinates,*

and therefore defined purely by the geometry (the mwc) and the discrete set \mathcal{E} .

In contrast, there is no 'natural' class of coordinate functions describing approach to $(0, 0)$ in \mathbb{R}^2 : Both $\sqrt{x^2 + y^2}$ and $\sqrt{x^2 + 2y^2}$ would be equally good candidates as 'defining functions of $(0, 0)$ ', but their quotient does not extend smoothly to $(0, 0)$.

This shows the special role played by mwc's and is one reason for their central role in the b -calculus.

2.5.2. *On invariance, II.* Because of the invariance of the blow-up construction, 2.5.1 can be generalized to types other than nice. For example, the following data:

- a compact mwc Z and a p -submanifold $Y \subset Z$ (see 2.3.2, point 4), and
- an index family \mathcal{E} on $[Z, Y]$, satisfying (2.3)

define the class of functions on $Z \setminus Y$ which have type $\beta : [Z, Y] \rightarrow Z$ with index family \mathcal{E} . Again, this is a piece of discrete data (\mathcal{E}) and a piece of geometric data (which is actually also discrete, since it is natural to consider diffeomorphic pairs (Z, Y) as equal).

2.5.3. *On 'regimes' and 'matching conditions'.* Characterizations like (2.6) are often expressed in terms of so-called *regimes*: In the regime $y/x < C, x < C$, u has a certain asymptotics and in the regime $x/y < C, y < C$ it has another. Of course, these two pieces of data are not independent: Since both asymptotics describe the same function, certain relations (called *matching conditions*) hold between their coefficients.

The notion of 'type β ' beautifully and economically combines regimes and matching conditions into a single geometric picture; in the case of $[\mathbb{R}_+^2, 0]$ this is the content of Remark 2.9.1.

The correspondence between the 'regime' language and the mwc picture can be described roughly as follows:

regime	\longleftrightarrow	minimal face
matching condition between regimes A,B	\longleftrightarrow	hypersurface containing the faces corresponding to A,B

(A face of a mwc is a non-empty intersection of hypersurfaces, and faces are ordered with respect to inclusion.)

EXAMPLE 2.13. In Example 2.1.5 there are three regimes (corresponding to each of the three terms being dominant), and these correspond to the three corners in the mwc used to resolve it (see Remark 2.10.3).

2.5.4. *How many blow-ups to make?* In a given problem (usually involving differential equations) one often expects certain type of asymptotic (or singular) behavior for the solution (for example, from making a model calculation). This may indicate on which blown-up space one should best consider the problem, in order to stay in the realm of nice functions.

However, one has to be careful not to blow up too much: Although nice functions remain nice after blow-up (Remark 2.10.3a), differential operators become 'worse'! Thus, one needs to find a balance between these forces. We will not address this important problem any further. This is one of the difficulties in solving the Main Problem in the Introduction. See the references given there for solutions in some cases.

3. Analysis

In this section we discuss two of the basic processes of analysis: pull-back and push-forward, and how they affect asymptotic behavior of smooth functions (as discussed in the previous section) and conormal distributions, which we also introduce.

What are pull-back and push-forward, and why are they important? Pull-back is composition, push-forward is integration. They are important since they may be used as building blocks for other operations. This allows to carry out recurring ugly calculations (e.g. those involving Fourier transform) once in the proof of theorems about pull-back and push-forward, and then never look at them again. Let us illustrate this in two simple but central examples:¹⁴

EXAMPLES 3.1.

Applying an operator to a function: Let $\pi_1, \pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projections onto the first and second coordinate. If v is a function on \mathbb{R} then its pull-back $\pi_2^*v = v \circ \pi_2$ is the function $(x, y) \mapsto v(y)$ on \mathbb{R}^2 . If u is a function on \mathbb{R}^2 then its push-forward $\pi_{1*}u$ is the function

$$(3.1) \quad \pi_{1*}u(x) = \int u(x, y)dy.$$

If A is an operator, acting on functions v on \mathbb{R} , with integral kernel $A(x, y)$ then

$$(3.2) \quad (Av)(x) = \int A(x, y)v(y) dy = \pi_{1*}(A \cdot \pi_2^*v).$$

¹⁴In the examples, \mathbb{R} may be replaced by any manifold, equipped with a fixed density. For the moment we naively neglect the distinction between functions, distributions and the respective densities; also, we neglect such tedious matters as integrability.

Though this may look like an exercise in formal nonsense, it shows that mapping properties of A may be read off from the structure (e.g. asymptotic type) of the function (distribution) A , if one understands how such structure is affected by pull-back and push-forward.¹⁵

Composition of operators: If A, B are operators, acting on functions on \mathbb{R} , with integral kernels $A(x, y), B(y, z)$, then $C = A \circ B$ has integral kernel $C(x, z) = \int A(x, y)B(y, z) dy$, i.e.

$$(3.3) \quad C = \pi_{2*}(\pi_3^* A \cdot \pi_1^* B)$$

where $\pi_1, \pi_2, \pi_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ are the projections leaving out the first, second and third variable, respectively. Again, understanding how pull-back, push-forward and product affect the structure of distributions allows to predict, for example, whether a class of operators with a given structure of its kernels is closed under composition.

Another place where one needs to understand the behavior of distributions under push-forward is in the 'specializations' mentioned in Footnote 2 in the Introduction, since many of them are obtained from the full kernel of Q by integration (i.e. push-forward).

The maps used for pull-back and push-forward in the examples are rather trivial projections, so it's legitimate to ask: Why be so formal, why not talk simply of 'integration in y ' instead of 'push-forward by π_1 '? The answer is given by:

Principle

The push-forward of a complicated function by a simple map should be analyzed by rewriting it as push-forward of a simple function by a complicated map.

The point is that the 'complication' of the map lies mainly in its global geometry, so by a partition of unity the problem can be reduced to the sum of relatively simple local problems. (In contrast, the 'complication' of the function is local, typically.) Melrose's Push-Forward Theorem gives the result of this analysis, for the case of smooth functions. In Subsection 3.1 we discuss all these matters, starting from an example. We also sketch the idea of a proof of the Push-Forward Theorem in the special case that the target space is \mathbb{R}_+ .

When integrating one needs measures. Therefore, push-forward is best defined as acting on measures (or densities) rather than functions. The push-forward of a smooth density may be not smooth, and (what's equivalent) the pull-back of a distribution is not always defined. For the reader unfamiliar with these matters, we give the precise definitions and a short discussion of pull-back and push-forward,

¹⁵Also, one needs to understand how structure is affected by multiplication. This is trivial for nice functions, geometrically non-trivial for functions with different asymptotic types, and analytically non-trivial for distributions. See Subsection 3.3 for the latter case.

and how they act on smooth functions, distributions and smooth and distributional densities, in the Appendix.¹⁶

In Subsection 3.2 we state Melrose's Pull-Back Theorem, which tells how pull-back by a b -map affects nice functions. This is rather trivial in comparison to the Push-Forward Theorem.

Finally, in Subsection 3.3 we introduce conormal distributions and discuss how pull-back, push-forward and multiplication affect them. As an illustration, we define pseudodifferential operators and study their composition.

3.1. Push-forward and asymptotic type. We begin by analyzing a few examples of push-forward under the projection $\mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $(x, y) \mapsto x$. In other words, we set

$$(3.4) \quad \tilde{u}(x) = \int_0^\infty u(x, y) dy, \quad x > 0.$$

Assuming that u is smooth in $(0, \infty)^2$ and $\text{supp } u$ is bounded, we ask how the behavior of u near the boundary of \mathbb{R}_+^2 affects the behavior of \tilde{u} near 0.

EXAMPLES 3.2.

- (1) If u is smooth up to the boundary then so is \tilde{u} (by first-year analysis). More generally (and just as easy),

$$u \text{ nice with index family } (E, F) \Rightarrow \tilde{u} \text{ nice with index set } F$$

if the integral (3.4) exists at all, i.e. if

$$(3.5) \quad \text{Re } z > -1 \text{ for } (z, p) \in E.$$

- (2) If $u(x, y) = y^{-1}v(x/y, y)$ with v smooth on \mathbb{R}_+^2 and compactly supported then

$$(3.6) \quad \tilde{u}(x) = \int_0^\infty v\left(\frac{x}{y}, y\right) \frac{dy}{y} \sim \sum_{i=0}^\infty (a_i x^i + b_i x^i \log x) \quad \text{as } x \rightarrow 0,$$

¹⁶The reader who prefers to neglect the distinction between functions and densities is invited to do so, but will probably begin to acknowledge their usefulness when making computations herself.

i.e., \tilde{u} is nice, but not smooth (the index set is $\mathbb{N}_0 \times \{0, 1\}$).¹⁷ More generally, if v has index sets (E, F) then \tilde{u} is nice with index set^{18 19}

$$(3.8) \quad E \cup F := E \cup F \cup \{(z, p' + p'' + 1) : (z, p') \in E, (z, p'') \in F\}.$$

(3) For $u(x, y) = \sqrt{x^2 + y^2}$ explicit integration shows (restricting to $y \leq 1$ for integrability – this does not affect the essential point)

$$(3.9) \quad \tilde{u}(x) = (\text{smooth near zero}) - \frac{1}{2}x^2 \log x.$$

Thus, \tilde{u} is nice, and again a logarithm appears.

The common feature of Examples 2 and 3 is that u has asymptotic type β , where β is the blow-up of 0 in \mathbb{R}_+^2 .

Claim: This already suffices to explain the similarity of the results (3.6) and (3.9).

PROOF. We first show this by a simple calculation and then explain how it may be seen directly by 'looking at pictures'.

Calculation: Consider any u of type β , so that $w = \beta^* u$ is nice on $W = [\mathbb{R}_+^2, 0]$, and assume w has no logarithmic terms in its expansions. We split up the integral

$$\int_0^\infty u(x, y) dy = A + B$$

'smoothly near $y = x$ '. That is, with any cut-off function $\phi \in C_0^\infty(\mathbb{R}_+)$ which equals one near 0, and with $\psi = 1 - \phi$, we set

$$A = \int_0^\infty u(x, y) \phi(y/x) dy = \int_0^\infty x u(x, x\eta_1) \phi(\eta_1) d\eta_1 = \int_0^\infty x w_1(x, \eta_1) \phi(\eta_1) d\eta_1$$

$$B = \int_0^\infty u(x, y) \psi(y/x) dy = \int_0^\infty w_2(x/y, y) \psi(y/x) y \frac{dy}{y}.$$

Here, w_1 is just w expressed in projective coordinates near the point A in Figure 4(a) (i.e. $w_1(\xi_1, \eta_1) = u(\xi_1, \xi_1 \eta_1)$ or $w_1 = \beta_1^* u$ with β_1 from (2.8c)); the integral A is like Example 1 (with $u(\xi, \eta) = \xi w_1(\xi, \eta) \phi(\eta)$). Similarly, w_2 is just w expressed

¹⁷Proof: Taylor expand $v(\xi, \eta)$ at $\xi = 0$ (for each fixed η), then Taylor expand each coefficient and the remainder at $\eta = 0$ to obtain, for any N ,

$$(3.7) \quad v(\xi, \eta) = \sum_{\alpha=0}^{N-1} \xi^\alpha \eta^N a_\alpha(\eta) + \sum_{\beta=0}^{N-1} \eta^\beta \xi^N b_\beta(\xi) + \sum_{\alpha, \beta=0}^{N-1} c_{\alpha, \beta} \xi^\alpha \eta^\beta + \xi^N \eta^N r(\xi, \eta)$$

with a_α, b_β, r smooth up to the boundary. Assume $\text{supp } v \subset [0, C]^2$. Then in the integral (3.6) one may replace \int_0^∞ by $\int_{x/C}^1$. To obtain the asymptotics, simply integrate (3.7) term by term, using the substitution $z = x/y$ in the second sum.

Note that the log-terms only come from the terms $\alpha = \beta$ in the third sum.

¹⁸Same proof, after the (non-trivial) analysis argument that our definition (2.5) of niceness implies an expansion like (3.7). Alternatively, one may define niceness by this expansion.

¹⁹Why did we write the integral (3.6) with dy/y instead of simply dy ? Since then the result (3.8) is beautifully symmetric! Cf. 'b-densities' below.

in projective coordinates near B in Figure 4(a) (i.e. $w_2(\xi_2, \eta_2) = u(\xi_2 \eta_2, \eta_2)$ or $w_2 = \beta_2^* u$); the integral B is like Example 2 (with $v(\xi, \eta) = w_2(\xi, \eta) \psi(\xi^{-1} \eta)$).

Since by assumption $w_{1/2}$ have no log's in their expansions and in Example 1 no logarithms are created, we conclude: The log terms in Examples 2 and 3 are of the same nature.²⁰

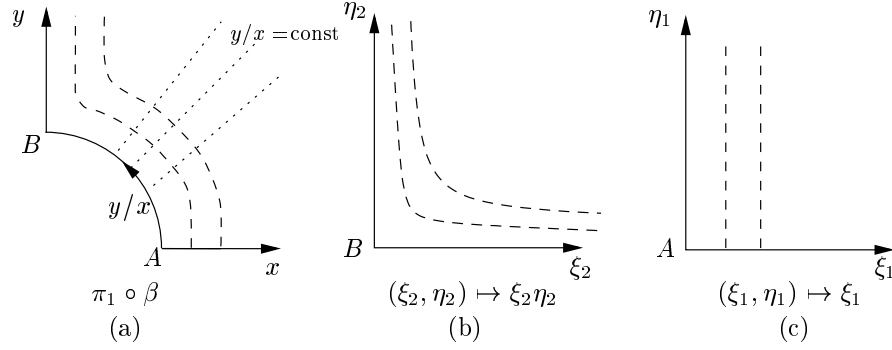


FIGURE 4. Level lines for push-forward under $[\mathbb{R}_+^2, (0, 0)] \rightarrow \mathbb{R}_+$

Pictures: We now show how the same result can be 'seen' geometrically. Since $\beta^* u = w$ and β is a diffeomorphism in the interior, we have $u = \beta_* w$, so

$$(3.10) \quad \tilde{u} = \pi_{1*} u = \pi_{1*} \beta_* w = f_* w$$

with $f = \pi_1 \circ \beta : W \rightarrow \mathbb{R}_+$. This says simply that $\tilde{u}(x)$ equals the integral of w over the fiber $f^{-1}(x)$ for each x (let's postpone the question of measures for a moment). This is clear since the values of w on $f^{-1}(x)$ are precisely the values of u on $\pi_1^{-1}(x)$, which are integrated to obtain $\tilde{u}(x)$.

Some fibers of f are shown in Figure 4(a), some of π_1 in Figure 4(c) and some of $g(\xi, \eta) = \xi \eta$ in Figure 4(b). Pictorially, we see:²¹

- Near A, Figure 4(a) looks like Figure 4(c),
- near B, Figure 4(a) looks like Figure 4(b).

Therefore, push-forward of w by f is the sum of push-forward (of w near A) by π_1 and push-forward (of w near B) by g , and this was precisely the calculation

²⁰But we also see that for general β -singular u infinitely many log terms will appear. For $u(x, y) = \sqrt{x^2 + y^2}$ only one log-term appears (see (3.9)); this is due to the fact that in $\sqrt{x^2 + y^2} = y \sqrt{1 + (x/y)^2} = x \xi_2^{-1} \sqrt{1 + \xi_2^2}$ only one power of x occurs. Such fine points are lost under (regular) coordinate changes and therefore invisible in the geometric setup of the Push-Forward Theorem.

²¹This can be made precise by expressing f in projective local coordinates (2.8):

- On $W \setminus \text{lb}$ ('near A') f is expressed as $f_1(\xi_1, \eta_1) = \xi_1$ (using β_1 in (2.8c)), i.e. $f_1 = \pi_1$,
- on $W \setminus \text{rb}$ ('near B') as $f_2(\xi_2, \eta_2) = \xi_2 \eta_2$ (using β_2), i.e. $f_2 = g$.

above. This also explains why the cut-off had to be chosen as a smooth function of y/x , see Figure 4(a).

In summary, we may say that the log terms in Examples 2 and 3 arise from the fact that the fibers of g and of $f = \pi_1 \circ \beta$ approach the corner as in Figure 4(b) for $x \rightarrow 0$. \square

In Melrose's Push-Forward Theorem these considerations are generalized to any b -map $f : W \rightarrow Z$: Under certain conditions on f , it says that the push-forward of a nice density μ on W is a nice density on Z , and computes the index sets of the latter from the index sets of the former and the 'boundary geometry' of f .

The conditions on f are best understood if we consider the special case $Z = \mathbb{R}_+$ first. Before we can state them, we need some definitions.

From now on, we assume that all index sets satisfy $(z, p) \in E \implies (z+1, p) \in E$, the condition ensuring coordinate invariance of niceness, and that bhs's are embedded and connected (see Definition 2.2 and Footnote 4).

Densities on manifolds with corners. If W is a mwc, then a density on W is, by definition, a density on the interior W° (concerning densities see the Appendix and Footnote 16). The notion of niceness carries over to densities immediately, e.g. on \mathbb{R}_+^2 :

DEFINITION 3.3. A density $\mu = u dx dy$ on \mathbb{R}_+^2 is *nice* with index sets E, F if u is nice with index sets E, F .²²

The following is a slight variant in book-keeping, which makes lots of things more transparent²³ (though it may seem artificial to the uninitiated):

DEFINITION 3.4. A *b-density* on \mathbb{R}_+^2 is just a density, except that we write it as $\mu = u(x, y) \frac{dx}{x} \frac{dy}{y}$ instead. When talking about smoothness or the index family of μ then we mean smoothness or the index family of u in such a representation.

Of course, a *b-density* on $\mathbb{R}_+ \times \mathbb{R}$ is of the form $u(x, y) \frac{dx}{x} dy$. That is, the $\frac{dx}{x}$ factor only occurs in the variables x defining some bhs. It is easy to see that the index family of a *b-density* is well-defined on any mwc.

Boundary geometry of a b-map $f : W \rightarrow \mathbb{R}_+$.

²²As usual, this should be checked for coordinate independence. But only under coordinate changes $(x, y) \mapsto (\tilde{x}, \tilde{y})$ for which \tilde{x}, \tilde{y} are still bdf's of the coordinate axes! Cf. 2.5.1.

²³Examples:

- (1) μ locally integrable $\iff \operatorname{Re} z > 0$ whenever $(z, p) \in E \cup F$ (rather than -1).
- (2) The transformation under projective coordinates becomes especially simple: Say $\xi_1 = x, \eta_1 = y/x$, then

$$(3.11) \quad \frac{dx}{x} \frac{dy}{y} = \frac{d\xi_1}{\xi_1} \frac{d\eta_1}{\eta_1}.$$

- (3) See Footnote 19 after Example 3.2.2.

DEFINITION 3.5. Let $f : W \rightarrow \mathbb{R}_+$ be a b -map. For any bhs G of W define $e_f(G)$ to be the order of vanishing of f at G .

In other words, in the local Definition 2.11 with $w \in G$, we set $e_f(G) = \alpha_{1j_0}$ if $f(w) = 0$ and x_{j_0} is a bdf for G , and $e_f(G) = 0$ if $f(w) \neq 0$. This is clearly locally constant and therefore constant on G by connectedness, so $e_f(G)$ is well-defined. Note that

$$(3.12) \quad f^{-1}(0) = \bigcup \{G : e_f(G) > 0\}.$$

THEOREM 3.6 (Push-Forward Theorem, special case $Z = \mathbb{R}_+$). *Let W be a manifold with corners and $f : W \rightarrow \mathbb{R}_+$ a b -map which is a fibration over $(0, \infty)$.²⁴ Let \mathcal{E} be an index family for W . Assume that f, \mathcal{E} satisfy the integrability condition (3.13) below.*

If μ is a compactly supported b -density on W , nice with index family \mathcal{E} , then f_μ is a b -density on \mathbb{R}_+ , nice with index family $f_{\#}\mathcal{E}$ (defined in (3.15) below).*

The integrability condition is:

$$(3.13) \quad \inf \mathcal{E}(G) > 0 \text{ whenever } e_f(G) = 0$$

where for any index set E

$$(3.14) \quad \inf E := \inf \{\operatorname{Re} z : (z, p) \in E\}$$

(which is actually a minimum).²⁵ To define $f_{\#}\mathcal{E}$, associate to every face F (i.e. non-empty intersection of boundary hypersurfaces) of W the index set

$$\tilde{\mathcal{E}}(F) = \overline{\bigcup}_G \left\{ \left(\frac{z}{e_f(G)}, p \right) : (z, p) \in \mathcal{E}(G) \right\}$$

where the extended union (defined in (3.8)) is over all bhs's G containing F and having $e_f(G) > 0$. Then, define

$$(3.15) \quad f_{\#}\mathcal{E} = \bigcup_F \tilde{\mathcal{E}}(F).$$

REMARKS 3.7.

- (1) f needs to be a fibration in the interior to ensure that $f_*\mu$ is smooth in the interior.
- (2) The definition of $f_{\#}\mathcal{E}$ given above is a little more precise than the one given in [25] (which may yield an index set that is 'too big'). But the

²⁴I.e. $f : f^{-1}((0, \infty)) \rightarrow (0, \infty)$ is a fibration in the sense of Footnote 65 in the Appendix, except that L is allowed to be a mwc.

²⁵Geometrically, $e_f(G) = 0$ means that $f > 0$ on G , so the fibers $f^{-1}(x)$, $x > 0$, will hit only these G , and actually transversally as in Figure 4(a) at the x -axis. So (3.13) generalizes (3.5) and comes from the fact that $\int_0^1 x^z \frac{dx}{x}$ exists iff $\operatorname{Re} z > 0$.

Push-Forward Theorem with this 'smaller' $f_{\#}\mathcal{E}$ follows directly from Melrose's by introducing a suitable partition of unity.²⁶

- (3) The *proof* was essentially done above: Localize as in the discussion of Example 3.2.3, this reduces to the cases of Examples 3.2.1/2 (modulo replacing x, y by powers x^ν, y^μ with $\nu, \mu > 0$ determined by the $e_f(G)$, and modulo straight-forward generalization to higher dimensions.)
- (4) See the article [6] in this book for a discussion of the relation of the Push-Forward Theorem (with $Z = \mathbb{R}_+$) and the 'Singular Asymptotics Lemma' by Brüning and Seeley ([2]).

Push-Forward Theorem with general target space. Here, some additional assumptions on the map f are needed. Before we can state these, we need to look a little closer at the geometry of b -maps:

Boundary geometry of b -maps. By definition, $f : W \rightarrow Z$ is a b -map iff

$$(3.16) \quad f_H := \rho_H \circ f : W \rightarrow \mathbb{R}_+$$

is a b -map for all bhs's H of Z , and bdf's ρ_H of H . So we can define:

DEFINITION 3.8. The *exponent matrix* of a b -map $f : W \rightarrow Z$ is the set of integers

$$e_f(G, H) = e_{f_H}(G), \quad G \text{ bhs of } W, H \text{ bhs of } Z.$$

Thus, $e_f(G, H) \neq 0$ iff $f(G) \subset H$, and in this case if $p \in W$ has small distance ε from G and distance $\geq \text{const} > 0$ from all other bhs's of W , then $f(p)$ has distance of order $\varepsilon^{e_f(G, H)}$ from H (say in Euclidean metric for any local coordinate systems based at points of G and H).

Referring to the Definition 2.11 of b -maps, we have $e_f(G, H) = \alpha_{ij}$ in (2.11) if $G = \{x_j = 0\}$ and $H = \{x'_i = 0\}$ locally.

Recall that a *face* of a mwc W is a non-empty intersection of boundary hypersurfaces, or W itself. Each face is a mwc. A b -map f induces a map

$$\bar{f} : \text{faces of } W \rightarrow \text{faces of } Z$$

characterized by

$$(3.17) \quad x \in F^\circ \implies f(x) \in (\bar{f}(F))^\circ.$$

Alternatively, $\bar{f}(F) =$ the intersection of the bhs's H of Z satisfying $f(F) \subset H$.

In summary, the 'combinatorics' of a b -map f can be described either by giving the pairs (G, H) with $f(G) \subset H$, or equivalently by the map \bar{f} , or (a little more refined) by the matrix e_f .

DEFINITION 3.9. A b -map $f : W \rightarrow Z$ is a *b -fibration* if for each face F of W ,

- (a) $\text{codim } \bar{f}(F) \leq \text{codim } F$ (it is enough to require this of bhs's F), and

²⁶Clearly, in (3.15) it is enough to take the union over all *minimal* faces (with respect to inclusion), for example the corners A, B in Figure 4(a). Thus, any 'regime' on W (see 2.5.3) contributes some asymptotic terms.

(b) f is a fibration $F^0 \rightarrow (\overline{f}(F))^0$.²⁷

For example, if $Z = \mathbb{R}_+$ then f is a b -fibration iff f is a fibration over $(0, \infty)$ (here (a) is empty). The polar-coordinate map is not a b -fibration (since it gets mapped to a codimension two face), nor is any other non-trivial blow-down map. An important example of a b -fibration where condition (a) is non-empty is given by the projection from the 'triple b -space' in (3.21).

For a b -map $f : W \rightarrow Z$ and an index family \mathcal{E} on W , define the index family $f_{\#}\mathcal{E}$ on Z by

$$(3.18) \quad f_{\#}\mathcal{E}(H) = (f_H)_{\#}\mathcal{E},$$

with the right hand side defined by (3.15) and (3.16). The integrability condition is:

$$(3.19) \quad \inf \mathcal{E}(G) > 0 \quad \text{whenever} \quad e_f(G, H) = 0 \quad \forall H.$$

(The latter condition means that $f(G) \not\subset \partial Y$.)

THEOREM 3.10 (Push-Forward Theorem). *Let W, Z be manifolds with corners and $f : W \rightarrow Z$ a b -fibration. Let \mathcal{E} be an index family for W , and assume (3.19).*

If μ is any compactly supported b -density μ on W , nice with index family \mathcal{E} , then the push-forward f_μ is a b -density on Z , nice with index family $f_{\#}\mathcal{E}$ (defined in (3.18)).*

See [25] for a proof.

REMARKS 3.11.

(1) *Why the b -fibration conditions are needed:* For example, (a) is violated for the polar-coordinate map, and this map does not preserve niceness (this was the reason for doing blow-ups in the first place!). Also, if $f_*\mu$ is to be nice then the expansion coefficients in the asymptotics at any face should be smooth in the interior of the face, so one should require a fibration condition here, which explains (b). (Thus, (a) ensures good behavior of $f_*\mu$ when approaching the boundary, while (b) does so *in* the boundary and locally in the interior.)

(2) *On determining the asymptotic type of $g_*\mu$ from the asymptotic type of μ ,* for a map $g : Z \rightarrow Z'$ between mwc's:

For $Z' = \mathbb{R}_+$ the answer is given essentially by Theorem 3.6: If $g : Z \rightarrow \mathbb{R}_+$ is a fibration in the interior and μ has type $\beta : W \rightarrow Z$ then applying the theorem to $f = g \circ \beta$ shows that $g_*\mu$ is nice on \mathbb{R}_+ . Compare (3.10) where $g = \pi_1$. Clearly, this only works if $(\beta^{-1})_*$ maps nice densities on Z to nice densities on W ; for blow-ups β this is clearly true, see (3.11) in Footnote 23.

²⁷Melrose's definition in [25] looks different, but is equivalent: As is easily seen, condition (a) is equivalent to his ' b -normality' (we assume all b -maps to be interior), and, assuming (a), condition (b) is equivalent to his ' b -submersion' condition, at least for proper maps (for which submersion \iff fibration), which is all that matters anyway.

For general Z' the problem is: Given $g : Z \rightarrow Z'$ and a blow-up $\beta : W \rightarrow Z$, find a blow-up $\beta' : W' \rightarrow Z'$ such that densities of type β are pushed forward to densities of type β' . By the Push-Forward Theorem, this would be satisfied if $\tilde{g} = (\beta')^{-1} \circ g \circ \beta : W \rightarrow W'$ was a b -fibration.

$$(3.20) \quad \begin{array}{ccc} W & \xrightarrow{\tilde{g}} & W' \\ \beta \downarrow & & \downarrow \beta' \\ Z & \xrightarrow{g} & Z' \end{array}$$

Note that even for \tilde{g} to be a well-defined map implies restrictions on β' . The problem when this is possible, and how to find β' , seems to be difficult.

- (3) The support condition on μ in Theorem 3.10 merely excludes problems of non-integrability at infinity. Clearly, it could be weakened to: f is proper on $\text{supp } \mu$. (We need this extension when we discuss Ψ DOs.)

3.2. Pull-back and asymptotic type. Though just as important as the Push-Forward Theorem, this is child's play in comparison. The main point was already mentioned in Remark 2.12.1. Doing the book-keeping on the index sets easily yields:

THEOREM 3.12 (Pull-Back Theorem). *Let $f : W \rightarrow Z$ be a b -map. Then, for any function v on Z which is nice with index family \mathcal{F} , the pull-back f^*v is a nice function on W with index family $f^\#\mathcal{F}$ defined by: If G is a bhs of W then*

$$f^\#\mathcal{F}(G) = \left\{ \left(q + \sum_H e_f(G, H) z_H, \sum_H p_H \right) : q \in \mathbb{N}_0 \text{ and} \right. \\ \left. \text{for each bhs } H \text{ of } Z : \begin{cases} (z_H, p_H) \in \mathcal{F}(H) & \text{if } e_f(G, H) \neq 0, \\ (z_H, p_H) = (0, 0) & \text{if } e_f(G, H) = 0. \end{cases} \right\}.$$

REMARKS 3.13.

- (1) *On determining the asymptotic type of g^*v from the asymptotic type of v .* Here one is given g and a blow-up β' in diagram (3.20), and needs to find a blow-up β such that g^*v has type β whenever v has type β' . By the Pull-Back Theorem, this is satisfied if \tilde{g} is well-defined and a b -map (and g is surjective). Note that $(\beta')^{-1} \circ g$ is usually only defined on the interior since β' is not a diffeomorphism on the boundary, so W has to be chosen 'big' enough so that \tilde{g} may extend continuously from the interior to all of W .
- (2) *The triple b -space.* As example consider the case relevant for composition in the b - Ψ DO calculus: $g = \pi_1 : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+^2$ is the projection $\pi_1(x_1, x_2, x_3) = (x_2, x_3)$. The solution is easy: If $\beta' : W' \rightarrow \mathbb{R}_+^2$ is any blow-up then let $W = \mathbb{R}_+ \times W'$, $\beta = \text{id}_{\mathbb{R}_+} \times \beta' : \mathbb{R}_+ \times W' \rightarrow \mathbb{R}_+ \times \mathbb{R}_+^2$. In

the case relevant for us, where $W' = [\mathbb{R}_+^2, 0]$ is just the blow-up of zero, W is the blow-up of the x_1 -axis.

However, in the composition problem W and β need to work for several maps g *simultaneously*, and this makes the problem more interesting. Let $\pi_i : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+^2$ be the projection that forgets the i 'th coordinate, for $i = 1, 2, 3$.

Problem: Find a blow-up $\beta : W \rightarrow \mathbb{R}_+^3$ such that whenever v has type $\beta' : [\mathbb{R}_+^2, 0] \rightarrow \mathbb{R}_+^2$ then $\pi_i^* v$ has type β for $i = 1, 2, 3$.

In other words, $\tilde{\pi}_i = (\beta')^{-1} \circ \pi_i \circ \beta : W \rightarrow [\mathbb{R}_+^2, 0]$ must be a b -map for $i = 1, 2, 3$. It is clear that at least all three coordinate axes must be blown up. The most naive thing to try is to blow up one axis (say the x_1 -axis) and then (the preimages of) the other two. However, it is easily seen that π_2 and π_3 are still not well-defined on the resulting space.

But there is a beautiful solution which even preserves the symmetry: First, blow up zero in \mathbb{R}_+^3 . Then, blow up the preimages of the three coordinate axes (in any order, since they are separated now!). the result is called triple b -space X_b^3 and shown in Figure 5.

Let us convince ourselves pictorially that the maps

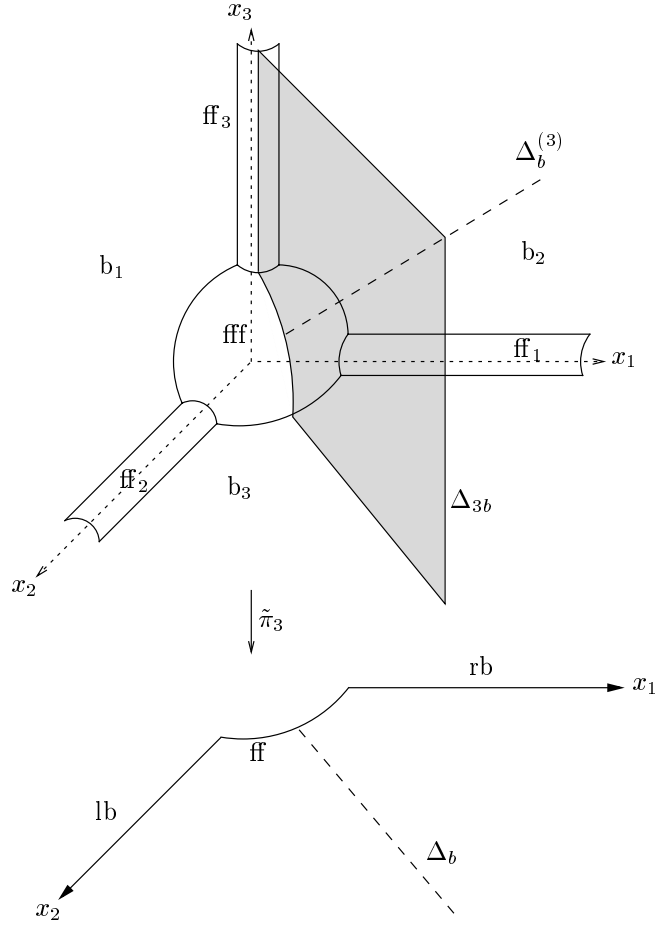
$$(3.21) \quad \tilde{\pi}_i : X_b^3 \rightarrow X_b^2 := [\mathbb{R}_+^2, (0, 0)]$$

are well-defined and b -fibrations. By symmetry, it is enough to consider $\tilde{\pi}_3$. It is well-defined since the x_3 -axis was blown up. Denote the bhs's of X_b^3 by b_1, b_2, b_3 (the 'old' bhs's from \mathbb{R}_+^3), $\text{ff}_1, \text{ff}_2, \text{ff}_3$ (the front faces of the axis blow-ups), and fff (the front face of the point blow-up) as in Figure 5. The bhs's are mapped as follows:

$$(3.22) \quad \begin{aligned} \text{ff}_2, b_1 &\mapsto \text{lb} \\ \text{ff}_1, b_2 &\mapsto \text{rb} \\ \text{fff}, \text{ff}_3 &\mapsto \text{ff} \\ b_3 &\mapsto X_b^2, \end{aligned}$$

and all these maps are onto. Also, $(X_b^3)^\circ \rightarrow (X_b^2)^\circ$. Therefore, the preimage of each bhs is a union of bhs's, which almost shows that $\tilde{\pi}_3$ is a b -map (see Remark 2.12.1; of course one may check the full condition (2.11) by direct calculation). Also, (3.22) defines the map $\overline{\tilde{\pi}}_3$ (see 3.17) on bhs's and this determines $\overline{\tilde{\pi}}_3$ for all faces. Since all faces on the right of (3.22) have codimension at most one, condition (a) in the Definition 3.9 of a b -fibration is satisfied.

Condition (b) is easily checked for each face: For example, $b_1^\circ \rightarrow \text{lb}^\circ$ is basically the same map as $[\mathbb{R}_+^2, (0, 0)] \rightarrow \mathbb{R}_+$ from 4, and so is $\text{fff}^\circ \rightarrow \text{ff}^\circ$ near the boundary. All maps from codimension two faces are either diffeomorphisms or constant, so they are fibrations trivially.

FIGURE 5. The triple b -space (and projection $\tilde{\pi}_3$)

3.3. Distributions. So far, all singular behavior occurred at the boundary. Now we turn to the description of singularities in the interior of a mwc Z . This means talking about distributions²⁸. In many situations only a very special class of distributions occurs, the 'step 1 polyhomogeneous conormal' (here called 'conormal') ones²⁹. They are smooth outside a submanifold, and at the submanifold have a special explicitly describable kind of singular behavior, which is in some sense similar to the behavior of a nice function at the boundary.

²⁸There are also distributions whose singular support is contained in the boundary. We will not discuss them here (although they are not really more difficult). See [22], for example.

²⁹This is not the most general kind of what's usually called conormal distributions, but they are easy to define and sufficient for many purposes.

In the case of manifolds most of this material is quite standard (see e.g. [11], [39]); we will briefly recall the definition, give some examples and state the push-forward and pull-back theorems. As an illustration, we use this to show that the set of (properly supported) classical pseudodifferential operators on \mathbb{R} is closed under composition. The extension of the definition and basic properties of conormal distributions to manifolds with corners is quite straight-forward if the singular submanifold hits the boundary in a 'product-type' way.

For lack of space we do not treat the transformation of the principal symbol under pull-back and push-forward. However, this is important for the composition formula for pseudodifferential operators (see the references above).

3.3.1. Conormal distributions on manifolds.

DEFINITION 3.14. Let Z be a manifold and $Y \subset Z$ a submanifold. A distribution $u \in \mathcal{D}'(Z)$ is *conormal* with respect to Y if, for some $m \in \mathbb{R}$,

- u is smooth on $Z \setminus Y$, and
- in any local coordinate system $x : U \subset Z \rightarrow \mathbb{R}^n$ sending $Y \cap U$ to $\mathbb{R}^k \times \{0\}^{n-k} \subset \mathbb{R}^n$ there is a representation

$$(3.23) \quad u(y, z) = \int_{\mathbb{R}^{n-k}} e^{iz\zeta} a(y, \zeta) d\zeta$$

where $y = (x_1, \dots, x_k)$, $z = (x_{k+1}, \dots, x_n)$ and a is a smooth function on $(Y \cap U) \times \mathbb{R}^{n-k}$ with asymptotics

$$(3.24) \quad a(y, \zeta) \sim \sum_{j=0}^{\infty} a_{m-j}(y, \zeta),$$

as $|\zeta| \rightarrow \infty$, where a_l is homogeneous of degree l in ζ , for each l .³⁰

Note that (3.23) is simply the inverse Fourier transform in z (i.e. 'transversal' to Y), with smooth dependence on the parameter $y \in Y$.

EXAMPLES 3.15.

- (1) For $Z = \mathbb{R}$ and $Y = \{0\}$ the distributions δ and $p.v.\frac{1}{x}$ are conormal, and also all of their derivatives and anti-derivatives, which include for example x_+^α for $\alpha \in \mathbb{N}_0$ (the function vanishing for $x \leq 0$ and equal to x^α for $x > 0$). Any (-1) -homogeneous distribution is a linear combination of δ and $p.v.\frac{1}{x}$, so a conormal distribution (with $m \in \mathbb{Z}$) may be thought of as 'series' of such terms, of increasing regularity.³¹

³⁰The meaning of the asymptotics is that, for any N , if $a^{(N)}$ is the sum up to the term a_{-N} then $|a(y, \zeta) - a^{(N)}(y, \zeta)| \leq C|\zeta|^{-N-1}$, plus analogous estimates for all derivatives in y and ζ .

The *order* of u is defined to be $m + (n - 2k)/4$, if $a_m \neq 0$.

³¹More generally, one can define conormality without reference to the Fourier transform: (3.23) and (3.24) are equivalent to the existence of distributions $u_s(y, \cdot) \in \mathcal{D}'(\mathbb{R}^{n-k}) \cap C^\infty(\mathbb{R}^{n-k} \setminus$

- (2) For $Z = \mathbb{R}^n$, $n > 1$, and $Y = \{0\}$ there is much more freedom since now the space of l -homogeneous distributions is infinite-dimensional for each l . The simplest example is δ again.
- (3) For $Z = \mathbb{R}^n \times \mathbb{R}^n$ (with coordinates $w, w' \in \mathbb{R}^n$) and $Y = \{w = w'\}$ (the diagonal) the conormal distributions are the integral kernels of classical pseudodifferential operators on \mathbb{R}^n since (3.23) precisely amounts to their 'usual' definition, in the coordinates $y = w$, $z = w - w'$, see for example [38], Section 3.7. (And similarly for Ψ DOs on any manifold.) The order of the conormal distribution is the order of the operator in the usual sense. For example, the differential operator $P = \sum_{\alpha} a_{\alpha}(w)(\partial/\partial w)^{\alpha}$ has kernel $P(w, w') = \sum_{\alpha} a_{\alpha}(w)\delta^{(\alpha)}(w - w')$.

REMARKS 3.16.

- It's not obvious, but the definition is actually independent of the chosen coordinate system x , see [11]. Of course, a will depend on the choice of coordinates, but its leading term a_m is invariant if considered as section of the conormal bundle of Y . It is called the *principal symbol* of u . One easily sees that it depends only on the restriction of u to arbitrarily small neighborhoods of Y .
- The definition carries over immediately to distribution densities or, more generally, to distributions with values in any bundle over Z .

We now consider push-forward and pull-back of distributions under a smooth map $f : W \rightarrow Z$. The proofs of the following theorems are quite easy, given the coordinate invariance of Definition 3.14. They can be found in [22]. The push-forward of any distribution density μ on W is a distribution density on Z (supposing, as usual, that f is proper on $\text{supp } \mu$), see the Appendix. The question arises whether conormality of μ with respect to a submanifold $X \subset W$ implies conormality of $f_*\mu$. The answer is no in general³²; it is a very tricky problem to determine precise conditions when it is true. It depends essentially on the behavior of the fibers of f and their tangency to X . We only consider the simplest case of a fibration whose fibers meet X transversally in isolated points only, which is enough for many purposes.

THEOREM 3.17 (Push-forward of conormal distributions). *Let $f : W \rightarrow Z$ be a fibration between manifolds. Let X be a submanifold of W such that for each $x \in X$, the tangent spaces to X and to the fiber $f^{-1}(f(x))$ through x intersect only*

0), homogeneous of degree s and depending smoothly on the parameter y , such that

$$u - \sum_{j=0}^N u_{j+n-k-m} \in C^{\infty}(\mathbb{R}^k, C^{N'}(\mathbb{R}^{n-k}))$$

(locally) for all N , where $N' = N - C$, with C only depending on n . This may look weaker than the definition above, but is actually equivalent (exercise!).

³²Example: If f is bijective and smooth then $\text{singsupp } f_*\mu \supset f(\text{singsupp } \mu)$. The latter need not be (contained in) a submanifold even if $\text{singsupp } \mu$ is, therefore $f_*\mu$ may be not conormal even if μ is.

in zero.³³ Let μ be a distribution density on W , conormal with respect to X , and assume f is proper on $\text{supp } \mu$.

- (a) If $f|_X$ is a diffeomorphism onto Z then $f_*\mu$ is smooth.
- (b) Otherwise, $f(X)$ is a proper submanifold of Z and $f_*\mu$ is conormal with respect to $f(X)$.

Thus, the 'vertical' (in fiber direction) singularities get integrated out, while the others remain, as in the simple example of $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $(x, y) \mapsto x$: For $X = \{(x, 0) : x \in \mathbb{R}\}$ one has, for example, $f_*(\delta(y) dx dy) = 1 \cdot dx$ (case (a)), and for $X = \{(0, 0)\}$ one has $f_*(\delta(x)\delta(y) dx dy) = \delta(x) dx$ (case (b)). (Calculations done using (App.1).) See Remark 3.11.3 concerning the support condition.

For pull-back the situation is different: While the pull-back for general distributions is only defined under fibrations (and conormality is always preserved under a fibration), a *weaker* condition on f already allows to pull back conormal distributions:

THEOREM 3.18 (Pull-back of conormal distributions). *Let $Y \subset Z$ be a submanifold, and assume that $f : W \rightarrow Z$ is transversal³⁴ to Y . Then $f^{-1}(Y)$ is a submanifold of W , and if u is a distribution on Z , conormal with respect to Y , then f^*u is a distribution on W which is conormal with respect to $f^{-1}(Y)$.*

We saw in Examples 3.1 that we also need to *multiply* distributions. A complete discussion of when this is possible would lead us too far astray, so we'll just sketch the procedure which allows to define a product in this context.

For two functions u_1, u_2 on a manifold Z , we can translate the trivial identity $u_1(x)u_2(x) = u_1(x)u_2(y)|_{x=y}$ into geometric terms as

$$u_1 u_2 = i^*(u_1 \times u_2)$$

where $i : Z \rightarrow Z \times Z$, $x \mapsto (x, x)$ is the diagonal inclusion and $(u_1 \times u_2)(x, y) = u_1(x)u_2(y)$ defines the direct product of u_1 and u_2 as function on $Z \times Z$.

When trying to generalize this to distributions u_1, u_2 on Z , we first note that the direct product is well-defined as a distribution on $Z \times Z$ (since u_1 and u_2 'depend on different sets of variables' in $Z \times Z$). The problem arises with the pull-back: i is certainly not a fibration (it's not even surjective), so one would hope to apply Theorem 3.18. But this fails since usually $u_1 \times u_2$ is not conormal, even if u_1 and u_2 are conormal; for example for $Z = \mathbb{R}$ and $u_1 = u_2 = p.v. \frac{1}{x}$ one gets $(p.v. \frac{1}{x})(p.v. \frac{1}{y})$, which is not conormal (since its singular support, the union of both coordinate axes, is not a manifold).

The following theorem allows a way out (at least in some situations):

THEOREM 3.19 (Direct product of conormal distributions). *Let u_i be distributions on manifolds Z_i , conormal with respect to submanifolds Y_i , for $i = 1, 2$. Then the direct product $u_1 \times u_2$ can be written $u_1 \times u_2 = v + w$ where v is conormal*

³³Equivalently, $d(f|_X)$ is injective.

³⁴I.e. if $z = f(x) \in Y$ then $T_z Z$ is spanned by $T_x Y$ and $df(T_x W)$.

with respect to $Y_1 \times Y_2$ and w has wave front set contained in any given conic neighborhood of $(Z_1 \times N^*Y_2) \cup (N^*Y_1 \times Z_2)$.

We will not define wave front sets here, see [10]. N^*X_i denotes the conormal bundle. The point is that in the applications we have in mind (Examples 3.1) the product is integrated in the end, and then the position of $WF(w)$ guarantees that the term resulting from w is smooth (by a generalization of Theorems 3.17(a) and 3.18), so the singularities are determined only by the conormal part v .

3.3.2. Composition of pseudodifferential operators. Let us check how these results show that the composition of two pseudodifferential operators on a manifold X is a pseudodifferential operator³⁵. For simplicity we take $X = \mathbb{R}$ although the general case works precisely the same way.

Thus, we are given distributions A, B on \mathbb{R}^2 , conormal with respect to the diagonal (see Example 3.15.3). To avoid confusion later on, we will write $A \in \mathcal{D}'(X_1 \times X_2)$, $B \in \mathcal{D}'(X_2 \times X_3)$ although $X_1 = X_2 = X_3 = \mathbb{R}$. The diagonals will be denoted $\Delta_A \subset X_1 \times X_2$ and $\Delta_B \subset X_2 \times X_3$.

The composition of A and B has integral kernel given by (3.3). Here, the product should be expanded, as explained above, as diagonal pull-back of the direct product. However, matters can be simplified slightly. The π_1, π_3 pull-backs can be omitted since clearly one also has

$$(3.25) \quad C = \pi_{2*}(i^*(A \times B))$$

where i is the embedding

$$i : X_1 \times X_2 \times X_3 \rightarrow X_1 \times X_2 \times X_2 \times X_3, \quad (x_1, x_2, x_3) \mapsto (x_1, x_2, x_2, x_3).$$

Using Theorem 3.19 we write

$$(3.26) \quad A \times B = v + w,$$

with v conormal with respect to $\Delta_A \times \Delta_B$ and $WF(w)$ close to $(X_1 \times X_2 \times N^*\Delta_B) \cup (N^*\Delta_A \times X_2 \times X_3)$.

We first analyze $\pi_{2*}(i^*v)$: i is transversal to $\Delta_A \times \Delta_B$ since the image of di is $\{(\alpha, \beta, \beta, \gamma)\}$ and the tangent space of $\Delta_A \times \Delta_B$ is $\{(\delta, \delta, \varepsilon, \varepsilon)\}$ (all free variables in braces range over \mathbb{R}), and these two subspaces clearly span \mathbb{R}^4 . Therefore, Theorem 3.18 shows that i^*v is conormal with respect to $\Delta' := i^{-1}(\Delta_A \times \Delta_B) = \{x_1 = x_2 = x_3\}$, the space diagonal. Finally, the tangent spaces to Δ' and the fiber of π_2 are $\{(\alpha, \alpha, \alpha)\}$ and $\{(0, \beta, 0)\}$, so they have zero intersection, and $\pi_2(\Delta') = \Delta_C$, the diagonal in $X_1 \times X_3$. Therefore, Theorem 3.17(b) applies, so

$$(3.27) \quad \pi_{2*}(i^*v) \text{ is conormal with respect to the diagonal.}$$

Finally, we analyze w , using standard results on wave front sets. First, by Theorem 8.2.4 in [10], the pull-back i^*w is defined as a distribution if $WF(w) \cap N^*(\text{Im } i) = \emptyset$, and then $WF(i^*w) \subset i^*(WF(w))$. Now $N^*(\text{Im } i) = \{(0, \alpha, -\alpha, 0)\}$

³⁵The support condition in Theorem 3.17 translates into the condition that at least one of the factors is properly supported (i.e. the two projections $X^2 \rightarrow X$ are proper on the support of the integral kernel). We will neglect this in the following discussion. See also Remark 4.4.3.

at every point, and this has non-zero angle with the fiber of $N^*\Delta_A \times X_2 \times X_3$, which is $\{(\alpha, -\alpha, 0, 0)\}$, and similarly with the fiber of $X_1 \times X_2 \times N^*\Delta_B$. Therefore, by choosing $WF(w)$ close enough to these latter sets, we may assume that $WF(w) \cap N^*(\text{Im } i) = \emptyset$. Also, $WF(i^*w)$ is contained in a small conic neighborhood of i^* of these sets, i.e. of (fiberwise) $\{(\alpha, -\alpha, 0)\} \cup \{(0, \alpha, -\alpha)\}$. By another standard theorem (see [39], ex. 6.7.8) the push-forward $\pi_{2*}u$ of a distribution u on $X_1 \times X_2 \times X_3$ is smooth unless $WF(u)$ hits the conormal space to the fiber of π_2 . Since the latter is $\{(\alpha, 0, \beta)\}$, this is clearly not the case for $u = i^*w$, so finally we obtain:

$$\pi_{2*}(i^*w) \text{ is smooth.}$$

This together with (3.27), (3.25) and (3.26) shows that C is conormal with respect to the diagonal, i.e. the integral kernel of a pseudodifferential operator.

3.3.3. Conormal distributions on manifolds with corners. The definition of conormal distributions depends in an essential way on the fact that 'normal slices' to Y in Z look the same at every point of Y . Therefore, it would be problematic to try to define conormality for $Z = \mathbb{R}_+^2$, $Y = \{(x, x) : x \in \mathbb{R}_+\}$: At zero, there is not even a reasonable candidate for a 'normal slice'!

However, if Z looks like $\mathbb{R}^{n-k} \times Y$ near Y then Definition 3.14 makes sense literally even if Y is a mwc, when we require the y -dependence to be smooth up to the boundary everywhere. For example, this is the case for $Z = \mathbb{R}_+ \times \mathbb{R}$, $Y = \mathbb{R}_+ \times \{0\}$ or $Z = [\mathbb{R}_+^2, 0]$, $Y = \Delta_b := \{\theta = \pi/4\}$ (using polar coordinates on Z , see Figure 3(b)).

Thus, we have defined *distributions on a mwc Z which are conormal with respect to an interior p -submanifold, smoothly at the boundary* (cf. 2.3.2.4; 'interior' means that $Y \not\subset \partial Z$). One may actually allow nice (rather than smooth) behavior at the boundary, with respect to a given index family. This gives *nice conormal distributions*. The precise definitions are quite straight-forward, we leave them as exercise to the reader. (See [22].)

The *push-forward* for nice conormal distributions may be analyzed by a combination of Theorems 3.10 and 3.17. Since the point of Theorem 3.10 was to allow maps f more general than fibrations, the assumption on f should be: f is a b -fibration, and a fibration (of mwc's) in some neighborhood of X , satisfying the additional transversality condition in Theorem 3.17. Then the result of the push-forward is nice conormal again. The proof is straightforward by use of a partition of unity, and is left as an exercise.

Similarly *pull-back* of nice conormal distributions is easy by combining Theorems 3.12 and 3.18. f needs to be a b -map transversal to Y , then the pull-back by f of a nice conormal function is nice conormal with respect to $f^{-1}(Y)$.

4. Partial Differential Equations

We now turn to the b -calculus in the narrow sense: the construction of parametrices for the 'simplest' class of non-uniformly elliptic differential operators, the b -differential (or 'totally characteristic') operators on a manifold with corners.

Since this is an introductory article, we only consider manifolds with boundary (and mostly even just \mathbb{R}_+). In this case, the operators are also called Fuchs type or cone operators. They have been studied by many authors, see e.g. [3], [13], [36]; some of them used pseudodifferential operator (Ψ DO) techniques. For a more complete list of references see [12].

In [31] and [26] the central results (on action and composition) were proved by direct calculation. There is a more systematic way to do this, using the Pull-Back and Push-Forward Theorem (as indicated in Figure 1 and Examples 3.1), and this shows more clearly the geometric reasons for the precise form of these results. Melrose alludes to this often in [26], and (probably) proceeds like this in the part of [22] that is not publicly available yet. Therefore, we will follow this systematic method here, but will be sketchy otherwise.

DEFINITION 4.1. A differential operator on a manifold with boundary X is a *b-differential operator* of order m , in symbols $P \in \text{Diff}_b^m(X)$, if it has smooth coefficients and, in any coordinate system around a boundary point in which x denotes a boundary defining function and $y = (y_1, \dots, y_{n-1})$ coordinates in the boundary, it has the form

$$(4.1) \quad P = \sum_{j+|\alpha| \leq m} a_{j\alpha}(x, y) (x\partial_x)^j \partial_y^\alpha$$

(α runs over multi-indices in \mathbb{N}^{n-1} , and $\partial_x = \partial/\partial x$ etc.) with functions $a_{j\alpha}$ that are smooth up to the boundary.³⁶

On manifolds without boundary the power of the Ψ DO calculus derives from two facts:

- The (principal) symbol of a Ψ DO describes the operator fairly precisely (up to lower order), and
- symbols are easy to invert.

We discuss this shortly in Subsection 4.1. Recall from Example 3.15.3 that a Ψ DO on a manifold X is just a distribution on $X \times X$ which is conormal with respect to

$$(4.2) \quad \Delta = \{(p, p) : p \in X\} \subset X \times X \quad (\text{the diagonal}),$$

and that the (principal) symbol is determined purely by the singular behavior at Δ (see Remark 3.16).

Therefore, for an extension of this theory to *b*-(pseudo)-differential operators (on \mathbb{R}_+ , say) one expects that it should be useful to define a class of distributions

³⁶For Fuchs type operators or cone operators one usually multiplies this with $x^{-\mu}$ for some positive number μ . This is inessential for parametrix constructions since the factor can be transferred to the parametrix. However, it makes an essential difference for the spectral theory (the analysis of the resolvent), which we don't consider here. See for example [3], [5], [15].

on $\mathbb{R}_+ \times \mathbb{R}_+$ by giving precise descriptions of their behavior in all possible limits. These are³⁷:

- (a) Approaching (and at) the interior of the diagonal Δ .
- (b) Approaching $\partial(\mathbb{R}_+^2) \setminus (0, 0)$.
- (c) Approaching the corner $(0, 0)$.

While there are obvious candidates for (a) and (b) (conormal singularity and complete asymptotics ('niceness' in Definition 2.3), respectively), it is unclear what a good description for (c) might be: Simultaneously one needs to describe the behavior of the conormal singularity on Δ when approaching zero, and of the coefficients in the asymptotics of (b).

This is most elegantly solved by blowing up the corner $(0, 0)$, which has the effect of separating the sets Δ and $\partial\mathbb{R}_+^2 \setminus (0, 0)$. Let us illustrate this by a simple example:

EXAMPLE 4.2. On \mathbb{R}_+ , consider the simplest non-trivial b -equation

$$\left(x \frac{d}{dx} + c\right) u(x) = v(x),$$

for some fixed $c \in \mathbb{C}$. Multiplying by x^{c-1} we get $\frac{d}{dx}(x^c u(x)) = x^{c-1} v(x)$, which can be integrated to yield

$$u(x) = x^{-c} a_0 + x^{-c} \int_0^x (x')^{c-1} v(x') dx'$$

with $a_0 = (x^c u(x))|_{x=0}$ (assuming $(x')^{c-1} v(x')$ is integrable near zero). For simplicity, consider only the solution with $a_0 = 0$. We can write it as

$$u(x) = \int_0^\infty K(x, x') v(x') \frac{dx'}{x'}$$

with

$$(4.3) \quad K(x, x') = \left(\frac{x'}{x}\right)^c H(x - x').$$

Here, $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t \leq 0$. Thus, K is (the kernel of) an inverse of the operator $x \frac{d}{dx} + c$ (on suitable function spaces), so it should be an example of a b -pseudodifferential operator.

As expected, K is singular at the two coordinate axes (actually only at $\{x' = 0\}$) and at the diagonal $\Delta = \{x = x'\} \subset \mathbb{R}_+^2$ (unless c happens to be a positive integer). Looking at K on the blow-up space $[\mathbb{R}_+^2, (0, 0)]$ (i.e. at $\beta^* K$ for $\beta : [\mathbb{R}_+^2, 0] \rightarrow \mathbb{R}_+^2$ the blow-down map) means rewriting K in terms of coordinates on this space; using projective coordinates x and

$$s = x'/x,$$

³⁷As already in Section 2 we always restrict attention to compact parts of the spaces involved. Therefore, we do not consider limits 'at ∞ '.

for example, we get:

$$\beta^* K(x, s) = s^c H(1 - s).$$

This is much nicer than (4.3) since:

- The submanifolds at which $\beta^* K$ is singular are disjoint.³⁸ (They are $\text{lb} = \{s = \infty\}$, $\text{rb} = \{s = 0\}$ and $\Delta_b = \{s = 1\}$, see Figure 3(b).)³⁹
- $\beta^* K$ is nice outside Δ_b .
- $\beta^* K$ has a conormal singularity at Δ_b , smoothly up to the boundary (i.e. up to $\{x = 0\}$).

Thus, we are lead to define (kernels of) b - Ψ DOs on a manifold X with boundary as a certain class of distributions on a blown-up space X_b^2 (where $X_b^2 = [\mathbb{R}_+^2, (0, 0)]$ for $X = \mathbb{R}_+$). These are then considered as kernels on X^2 by use of the identification of the interiors of X_b^2 and X^2 via the blow-down map.

What should we expect the *symbol* of a b - Ψ DO to be? X_b^2 has four distinguished submanifolds: $\text{lb}, \text{rb}, \text{ff}$ and

$$(4.4) \quad \Delta_b = \overline{\beta^{-1}(\text{interior}(\Delta))} = \{s = 1\},$$

and b - Ψ DOs are characterized by their behavior at them. Just as the symbol in the boundaryless case describes the leading behavior at Δ (which is the only distinguished submanifold then), one might expect the symbol in the b -calculus to describe the leading behavior at $\Delta_b, \text{ff}, \text{lb}, \text{rb}$. In fact, it turns out that only the former two are needed since their vanishing implies compactness (between spaces that are determined by the latter two!).

The b -calculus is introduced in two steps. They are motivated by the construction of a parametrix of an elliptic b -differential operator P . First, the 'small calculus' Ψ_b^* is constructed; it allows inversion of the symbol on Δ_b and thus, by the usual iterative procedure, inversion of P modulo errors that are smooth on Δ_b . Since this game is played away from lb, rb , operators in Ψ_b^* are assumed to vanish to infinite order there. However, as we saw in the example above, the inverse of even the simplest b -differential operator is not of this type. This is reflected in the fact that the 'remainders' in the parametrix construction, i.e. elements of $\Psi_b^{-\infty}$, are not compact operators, even if X is compact. Therefore, in a second step the 'full calculus' is introduced. This allows inversion of the symbol at ff (the 'conormal

³⁸This is better than needed. 'Normal crossings' (i.e. locally looking like coordinate subspaces in a suitable coordinate system) would be enough. This is satisfied by the boundary faces of a mwc (and by the four distinguished submanifolds $\text{lb}, \text{rb}, \text{ff}, \Delta_b$ of $[\mathbb{R}_+^2, (0, 0)]$, see below), but not by the singular support of K .

³⁹Strictly speaking, one should also check $\beta^* K$ in the $x', x/x'$ coordinates (in the sequel we will neglect this when it gives no information). Alternatively, you may use instead the coordinates $\rho = x + x', \tau = (x - x')/(x + x')$ (see 2.3), then

$$\beta^* K(\rho, \tau) = \left(\frac{1 - \tau}{1 + \tau} \right)^c H(\tau)$$

and $\text{lb} = \{\tau = -1\}$, $\text{rb} = \{\tau = 1\}$, $\Delta_b = \{\tau = 0\}$.

symbol' or 'indicial operator'); the price to pay is non-trivial asymptotic behavior at lb, rb.

For the sake of presentation we mostly work with the simplest manifold with boundary, \mathbb{R}_+ ; most ideas may be understood already in this case.⁴⁰ At the end of the chapter we add some remarks on the changes necessary when dealing with a general manifold with boundary.

4.1. Classical pseudodifferential operators. We shortly summarize the essential ingredients of the classical Ψ DO calculus, and how they are used to find parametrices, i.e. approximate inverses, for elliptic (pseudo-)differential operators on a compact manifold X . Extensive treatments can be found in [11] and [38], for example. A similar (and more general) axiomatic treatment was given in [35].

Given, for each $m \in \mathbb{R}$:

(Op) Classes Ψ^m of distributions on X^2 , with $\Psi^m \subset \Psi^{m+1}$. (Ψ^m is taken to be the set of distributions on X^2 conormal with respect to the diagonal Δ , of order m .)

(Symb) Classes S^m of symbols, with $S^m \subset S^{m+1}$, and symbol maps $\sigma : \Psi^m \rightarrow S^{[m]} := S^m/S^{m-1}$. (S^m is taken to be the set of smooth functions on T^*X with complete asymptotic expansions in homogeneous components of order $\leq m$. Homogeneity refers to the covariable ξ , and the asymptotics is for $\xi \rightarrow \infty$.)

$\sigma(u)$ is defined by the principal symbol of $u \in \Psi^m$, which is a function on $N^*\Delta$ (see Remark 3.16), using the canonical identification $N^*\Delta \cong T^*X$. Note that $S^{[m]} \cong$ smooth functions on $T^*X \setminus 0$, homogeneous in ξ of order m .)

The essential properties of these objects are:

(Alg) $\bigcup_m \Psi^m$ and $\bigcup_m S^m$ are graded algebras, and σ is an algebra homomorphism. (The products are taken as composition, defined by (3.3), and pointwise multiplication, respectively, and 'graded' means $P \in \Psi^m, Q \in \Psi^l \implies PQ \in \Psi^{m+l}$, and similarly for symbols; the main point is that σ respects products.)

(Exact) The sequence

$$(4.5) \quad 0 \rightarrow \Psi^{m-1} \hookrightarrow \Psi^m \xrightarrow{\sigma} S^{[m]} \rightarrow 0$$

is exact for every m . This means:

(E1) For each $a \in S^{[m]}$ there is $P \in \Psi^m$ with $\sigma(P) = a \pmod{S^{m-1}}$.

(E2) If $P \in \Psi^m$ and $\sigma(P) = 0$ then $P \in \Psi^{m-1}$.

Part of (Alg) was checked in 3.3.2, and (Exact) is straight-forward from the definitions.

⁴⁰Of course, one must resist the temptation to use simpler arguments only suited for ordinary differential equations.

Finally, we define: An element $P \in \Psi^m$ is *elliptic* if $\sigma(P)$ is invertible (then its inverse lies in S^{-m} necessarily). A *parametrix* of order k for $P \in \Psi^m$ is a $Q \in \Psi^{-m}$ such that both $PQ - \text{Id}$ and $QP - \text{Id}$ lie in Ψ^{-k} .

The main fact is:

THEOREM 4.3 (Parametrix construction for elliptic Ψ DO). *(Alg) and (Exact) above imply: If $P \in \Psi^m$ is elliptic then it has a parametrix of any order.*

Let us quickly recall the proof: By ellipticity of P , $\sigma(P)^{-1}$ is invertible with inverse in S^{-m} . By (E1), there is $Q \in \Psi^{-m}$ with $\sigma(Q) = \sigma(P)^{-1}$. Then, by (Alg), $\sigma(PQ - \text{Id}) = \sigma(P)\sigma(Q) - \sigma(\text{Id}) = 0$, so by (E2) (with $m = 0$) we have $R := \text{Id} - PQ \in \Psi^{-1}$. Thus, Q is a 'right' parametrix of order 1. Set $Q_k = Q(\text{Id} + R + \dots + R^{k-1})$, then $PQ_k = (\text{Id} - R)(\text{Id} + R + \dots + R^{k-1}) = \text{Id} - R^k$ and $R^k \in \Psi^{-k}$, so Q_k is a right parametrix of order k . By the same procedure we get a left parametrix Q'_k order k . Evaluating $Q'_k PQ_k$ in two ways one obtains that $Q_k - Q'_k \in \Psi^{-m-k}$, and from this that Q_k is also a left parametrix of order k . \square

This may be refined slightly: One also has

(AC) Asymptotic completeness: If $P_i \in \Psi^{m-i}$ for $i \in \mathbb{N}_0$ then there is $P \in \Psi^m$ with $P - \sum_{i=0}^N P_i \in \Psi^{m-N-1}$ for all N .

This clearly implies that elliptic elements have parametrices of order ∞ (usually just called parametrices). This improvement is mainly cosmetic and not needed in most applications.

Note that these arguments were purely formal and did not use any properties beyond (Alg), (Exact) and (AC). Therefore, the same result holds with different choices of Ψ^* , S^* , σ .

However, in order to apply Theorem 4.3 to problems of differential equations, one needs:

(Diff) $\text{Diff}^m \subset \Psi^m$, where Diff^m denotes the differential operators of order m (with smooth coefficients).

(EII) The 'usual' elliptic operators one is usually interested in (Dirac, Laplace) are elliptic in the sense above.⁴¹

Finally, in order to make all of this useful for analysis (e.g. for proving regularity of solutions of elliptic PDE) one needs:

(Map) Mapping properties of $P \in \Psi^m$ (e.g. continuity on Sobolev spaces).

(Neg) The remainders, i.e. the elements in $\Psi^{-\infty}$, are actually 'negligible' (e.g. compact, trace class, smoothing, etc.).

Of course, the usual Ψ DO calculus has all of these properties.

REMARKS 4.4.

⁴¹For example, this is not satisfied if one takes Ψ^m as above but lets $S^m = \Psi^m$, $\sigma = \text{id}$, which satisfies all other requirements!

- (1) The motivation for the definition of Ψ^* lies in solving differential equations by Fourier transform, which gives precise solutions for constant coefficient equations. In this case, it suffices to invert the symbol. The rest is just the algebra that's needed to make this method work for non-constant coefficient equations.⁴² Note that the Fourier transform does not appear explicitly. It is stowed away in the definition of conormal distributions, and is only used in the proofs of pull-back and push-forward theorem for these (cf. the hierarchy in Figure 1).
- (2) When trying to construct Ψ^* , one has to find a compromise between opposing forces: It has to be large enough to contain elliptic differential operators and their parametrices, but small enough for $\Psi^{-\infty}$ to be actually negligible.
- (3) If X is not compact then composition of $P, Q \in \Psi^*$ may be undefined (since the integral $\int P(x, y) Q(y, z) dy$ may diverge 'at ∞ '). The simplest remedy is to replace Ψ^* by

$$\Psi_{\text{prop}}^* = \{P \in \Psi^* : K \Subset X \implies (\text{supp } P) \cap (X \times K) \text{ is compact}\},$$

the set of properly supported Ψ DOs. Then everything goes through as before, except that remainders are still smoothing but not compact, so that (Neg) should be localized:

(Neg') If $P \in \Psi_{\text{prop}}^{-\infty}$ and $\phi \in C_0^\infty(X)$ then ϕP (i.e. P followed by multiplication by ϕ) is negligible (i.e. compact, trace class, etc.).

4.2. The small b -calculus. The small b -calculus takes care of the conormal singularity on the diagonal and its behavior near the boundary of the diagonal. However, it does not admit non-trivial asymptotics near the boundary of X^2 away from the corner $(0, 0)$.

We will first motivate the definition of the small b -calculus by calculating the kernels of b -differential operators as distributions on X_b^2 , then discuss the annoying but non-negligible question of (half-)densities (but see Footnote 16), and finally define the small calculus and check its properties, in particular composition.

Recall that we work on $X = \mathbb{R}_+$, and that $X_b^2 = [\mathbb{R}_+^2, (0, 0)]$.

4.2.1. *Kernels of b -differential operators.* For a start, let's be naive and consider as integral kernel of an operator P a distribution $K_P(x, x')$ on X^2 such that

$$(4.6) \quad (Pu)(x) = \int K_P(x, x')u(x') dx' \quad (\text{'naive kernel'}).$$

The simplest b -differential operator is the identity Id. Its kernel (on X^2) is $\delta(x-x')$. To obtain the kernel as distribution on X_b^2 , we simply rewrite this in terms of

⁴²In other words, inversion of the symbol $\sigma(P)(x_0, \xi)$ corresponds to inversion of (the principal part of) the constant coefficient operator $P(x_0, D)$ obtained by freezing coefficients at x_0 (which acts on $T_{x_0}X$), and the parametrix construction shows how to patch these local inverses together.

coordinates on X_b^2 , for example of the projective coordinates $x, s = x'/x$. Since δ is homogeneous of degree -1 , we get

$$K_{\text{Id}} = \delta(x - x') = \delta(x(1 - \frac{x'}{x})) = \frac{1}{x}\delta(1 - s).$$

Next, let us consider $x\partial_x$. Since δ' is homogeneous of degree -2 , we get

$$K_{x\partial_x} = x\delta'(x - x') = \frac{1}{x}\delta'(1 - s).$$

Iterating this, one sees easily:

THEOREM 4.5 (*b*-differential operators as naive kernels). *The 'naive' kernels (as in (4.6)) of b-differential operators on \mathbb{R}_+ are precisely the distributions of the form*

$$(4.7) \quad \frac{1}{x} \sum_{\text{finite}} a_j(x)\delta^{(j)}(1 - s)$$

with a_j smooth up to $x = 0$.

4.2.2. Densities, half-densities and their b-rethren. Since we want to switch between different coordinate systems (as above), we have to know how objects transform under such changes; to put it differently, we should define things invariantly. The idea of 'integral kernel' involves integration and therefore measures. There are three obvious possibilities to take care of these:

- Integral kernel is a function, acts on densities.
- Integral kernel is a density, acts on functions.
- Integral kernel is a 'density in x' ', acts on functions.

In the first case, the result $\int K(x, x')\mu(x')$ would be a function. Therefore, this would describe an operator mapping densities to functions. But then two such operators cannot be composed⁴³. The same problem occurs in the second case: Here functions are mapped to densities. The third possibility avoids this problem, here functions are mapped to functions, but now the symmetry between x and x' is broken. There is a way to overcome this flaw as well:

- Integral kernel is a half-density, acts on half-densities.

For a very short introduction to half-densities, see the Appendix. Now there is only one kind of objects: Operator kernels are half-densities, and they map half-densities to half-densities, so they can be composed without making additional choices. Also, an integral kernel is a 'symmetric' object, i.e. x, x' can be interchanged freely.⁴⁴

⁴³unless one identifies functions with densities by (non-canonical) choice of a fixed density, which is what we wanted to avoid in the first place!

⁴⁴The pedantic reader will notice that now the formula (3.2) translating action into the pull-back/push-forward world does not make sense directly: Although π_2^*v is well-defined for a half-density v since π_2 is a fibration, the product $A \cdot \pi_2^*v$ is not quite a density on X^2 (a half-density in the first factor is missing), so its push-forward is not defined. This can be remedied easily: Fix any non-vanishing half-density μ on X . Then the push-forward $\pi_{1*}(\pi_1^*\mu \cdot A \cdot \pi_2^*v)$ is a well-defined density on the first factor X , and dividing it through μ one obtains the result, which

Since (half-)densities are differential objects, we should, by Principle 4 in the Introduction, use b -half-densities instead. Let us illustrate this by an example: Consider the identity again. Now using the same letter for kernels and operators, we have

$$\begin{aligned} \text{Id} &= \delta(x - x') |dx dx'|^{1/2} \quad \text{as half-density} \\ &= x \delta(x - x') \left| \frac{dx}{x} \frac{dx'}{x'} \right|^{1/2} \quad \text{as } b\text{-half-densities} \\ &= \delta(1 - s) \left| \frac{dx}{x} \frac{ds}{s} \right|^{1/2} \quad \text{as } b\text{-half-densities on } X_b^2 \end{aligned}$$

where in the second line we used $(x')^{1/2} \delta(x - x') = x^{1/2} \delta(x - x')$ and in the last line the calculation before Theorem 4.5, together with (3.11). In general, one sees easily that the x^{-1} factor in Theorem 4.5 is always canceled when we use half-densities, i.e.

$$(4.8) \quad \sum_j a_j(x) (x \partial_x)^j = \sum_j a_j(x) \delta^{(j)}(1 - s) \left| \frac{dx}{x} \frac{ds}{s} \right|^{1/2}.$$

Recall that $\{s = 1\}$ is just the b -diagonal Δ_b . An expression like $\sum a_j(x) \delta^{(j)}(1 - s)$ is called a Dirac distribution on Δ_b . We may now restate Theorem 4.5 in a very simple form (Lemma 4.21 in [26]):

THEOREM 4.6 (*b -differential operators as b -half-density kernels*). *When considered as b -half-densities on X_b^2 , the kernels of b -differential operators on $X = \mathbb{R}_+$ are precisely the Dirac distributions on Δ_b which are smooth up to the boundary.*

4.2.3. *Definition and properties of the small b -calculus.* Theorem 4.6 suggests:

DEFINITION 4.7. The *small b -calculus* $\Psi_b^m(X)$, $m \in \mathbb{R}$, is defined as the set of (b -half-density-valued) distributions u on X_b^2 satisfying

- (a) u is conormal of order m with respect to Δ_b , smoothly up to the boundary ff ,
- (b) u vanishes to infinite order at lb and rb .⁴⁵

We now want to see that the ' Ψ DO machine' from Subsection 4.1 works for Ψ_b^* as well. We address the essential properties listed there, in varying degrees of completeness.

(Symb) For a b -differential operator as in (4.1) the principal symbol is defined as

$$p(x, y, \lambda, \eta) = \sum_{j+|\alpha|=m} a_{j\alpha}(x, y) \lambda^j \eta^\alpha.$$

is immediately checked to be independent of the choice of μ (see [22]). A similar remark applies to composition (3.3).

⁴⁵Requiring u to vanish in a neighborhood of lb and rb would work as well, but would be somewhat less natural in the context of the full calculus.

We don't give the definition for general b - Ψ DOs here. We only remark that, in order to make this defined invariantly, it should be considered as function on a bundle called ${}^bT^*X$ (the b -cotangent bundle) by Melrose. See [26], Sections 2.2 and 4.10.

Ellipticity means that $p(x, y, \lambda, \eta) \neq 0$ whenever $(\lambda, \eta) \neq 0, x \geq 0$.⁴⁶ **(Alg)** Let us check that Ψ_b^* is closed under composition.⁴⁷ (That the symbol map preserves products is then done precisely as in the classical case.) As we discussed in Remark 3.13.2, for a systematic analysis of composition one needs to blow-up the space X^3 , and a good way to do this is to first blow up zero and then the preimages of the three coordinate axes. The resulting space is called 'triple b -space' X_b^3 , see Figure 5. Recall that the three projections $\pi_i : X^3 \rightarrow X^2, i = 1, 2, 3$ lift to b -fibrations $\pi_{ib} : X_b^3 \rightarrow X_b^2$ (called $\tilde{\pi}_i$ before). The composition formula (3.3), rewritten in terms of b -spaces, reads:

$$(4.9) \quad P \circ Q = \pi_{2b*}(\pi_{3b}^* P \cdot \pi_{1b}^* Q).$$

We give the argument purely in geometric terms. Some details are left to the reader. Let P, Q be distributions on X_b^2 , conormal with respect to Δ_b and supported near Δ_b . We only check condition (a) in Definition 4.7. Condition (b) will be discussed shortly in the context of the full calculus (see the proof of Theorem 4.9).

- (1) The maps π_{ib} are transversal to Δ_b . By the pull-back theorem for conormal distributions (see 3.3.3) $\Delta_{ib} := \pi_{ib}^{-1}(\Delta_b)$ are p-submanifolds (see Figure 5 for Δ_{3b}), and $\pi_{3b}^* P, \pi_{1b}^* Q$ are conormal with respect to Δ_{3b}, Δ_{1b} , respectively.
- (2) The behavior of the space X_b^3 and the maps π_{ib} transversal to $\Delta_b^{(3)} := \Delta_{1b} \cap \Delta_{3b}$ (the space diagonal in X_b^3) remains the same at the boundary (i.e. at fff) as in the interior (cf. the discussion in 3.3.3), and $\pi_{2b}(\Delta_b^{(3)}) = \Delta_b$. Therefore, the discussion of product and push-forward in 3.3 carries over literally, as far as the part v in (3.26) is concerned.
- (3) The map π_{2b} is a fibration near Δ_{1b} and Δ_{3b} and satisfies the transversality condition in the push-forward Theorem 3.17. Case (a) of that theorem and the arguments after (3.27) apply, therefore part w in (3.26) is smooth. Since P and Q are supported near Δ_b , $P \circ Q$ is supported near Δ_b .
- (4) It remains to check the smoothness of the conormal singularity of $P \circ Q$ up to the boundary, i.e. at ff. By the Pull-Back Theorem 3.12 (or rather its trivial extension to the present context), $\pi_{3b}^* P$ and $\pi_{1b}^* Q$ have index set 0

⁴⁶Schulze's definition (see [37]) of ellipticity requires, in addition, that the 'conormal operator' (the operator family on the boundary given locally by $\sum_{j,\alpha} a_{j\alpha}(0, y) \lambda^j \partial_y^\alpha, \lambda \in \mathbb{C}$) be invertible on a 'weight line' $\operatorname{Re} \lambda = \frac{n}{2} - \gamma$, where γ is a parameter. This condition will be imposed also in the parametrix construction in the full calculus, see Remarks 4.10 below. It is satisfied for all but countably many values of γ and ensures that P is Fredholm between suitable γ -weighted Sobolev spaces (if X is compact). Lesch shows ([13]) that Fredholmness holds even without this condition, for any closed extension of P (whose domain may then not be a Sobolev space).

⁴⁷Assuming, as in 3.3.2, that there is no problem with integrability at infinity, i.e. that at least one factor is properly supported. Compare Footnote 35 and Remark 4.4.3.

at fff (recall that 0 stands for smooth behavior, see Remark 2.4.3)⁴⁸, then so does their product, and then the Push-Forward Theorem 3.10 shows that $P \circ Q$ has index set 0 at ff .

(Exact) Checking the short exact sequence (4.5) is almost trivial (as in the classical case), once symbols are defined.

(AC) Asymptotic completeness is easy, just the same as in the classical case.

(Diff) We have $\text{Diff}_b^m \subset \Psi_b^m$ by construction.

(Map) See the section on the full calculus.

(Ell) Typical elliptic b -operators are the Laplacian and Dirac operators on Riemannian manifolds with infinite cylindrical ends, or (up to a conformal factor) with conical points.

(Neg) This is the main difference between the small calculus and the classical Ψ DO calculus, and the point that makes an extension (the full calculus) necessary:

$$(4.10) \quad \text{Elements in } \Psi_b^{-\infty} \text{ are not necessarily compact.}$$

This is due to the $1/x$ factor that always occurs in b -densities⁴⁹. More precisely, since $X = \mathbb{R}_+$ is not compact we should look at (Neg') instead: Choose $\phi \in C_0^\infty(\mathbb{R}_+)$ with $\phi(0) = 1$. Then:

$$(4.11) \quad \phi P \in \Psi_b^{-\infty} \text{ is compact} \iff P_{\text{ff}} \equiv 0.$$

This is not hard to check directly, see [26], Section 4.14. We'll just check the even easier:

$$\phi P \in \Psi_b^{-\infty} \text{ is Hilbert-Schmidt} \iff P_{\text{ff}} \equiv 0.$$

PROOF. Note that the condition on a half-density α to be L^2 (square-integrable) is independent of the choice of a measure, since $|\alpha|^2$ is a density. Similarly, for an operator A acting on half-densities on a space X , being Hilbert-Schmidt is equivalent to $\int_{X^2} |A|^2 < \infty$ for the Schwartz kernel. Assume for (notational) simplicity that P is supported in a compact set disjoint from lb and rb . When we use projective coordinates $x, s = x'/x$ on X_b^2 then $P \in \Psi_b^{-\infty}$ means $P(x, s) = p(x, s) \left| \frac{dx}{x} \frac{ds}{s} \right|^{1/2}$ with p smooth in $x \in \mathbb{R}_+, s \in \mathbb{R}_+$; ϕP has kernel $\phi(x)p(x, s)$, which by assumption is supported in $x \leq C, C^{-1} \leq s \leq C$, for some positive C . Therefore,

$$(4.12) \quad \int_{X^2} |\phi P|^2 = \int_0^C \left(\int_{C^{-1}}^C |\phi(x)p(x, s)|^2 \frac{ds}{s} \right) \frac{dx}{x},$$

and this is finite iff the smooth function in parentheses vanishes at $x = 0$, i.e. iff $p(0, s) = 0$ for all s . \square

⁴⁸Here one needs to check that the (half-)density factors do not introduce extra powers of bdf's. Compare Footnote (3.11).

⁴⁹in other words, the $1/x$ -factor in Theorem 4.5, so this is not an artifact of the b -formalism!

In summary, we have seen that the small calculus has all properties required for the Ψ DO machine described in Subsection 4.1, except (Neg). Therefore, applying the machine one can find a parametrix $Q \in \Psi_b^{-m}$ for an elliptic element $P \in \Psi_b^m$, that is an inverse modulo $\Psi_b^{-\infty}$. However, since this error term is not compact (even after localization), this is not enough to draw many conclusions about the analytic properties of P .

Therefore, more work is needed: The full calculus.

4.3. The full b -calculus. Introducing the full b -calculus in detail would exceed the scope of this article. We refer the reader to [26], chapter 5, for an extensive treatment. Here we will only state its definition and main properties, check how it acts on nice function (which gives, again, a nice illustration of Pull-Back and Push-Forward Theorem), and then outline how this definition arises when one tries to improve the parametrix construction for elliptic b -differential operators.

DEFINITION 4.8. The *full b -calculus* on $X = \mathbb{R}_+$ is the collection of spaces $\Psi_b^{m,\mathcal{E}}$ defined as follows: Let $m \in \mathbb{R}$ and let $\mathcal{E} = (E_{\text{lb}}, E_{\text{rb}})$ be an index family for X^2 . Then a distribution u on X_b^2 is in $\Psi_b^{m,\mathcal{E}}$ iff $u = u_1 + u_2 + u_3$ with

- (a) $u_1 \in \Psi_b^m$, the small calculus,
- (b) u_2 is nice on X_b^2 , with index family $(E_{\text{lb}}, 0, E_{\text{rb}})$ at $(\text{lb}, \text{ff}, \text{rb})$ (0 is the 'smooth' index set, see Remark 2.4.3),
- (c) $u_3 = \beta^*v$, where $\beta : X_b^2 \rightarrow X^2$ is the blow-down map and v is nice on X^2 with index family \mathcal{E} .

Thus, the conormal singularity at Δ_b is the same as in the small calculus, but in addition one allows non-trivial asymptotic expansions at lb and rb, plus an additional 'residual' term which is even nice on X^2 .⁵⁰

The main properties of the full calculus are:

THEOREM 4.9. *The full calculus acts on nice functions and is conditionally closed under composition, more precisely:*

Action: Let $P \in \Psi_b^{m,\mathcal{E}}$ and w be a nice function on \mathbb{R}_+ with index set F .
If

$$\inf E_{\text{rb}} + \inf F > 0$$

then Pw is nice with index set $E_{\text{lb}} \cup F$.

Composition: Let $P \in \Psi_b^{m,\mathcal{E}}$, $Q \in \Psi_b^{m',\mathcal{F}}$ and $\mathcal{F} = (F_{\text{lb}}, F_{\text{rb}})$. If

$$\inf E_{\text{rb}} + \inf F_{\text{lb}} > 0$$

then $P \circ Q \in \Psi_b^{m+m', (E_{\text{lb}} \cup F_{\text{lb}}, E_{\text{rb}} \cup F_{\text{rb}})}$.

(To avoid problems at infinity, assume P to be properly supported.)

⁵⁰ u_3 can not be absorbed into u_2 since it has index family $(E_{\text{lb}}, E_{\text{lb}} \cup E_{\text{rb}}, E_{\text{rb}})$ by the Pull-Back Theorem; however, it is much better than just any function with these index sets.

PROOF. These are Proposition 5.52 and Theorem 5.53 in [26]. The proofs there avoid the systematic use of Pull-Back and Push-Forward Theorem. Let us check the statement on action in the systematic way. Write the kernel of P as $u_1 + u_2 + u_3$ as in Definition 4.8. Let us assume $P = u_2$ for simplicity, this is the most interesting part.

Let $\pi_{1/2} : X^2 \rightarrow X$ be the projections onto the first and second factor and $\pi_{ib} = \pi_i \circ \beta : X_b^2 \rightarrow X$ their analogues on X_b^2 . Then, formula (3.2) becomes

$$Pw = \pi_{1b*}(P \cdot \pi_{2b}^* w).$$

Using the Pull-Back and Push-Forward Theorem, we can now read off the result from Figure 6:

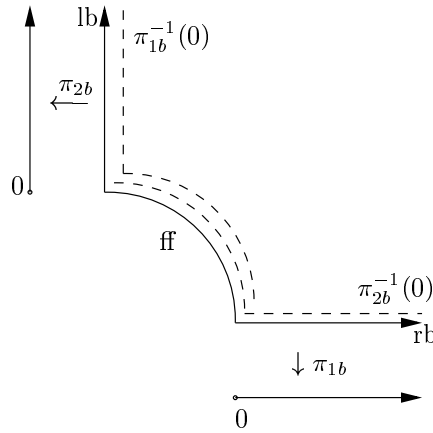


FIGURE 6. Applying an operator to a function in the full calculus

Since $\pi_{2b}^{-1}(0) = \text{ff} \cup \text{rb}$, the Pull-Back Theorem shows that $\pi_{2b}^* w$ has index family $(0, F, F)$ (at $(\text{lb}, \text{ff}, \text{rb})$); therefore, $P \cdot \pi_{2b}^* w$ has index family $(E_{\text{lb}}, F, E_{\text{rb}} + F)$.⁵¹ Finally, since π_{1b} is a b -fibration and $\pi_{1b}^{-1}(0) = \text{lb} \cup \text{ff}$, the Push-Forward Theorem shows that $\pi_{1b*}(P \cdot \pi_{2b}^* w)$ is nice with index set $E_{\text{lb}} \cup F$, provided the integrability condition $\inf(E_{\text{rb}} + F) > 0$ holds (since rb is the only face which is not mapped to 0 under π_{1b}).

The proof for composition proceeds similarly. Here one needs to know in addition that $\tilde{\pi}_2 : X_b^3 \rightarrow X_b^2$, the lift of the 'middle' projection $X^3 \rightarrow X^2$ already used in (4.9), is a b -fibration. This was checked in Remark 3.13.2. \square

REMARKS 4.10 (Parametrix, why full calculus, etc.).

⁵¹Addition of index sets E, F is defined in the obvious way:

$$E + F = \{(z + w, k + l) : (z, k) \in E, (w, l) \in F\}.$$

If u, v are nice with index families \mathcal{E}, \mathcal{F} then clearly uv is nice with index family $\mathcal{E} + \mathcal{F}$.

- (1) Let us see how terms of type u_2 arise from improving the parametrix construction in the small calculus. Let $P \in \text{Diff}_b$ be elliptic⁵², and Q_1 a parametrix in the small calculus, i.e. $PQ_1 = \text{Id} + R$ with $R \in \Psi_b^{-\infty}$ (and similarly for Q_1P).

- As we saw in (4.11), the obstruction to compactness of the remainder R is the restriction of its Schwartz kernel to the front face ff . Therefore, in order to improve the parametrix we must 'cancel' this obstruction.
- For any $A \in \Psi_b^*$ define the *indicial operator* $I(A) \in \Psi_b^*$ by

$$(4.13) \quad A = A(x, s) \left| \frac{dx ds}{x s} \right|^{1/2} \implies I(A) = A(0, s) \left| \frac{dx ds}{x s} \right|^{1/2},$$

i.e. by 'freezing coefficients' at $x = 0$.⁵³ Since $\text{ff} = \{x = 0\}$, we have

$$A|_{\text{ff}} = 0 \iff I(A) = 0.$$

If P is a b -differential operator then (4.8) shows that

$$(4.14) \quad P = \sum_j a_j(x)(x\partial_x)^j \implies I(P) = \sum_j a_j(0)(x\partial_x)^j,$$

i.e. $I(P)$ has constant coefficients as a b -operator.

- Since $I(P)$ has constant coefficients it can be inverted easily: Substitute $x = e^t$, so that $x\partial_x = \partial_t$, and then solve a constant coefficient ordinary differential equation.

Alternatively, this may be done using the Mellin transform (which is just the Fourier transform under this change of variables), since it transforms $I(P)$ into a multiplication operator. The latter method works for any elliptic $A \in \Psi_b^*$ instead of P .

The calculation shows (see [26], equation (5.28) and Lemma 5.16) that the inverse of $I(P)$ (for elliptic $P \in \text{Diff}_b^*$) has kernel of the type $u_1 + u_2$. Two important points are:

- For the index sets of u_2 one has⁵⁴

$$(4.15) \quad \begin{aligned} E_{\text{lb}} \cup (-E_{\text{rb}}) &= \text{Spec}_b I(P) := \\ &:= \{(z, l) : \sum_j a_j(0)z^j \text{ has a zero of order } \geq l + 1 \text{ at } z\} \end{aligned}$$

(with $-E_{\text{rb}} := \{(-z, p) : (z, p) \in E_{\text{rb}}\}$).

⁵²In the case $X = \mathbb{R}_+$ this simply means $a_m(x) \neq 0$ for all x , in (4.1).

⁵³For a satisfying discussion of $I(A)$ it now actually matters that its kernel is not properly supported. Melrose deals with this by compactifying, i.e. adding suitable points at $x = \infty$, see Section 4.15 in [26]; we neglect this here.

⁵⁴Melrose's definition of Spec_b , (5.10) in [26], differs from this by a 90 degree rotation. Our convention fits better to our definition of index sets (which is the same as Melrose's in [26]; but in [22] index sets are also rotated by 90 degrees; this is more consistent with conventions in (mathematical literature on) scattering theory).

- There is not a unique inverse, but a whole family, $(I(P)^{-1})_\gamma$, parametrized by a real parameter γ . In terms of kernels, γ determines how $\text{Spec}_b I(P)$ splits up into E_{1b} and E_{rb} . Functional analytically, γ is the weight of a pair of suitable Sobolev spaces on which $I(P)$ is actually invertible with inverse $(I(P)^{-1})_\gamma$. γ is restricted to lie in $\mathbb{R} \setminus \{\text{Re } z : (z, 0) \in \text{Spec}_b I(P)\}$, and $(I(P)^{-1})_\gamma$ is locally constant on this set (as a distribution). See [26], Proposition 5.15, and Footnote 46.⁵⁵
 - Finally, a parametrix for P may be constructed as follows: Set $Q = Q_1 + Q_2$, where Q_2 is any (compactly supported) operator with $I(Q_2) = -I(P)^{-1}I(R)$. Then one checks easily that $PQ = \text{Id} + R'$ with $I(R') = 0$.⁵⁶ Since R is smoothing, Q_2 is of the type u_2 . R' is in $\Psi_b^{-\infty, \mathcal{E}\gamma}$ and vanishes on ff , so it is compact by a similar argument as before (after localization). More precisely, one has a parametrix for each admissible parameter γ , and R' is compact on the Sobolev spaces with weight γ .
- (2) This parametrix construction may be formalized analogous to the classical Ψ DO calculus in 4.1: Regard the set of constant coefficient operators as second symbol space (just like for the usual symbols, their inversion is easier than that of P itself) and I as second symbol map. The central fact is again a short exact sequence:

$$0 \rightarrow \rho \Psi_b^{m, \mathcal{E}} \hookrightarrow \Psi_b^{m, \mathcal{E}} \xrightarrow{I} \Psi_{b, \text{const}}^{m, \mathcal{E}} \rightarrow 0,$$

where now ρ is a boundary defining function for ff , and $m \in \mathbb{R}$ and the index family \mathcal{E} are (almost) arbitrary (see (5.160) in [26]). The parametrix is obtained by combined use of both symbol maps and both short exact sequences. (The first symbol is defined from the singularity of u_1 at Δ_b as in the small calculus.)

- (3) The parametrix constructed above does not contain a term of type u_3 . But u_3 is contained in the calculus since it arises when composing two terms of type u_2 . In particular, such compositions are necessary when improving the parametrix further (for example, making R vanish to higher than first order at ff , see the proof of Theorem 4.3). The index set \mathcal{E} will have to be enlarged in this process. Such a more precise parametrix is constructed in [26], Sections 5.18-5.25.

⁵⁵In Example 4.2, we have $I(P) = P$ and $\text{Spec}_b I(P) = \{(-c, 0)\}$. The inverse we constructed there had $E_{1b} = \emptyset$, $E_{rb} = \{(c, 0)\}$. It corresponds to $\gamma > -\text{Re } c$. As an exercise, check that it maps $x^\gamma L_{\text{comp}}^2(dx/x) \rightarrow x^\gamma L_{\text{loc}}^2(dx/x)$. The latter space may be improved to a suitably defined H^1 -Sobolev space.

⁵⁶Here one needs that I preserves products. This is clear from (4.14) for b -differential operators, but requires a little work in general.

In fact, this is the reason why we identify $P|_{\text{ff}}$ with the operator $I(P)$: This is done precisely in order to make $P \mapsto P|_{\text{ff}}$ an algebra homomorphism (the 'second symbol map', see below)!

4.4. General manifolds with boundary. We describe shortly the changes needed in small and full calculus when considering a general manifold with boundary X instead of \mathbb{R}_+ . For simplicity, we assume ∂X to be connected. x, y will denote local coordinates as in (4.1):

- *Definition of X_b^2 and X_b^3 :* In X^2 , one has boundary defining functions x (for the boundary of the first factor) and x' (for the second). The 'corner' in X^2 is the submanifold of codimension two $(\partial X)^2 = \{x = x' = 0\}$. Then

$$X_b^2 := [X^2, (\partial X)^2].$$

If y, y' are local coordinates in the boundary of the first and second factor, then local coordinates on X_b^2 are $x, s = x'/x, y, y'$ (and $x', x/x', y, y'$). In other words, everything is as before, only with y, y' as parameters. The b -diagonal is defined by the first equation in (4.4), in coordinates

$$\Delta_b = \{s = 1, y = y'\}.$$

X_b^2 again has three boundary hypersurfaces, denoted lb, rb, ff as before and locally given by $\{s = \infty\}, \{s = 0\}, \{x = 0\}$, respectively.

The triple b -space X_b^3 is defined by first blowing up $(\partial X)^3$ in X^3 and then the (now disjoint) preimages of $(\partial X)^2 \times X, \partial X \times X \times \partial X, X \times (\partial X)^2$. Again, this means doing the same as for \mathbb{R}_+ , carrying the y -variables along as parameters.

- *Small calculus:* Theorem 4.6 and Definition 4.7 extend literally, and also the discussion of properties (except that in (4.12) an additional $dy dy'$ integration is needed).
- *Full calculus:* Definition 4.8 and Theorem 4.9 (and its proof) and the first and last point of Remark 4.10.1 (the parametrix construction) extend literally, as well as Remarks 4.10.2 and 3. $I(A)$ is now defined on $\mathbb{R}_+ \times \partial X$, and defined as in (4.13), except that A depends on y, y' also (in local coordinates on the boundary) and the half-density factor is $\left| \frac{dx}{x} \frac{ds}{s} dy dy' \right|^{1/2}$. Similarly, in (4.14) the a_j depend on y also and in $I(P)$ are replaced by $a_j(0, y)$. But now $I(P)$ has constant coefficients only in x , not in y !⁵⁷ Therefore, it should be considered as ordinary differential operator (in x) whose coefficients are partial differential operators on ∂X :

$$I(P) = \sum A_j \cdot (x\partial_x)^j, \quad A_j \in \text{Diff}(\partial X).$$

Then the inversion using Mellin transform works as before.

Spec_b is defined as in (4.15), with $a_j(0)$ replaced by A_j . Since $I(P, z) := \sum_j A_j z^j$ is now an operator⁵⁸ on ∂X for every z , the notion of 'zero' must be interpreted suitably: as a point z where $I(P, z)$ is not invertible. ('Order' may also be defined easily, see Section 5.2 in [26].) The role of $\text{Spec}_b I(P)$ in determining the asymptotic type of a parametrix at lb and

⁵⁷This is the 'partial freezing of coefficients' mentioned in the Introduction.

⁵⁸ $I(P, z)$ is called 'conormal symbol' by some authors, e.g. Schulze [37], see Footnote 46.

rb is as before (except that, for coordinate invariance, E_{1b}, E_{rb} have to be 'completed', cf. Footnote 7). The only *essentially new features* are:

- $\text{Spec}_b I(P)$ may be an infinite set (but – in case ∂X is compact – it is still discrete and finite for $\text{Re } z$ bounded, which is proved by 'analytic Fredholm theory').
- $\text{Spec}_b I(P)$ is global on the boundary, i.e. determined by the (global) solvability of some partial differential equation on ∂X .
- The algebra of symbols is not commutative. (But this does not matter in the parametrix construction since commutativity was never used.)

Appendix: Pull-back, push-forward, densities etc.

Let $f : M \rightarrow N$ be a smooth map between manifolds.

The *pull-back* by f of a function v on N is the function

$$f^*v = v \circ f$$

on M . Clearly, f^*v is smooth if v is.

Pull-backs appear everywhere. Depending on context and personal taste, they may be interpreted as 'plugging in', 'reinterpretation', or 'distortion'. For example, $v(xy)$ (plugging in xy into v) is the pull-back of v under the map $f(x, y) = xy$; π_2^*v from Example 3.1.1 is just v reinterpreted as function on \mathbb{R}^2 ; and if f is a diffeomorphism, then f^*v is just v looked at through the 'distortion lens' f . (For example, if $f : (0, \infty) \rightarrow (0, \infty), x \rightarrow x^2$ then the graph of f^*v is obtained from the graph of v by a stretching for $x < 1$ and a compression for $x > 1$.)

Related, though quite different at first glance, is *push-forward* by f . The idea is that, for a function u on M , $(f_*u)(y)$ for $y \in N$ should be 'the integral of u over the fiber $f^{-1}(y)$ '. Now this clearly depends on the choice of a measure⁵⁹ on this fiber (e.g. dy in (3.1)). Rather than to consider u and this measure separately, or to consider a measure on each fiber, it is more convenient to start with a measure μ (Borel, complex) on all of M . For measures, push-forward is a standard operation: $f_*\mu$ is the measure on N defined by $(f_*\mu)(V) = \mu(f^{-1}(V))$ (= measure of the union of all fibers over V), $V \subset N$.⁶⁰ In terms of integrals, this is equivalent to

$$\text{(App.1)} \quad \int_N (f_*\mu) \phi = \int_M \mu f^*\phi$$

for all $\phi \in C_0^\infty$. If $f = \pi_1$ as in Example 3.1.1 then this easily gives

$$\pi_{1*}(u(x, y) dx dy) = \left(\int_{\mathbb{R}} u(x, y) dy \right) dx,$$

⁵⁹We are a little sloppy about the use of the word 'measure': Contrary to standard usage we only require that a measure be defined on bounded Borel sets (i.e. those contained in a compact set). Thus, $u(x)dx$ is a measure on \mathbb{R} for any *locally* integrable function u .

⁶⁰With our use of the word 'measure' one needs to require 'integrability' here. This is guaranteed for example when f is proper on the support of μ . We always assume this tacitly.

which shows that push-forward in the sense of measure theory is integration over the fiber, as we intended. The additional factor dx may look cumbersome, but this is the only way to have invariance with respect to coordinate changes on both M and N .

(App.1) shows that push-forward f_* is dual to pull-back f^* , under the duality of functions and measures. Also, (App.1) is not just formal nonsense but actually a recipe for calculation:

EXAMPLE A.1. Let $f : (0, \infty)^2 \rightarrow (0, \infty)$ be such that $f(x, y) = xy$, and let $\mu(x, y) = u(x, y) dx dy$. Then $(f^* \phi)(x, y) = \phi(xy)$, so

$$\begin{aligned} \int_{(0, \infty)^2} u f^* \phi &= \int_0^\infty \int_0^\infty u(x, y) \phi(xy) dy dx \\ &= \int_0^\infty \int_0^\infty u(x, \frac{t}{x}) \phi(t) \frac{dt}{x} dx, \end{aligned}$$

(using Fubini and changing variables $t = xy$ in the inner integral) and comparison with (App.1) gives⁶¹

$$f_* u(t) = \left(\frac{1}{x} \int_0^\infty u(x, t/x) dt \right) dx.$$

In the smooth context we usually deal not with arbitrary measures, but rather with the more special densities ('smooth measures') and with the more general distributional densities, which we introduce next.

A (*smooth*) *density* μ on a manifold M is a measure such that for any local coordinate system $x : U \subset M \rightarrow \mathbb{R}^n$ one can find a smooth function u on U such that $\mu(U') = \int_{U'} u(x) dx$ for all measurable $U' \subset U$. In this case we write $\mu = u(x) dx$ for short.⁶²

A *distribution density* on M is an object which on every coordinate patch $x : U \subset M \rightarrow \mathbb{R}^n$ looks like $u(x) dx$ for a distribution $u(x)$ on U , where u transforms as for densities.⁶³

The push-forward of a distribution density μ under the map f may be defined by Equation (App.1) again (where \int is interpreted as the usual pairing of distributions and functions), which shows that $f_* \mu$ is a distribution density again.

So far, everything was quite straight-forward, the main problem was keeping the dualities straight. Here comes a more substantial point:

The push-forward of a smooth density need not be a smooth density!

⁶¹Anyone who is still sceptical of densities should once try to calculate (or just make sense of) the notion of integrating a function over the hyperbola $xy = t$!

⁶²This definition shows how u transforms under a change of coordinates. Of course one can use this to define a line bundle over M , usually denoted Ω_M , such that densities are just sections of this line bundle, see [26], Section 4.5.

⁶³If you want to define this more formally, you can exploit the idea of duality; then a distributional density on M is simply an element in the dual space of $C_0^\infty(M)$.

For example, if $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^3$ then $f_*dx = \frac{1}{3}y^{-2/3} dy$.⁶⁴

However, if f is a *fibration* then $f_*\mu$ is a smooth density whenever μ is (assuming integrability).⁶⁵

A *distribution* is a continuous functional on the set of (compactly supported) smooth densities. Therefore, the pull-back f^*u is defined for a distribution u if f is a fibration (as the adjoint operation to f_*).⁶⁶ Actually, here it suffices that f be a *submersion*, i.e. have surjective differential at every point. (This is weaker than, and the local analogue of, f being a fibration.) f^*u may then be defined by approximation of u by smooth functions (see [10], chapter VI.1).

⁶⁴Proof: $\int dx(f^*\phi)(x) = \int \phi(x^3)dx = \int \phi(y)y^{-2/3} dy/3$, and use (App.1).

⁶⁵Recall the definition of a fibration: For any x the preimage $L = f^{-1}(x)$ is a manifold and a neighborhood of $L \subset M$ can be identified (via a diffeomorphism) with $U \times L$, for some neighborhood U of x , such that $f(x', l) = x'$ for all $l \in L, x' \in U$. I.e., locally f looks like (a higher-dimensional version of) π_1 in Example 3.1.1, and then the assertion is clear.

⁶⁶If f is not a fibration, then f^*u may not make sense at all. For example, expressions like $\delta(x^3)$ or $\delta(xy)$ make no sense.

Table 1 gives an overview of push-forward and pull-back compatibilities.

	α smooth	α distributional	α conormal distributional
Push-forward $f_*\alpha$	f fibration	any f	f fibration, transversality condition
Pull-back $f^*\alpha$	any f	f submersion	$f(M)$ transversal to $\text{singsupp } \alpha$

TABLE 1. Conditions on a smooth map $f : M \rightarrow N$ to define push-forward or pull-back of a density (resp. function) α , resulting in an object of same type as α . For push-forward, it is always assumed that f is proper on $\text{supp } \alpha$.

Finally, when talking about operators it is useful to have (smooth or distributional) *half-densities*. These are defined as objects which look like

$$u(x)|dx_1 \cdots dx_n|^{1/2}$$

in any coordinate system x , with u smooth respectively a distribution. Accordingly, u transforms with the square root of the Jacobian under coordinate changes. The reader who prefers more formal definitions may consult [26], Section 4.5 for the definition of a (trivial) line bundle of which half-densities are sections.

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