Singular Asymptotics Lemma and Push–Forward Theorem

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ABSTRACT. The Singular Asymptotics Lemma by Brüning and Seeley and the Push-Forward Theorem by Melrose lie at the very heart of their respective approaches to singular analysis. We review both and show that they deal with the same basic problem, giving solutions that emphasize different aspects of it. This also points to a possible common extension.

1. An Example

As a simple example, consider

$$f : \mathbb{R}_+^2 \to \mathbb{R}_+, (x,y) \mapsto xy. \tag{1.1}$$

The level lines $\{f = c\}$ of f are shown in Figure 1. As c tends to 0, the corresponding level line approaches the boundary $\{0\} \times \mathbb{R}_+ \cup \mathbb{R}_+ \times \{0\}$ (where we write $\mathbb{R}_+ = [0, \infty)$).

We wish to integrate a compactly supported smooth density over the level lines of f, i.e. to compute the push-forward of $u\,dxdy$ under f, for some $u \in C_c^\infty(\mathbb{R}_+^2)$ (that is, u is the restriction to $\mathbb{R}_+^2$ of a compactly supported function on $\mathbb{R}^2$): we pair with $v \in \mathbb{C}_c^\infty(\mathbb{R}_+)$ to get

$$\begin{aligned} \langle f_*(u\,dxdy), v\rangle = \langle u\,dxdy, f^*v\rangle &= \iint_{\mathbb{R}_+^2} u(x,y)v(xy)\,dxdy \\ &= \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} u\left(x, \frac{t}{x}\right) \frac{1}{x}\,dx \right) v(t)\,dt, \qquad (1.2) \end{aligned}$$

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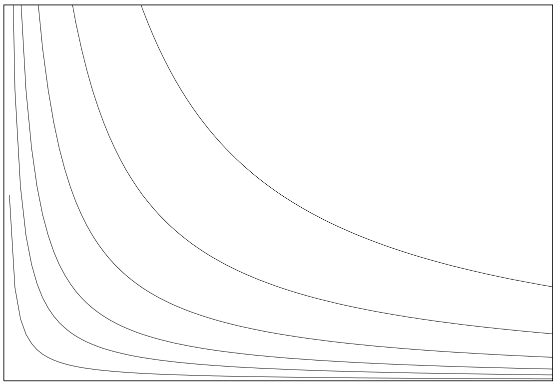

FIGURE 1. Level lines of $f : (x, y) \mapsto xy$

so that

$$f_* u(t) = \int_{\mathbb{R}_+} u\left(x, \frac{t}{x}\right) \frac{1}{x}\, dx \tag{1.3}$$

in distributional sense[1]. u is smooth by assumption and has therefore an asymptotic expansion at the boundary of $\mathbb{R}_+^2$ given by the Taylor series, but $f_* u$ may be non-smooth: Choose a smooth cut-off function $\rho_\varepsilon : \mathbb{R}_+ \to [0, 1]$ which is 1 on $[0, 1]$ and 0 on $[1 + \varepsilon, \infty)$. If we set $u_\varepsilon(x, y) = \rho_\varepsilon(x)\rho_\varepsilon(y)$, then the push-forward satisfies:

$$f_* u_\varepsilon(t) = \int_{t/(1+\varepsilon)}^{(1+\varepsilon)} \rho_\varepsilon(x)\rho_\varepsilon\left(\frac{t}{x}\right) \frac{1}{x}\, dx \begin{cases} \le (1+\varepsilon)\ln(1+\varepsilon) - \ln t \\ \ge -\ln t \end{cases} \tag{1.4}$$

for $t > 0$. In other words: Already the innocent constant 1 (cut off by $\rho_\varepsilon(x)\rho_\varepsilon(y)$) produces a logarithmic singularity, when pushed forward under the seemingly innocent f. Note that $f_* u_\varepsilon(t)$ tends to $-\ln t$ pointwise as ε tends to 0, and also in the distributional sense. Therefore, $f_* u_0(t) = -\ln t$, where u_0 is the characteristic function of $[0, 1]^2$.

This raises the following questions:

- What happens for more general u?
- What happens for more general f?

The Push-Forward Theorem (PFT) answers these questions by describing the type of asymptotic behavior of the push-forward which can occur when f is a b-fibration

[1] Recall that the push-forward of a distribution (under a smooth proper map) always exists as a distribution, and that the push-forward of a smooth function may be non-smooth. On the other hand, the pull-back of a smooth function is always smooth, but not every distribution can be pulled back under smooth proper maps.

In the following we identify densities with functions by using the standard (Lebesgue) measure on $\mathbb{R}^2$ and $\mathbb{R}$, as we did already in (1.3).

and u is a polyhomogeneous conormal distribution on a compact manifold with corners.

Another way to look at (1.3) is to interpret it as integration of asymptotics: Set

$$\sigma(x,\zeta) = u\left(x, \frac{1}{\zeta}\right)\frac{1}{\zeta} \tag{1.5}$$

and $z = 1/t$, then

$$f_* u(t) = z \int \sigma(x, zx)\, dx. \tag{1.6}$$

Since $u \in C_c^\infty(\mathbb{R}_+^2)$, the asymptotic behavior of $\sigma(x,\zeta)$ for small x and for large ζ is given by the Taylor expansion of $u(x,y)$ at $x = 0$ and at $y = 0$. One might ask whether this asymptotic expansion can be integrated to get the asymptotic behavior of $f_*u(t)$ for small t, i.e. to get an expansion of (1.6) for large z. For $u \equiv u_0$ as above, i.e. $\sigma(x,y) = 1/y$ on $[0,1] \times [1,\infty)$ and $\sigma(x,y) = 0$ otherwise, one recovers the logarithmic singularity, of course.

This raises the following questions:

- What happens for more general u (or σ)?
- What is the structure of the coefficients in the expansion?

The Singular Asymptotics Lemma (SAL) answers these questions by computing the coefficients in terms of derivatives and generalized integrals of σ, for σ having quite general asymptotics.

In Section 2 we review the SAL and in Section 3 the PFT. In Section 4 we analyze the example above from both the SAL and PFT point of view. In Section 5 we drop the assumption of compact y-support for u and allow more general asymptotics as $y \to 0$. This is essentially the generality treated by SAL. We construct the corresponding PFT setting and show that the 'integrability conditions' in SAL and PFT correspond. A short summary and some remarks in Section 6 on the different perspectives of SAL and PFT conclude the paper.

2. The Singular Asymptotics Lemma

This lemma was set up in [**BS85**] and used to study singular differential operators defined by the Friedrichs extension of $-\partial_x^2 + x^{-2}a$ on $L^2(0,\infty)$, where a is smooth and $a \geq -\frac{1}{4}$, $|a(x)| \leq C(1+x)$. In the sequel this has been used to determine heat trace asymptotics for conical singularities ([**BS87**]), which corresponds to allowing a to be a family of unbounded operators. For this, and further references, as well as generalizations to operators of higher orders, we refer to [**Les97**], which we follow closely in this section. We outline the main ideas, but do not reproduce the proofs.

To state SAL's assertion about the coefficients appearing in the expansion we need to define a regularized integral. We do this for functions having finite asymptotic expansions in the L^1-sense:

Definition 2.1. Let $p, q \in \mathbb{R}$, $p > 0, q > 0$. We define $L_{p,q}(\mathbb{R}_+^\times)$, where $\mathbb{R}_+^\times = (0,\infty)$, to be the set of functions f for which there are complex numbers $\alpha_j, \beta_j \in \mathbb{C}$ and (nontrivial) polynomials $p_{\alpha_j}, q_{\beta_j} \in \mathbb{C}[\ln x]$ such that

$$f(x) = \sum_{j=1}^{N} x^{\alpha_j} p_{\alpha_j}(\ln x) + x^p f_1(x) \tag{2.1a}$$

$$= \sum_{j=1}^{M} x^{\beta_j} q_{\beta_j}(\ln x) + x^{-q} f_2(x) \tag{2.1b}$$

for some $f_1 \in L^1_{loc}([0,\infty))$, $f_2 \in L^1([1,\infty))$. We assume that the real parts $\Re\alpha_j$ are increasing and $\le p-1$, and that $\Re\beta_j$ is decreasing and $\ge -q-1$. Furthermore, we define the set of exponents as

$$\operatorname{spec}_0(f) = \{\alpha_j \mid j = 1, \ldots, N\}, \tag{2.2a}$$

$$\operatorname{spec}_\infty(f) = \{\beta_j \mid j = 1, \ldots, M\}. \tag{2.2b}$$

Finally we assume that all α_j are distinct and set, for arbitrary $\alpha \in \mathbb{C}$, $p_\alpha \equiv p_{\alpha_j}$ if $\alpha = \alpha_j$ and $p_\alpha \equiv 0$ if there is no such α_j (analogous for q_β).

Now, the shortest way to define the regularized integral of functions in $L_{p,q}(\mathbb{R}_+^\times)$ is by using the following limit in the mean:

Definition 2.2. For $f \in L_{p,q}(\mathbb{R}_+^\times)$, $p > 0, q > 0$, with asymptotics (2.1a) and (2.1b) we define

$$\operatorname*{LIM}_{x\to 0} f(x) := p_0(0), \tag{2.3a}$$

$$\operatorname*{LIM}_{x\to \infty} f(x) := q_0(0). \tag{2.3b}$$

In other words: the limit in the mean is defined as the constant term in the asymptotic expansion.

Definition and Proposition 2.3. *Let $p > 0, q > 0$, $f \in L_{p,q}(\mathbb{R}_+^\times)$. If we define*

$$F(x) := \int_1^x f(t)\,dt$$

then $F \in L_{p+1-\varepsilon, q+1-\varepsilon}(\mathbb{R}_+^\times)$ for every $\varepsilon > 0$.

We define the regularized integral $⨍ f$ of f by

$$⨍_0^\infty f(x)\,dx := \operatorname*{LIM}_{x\to\infty} F(x) - \operatorname*{LIM}_{x\to 0} F(x). \tag{2.4}$$

Remark 2.4.

- A more conceptual approach is to use the Mellin transform $\mathcal{M}f(z) = \int_0^\infty x^{z-1} f(x)\,dx$. A priori it is defined for arbitrary $z \in \mathbb{C}$ in case of $f \in C_c^\infty(\mathbb{R}_+^\times)$, and $\mathcal{M}f(1) = \int_0^\infty f$.

  As it turns out, functions $f \in L_{p,q}(\mathbb{R}_+^\times)$ still have a well-defined Mellin transform which is meromorphic for $1-p < \Re z < 1+q$, and has at most poles of order $\deg p_\alpha + 1$ in $-\alpha$ for $\alpha \in \operatorname{spec}_0(f)$, and poles of

order $\deg q_\beta + 1$ in $-\beta$ for $\beta \in \operatorname{spec}_\infty(f)$. Then it is natural to regularize $\int$ by defining $⨍ f$ to be the zeroth order residuum of $\mathcal{M}f(z)$ at $z = 1$. This leads to the same definition as above.
- An easy calculation shows $⨍_0^\infty x^\alpha \ln^k x\, dx = 0$, since the contributions at 0 and at ∞ just cancel.

From the definition, $⨍$ shares the linearity properties with $\int$. But a fundamental difference appears in the substitution rule:

Lemma 2.5. *Let $p > 0, q > 0$, $f \in L_{p,q}(\mathbb{R}_+^\times)$, $t > 0$. Set $P_{-1}(\xi) := \int_0^\xi p_{-1}(s)\, ds$ and $Q_{-1}(\xi) := \int_0^\xi q_{-1}(s)\, ds$ for the coefficient polynomials p_{-1}, q_{-1} from the asymptotic expansions* (2.1a) *and* (2.1b). *Then*

$$⨍_0^\infty f(tx)\, dx = \frac{1}{t}\left(⨍_0^\infty f(x)\, dx + Q_{-1}(\ln t) - P_{-1}(\ln t)\right). \tag{2.5}$$

By applying this lemma to a function $\varphi(tx)f(x)$, where f is as above and φ is a Schwartz function (and therefore has asymptotics given by its Taylor series), we arrive at the first SAL–like result:

Proposition 2.6. *Let $p > 0, q > 0$, $f \in L_{p,q}(\mathbb{R}_+^\times)$, $\varphi \in \mathcal{S}(\mathbb{R})$. Then, as $t \to 0$,*

$$\begin{aligned} ⨍_0^\infty \varphi(tx) f(x)\, dx &= \sum_{j<q} \frac{\varphi^{(j)}(0)}{j!} ⨍_0^\infty x^j f(x)\, dx \cdot t^j \\ &+ \sum_{\substack{\beta \in \operatorname{spec}_\infty(f) \\ \Re\beta \geq -q-1}} ⨍_0^\infty \varphi(x) x^\beta q_\beta\left(\ln \frac{x}{t}\right) dx \cdot t^{-\beta-1} \\ &+ \sum_{\substack{\beta \in \operatorname{spec}_\infty(f)\cap\mathbb{Z} \\ -1\geq\beta\geq -q-1}} \frac{\varphi^{(-\beta-1)}(0)}{(-\beta-1)!} Q_\beta(-\ln t) \cdot t^{-\beta-1} + \mathcal{O}(t^q). \end{aligned} \tag{2.6}$$

The poles from $\operatorname{spec}_0(f)$ produce additional terms if we study $\varphi(x) f(\frac{x}{t})$ by applying the substitution lemma to the previous proposition:

COROLLARY 2.7. *Let φ, f be as above. Then, as $t \to 0$,*

$$
\begin{aligned}
\int_0^\infty \varphi(x) f\left(\frac{x}{t}\right) dx &= \sum_{j<q} \frac{\varphi^{(j)}(0)}{j!} \int_0^\infty x^j f(x)\, dx \cdot t^{j+1} \\
&+ \sum_{\substack{\beta \in \operatorname{spec}_\infty(f) \\ \Re\beta \geq -q-1}} \int_0^\infty \varphi(x) x^\beta q_\beta \left(\ln \frac{x}{t}\right) dx \cdot t^{-\beta} \\
&+ \sum_{\substack{\beta \in \operatorname{spec}_\infty(f) \cap \mathbb{Z} \\ -1 \geq \beta \geq -q-1}} \frac{\varphi^{(-\beta-1)}(0)}{(-\beta-1)!} Q_\beta(-\ln t) \cdot t^{-\beta} \\
&- \sum_{\substack{\alpha \in \operatorname{spec}_0(f) \cap \mathbb{Z} \\ -1 \geq \alpha \geq -q-1}} \frac{\varphi^{(-\alpha-1)}(0)}{(-\alpha-1)!} P_\alpha(-\ln t) \cdot t^{-\alpha} + \mathcal{O}(t^{q+1}).
\end{aligned}
\tag{2.7}
$$

If we pass now from $\varphi \otimes f$ to more general functions σ of two variables, then it is easy to derive

THEOREM 2.8 (Singular Asymptotics Lemma). *Let C be the sector $\{\xi \in \mathbb{C}^\times \mid |\arg \xi| < \pi - \varepsilon\}$ for some $\varepsilon > 0$. Assume that $\sigma : \mathbb{R} \times C \to \mathbb{C}$ fulfills the following properties for some p:*

1. *σ is $(p-1)$-times continuously differentiable in the first variable x, and $\partial_x^{(p-1)} \sigma(\cdot, \zeta)$ is absolutely continuous on $[0, \infty)$.*
2. *All derivatives up to order p are analytic*[2] *(resp. measurable*[3]*) in the second variable ζ.*
3. *There are polynomials $p_\alpha \in \mathcal{S}(\mathbb{R})[\ln \zeta]$ with Schwartz functions as coefficients, and constants C_{JK} and r such that*

$$
\left| x^J \partial_1^K \left(\sigma(x, \zeta) - \sum_{\Re\alpha > -p-1} \zeta^\alpha p_\alpha(x, \ln \zeta) \right) \right| \leq C_{JK} |\zeta|^{-p-1} |\ln^r \zeta|
\tag{2.8}
$$

   for $|\zeta| \geq 1$, $0 < x \leq |\zeta|$, $K \leq p$, $J \in \mathbb{Z}_+$

4. *There is $f_p : (0, 1] \to \mathbb{R}$ such that one has:*

$$
\int_0^1 \left| \zeta^j \partial_1^j \sigma(0, \zeta) \right| d\zeta < \infty, \qquad j = 0, \ldots, p-1,
\tag{2.9a}
$$

$$
\int_0^1 \int_0^1 s^p \left| \partial_1^p \sigma(\theta s t, s\xi) \right| ds\, dt \leq f_p(\theta), \qquad \text{if } p > 0,
\tag{2.9b}
$$

$$
\int_0^1 \left| \sigma(\theta s, s\xi) \right| ds \leq f_0(\theta), \qquad \text{if } p = 0,
\tag{2.9b'}
$$

[2] Even in the analytic case assumption (2.9a) below is a non-empty condition, since the sector C does not contain 0.

[3] In the measurable case the asymptotics will only hold along the real line $z \to \infty$, not inside the whole sector $z \in C$.

for $0 < \theta \le 1$, uniformly for $|\xi| = 1$.

Then, as $z \to \infty$ in C (resp. for $z \ge 0$), one has

$$
\begin{aligned}
\int_0^\infty \sigma(x, xz)\, dx = &\sum_{j=0}^{p-1} \fint_0^\infty \frac{\zeta^j}{j!} \partial_1^j \sigma(0,\zeta)\, d\zeta \cdot z^{-j-1} \\
&+ \sum_{\Re\alpha > -p-1} \fint_0^\infty x^\alpha p_\alpha(x, \ln xz)\, dx \cdot z^\alpha \\
&+ \sum_{\alpha=-1}^{-p} \frac{z^\alpha}{(-\alpha-1)!} \partial_1^{(-\alpha-1)} P_\alpha(0, \ln z) \\
&+ \mathcal{O}\left(z^{-p-1} \ln^{r+1} z\right) + \mathcal{O}\left(z^{-p-1} f_p(|z|^{-1})\right)
\end{aligned} \tag{2.10}
$$

Here, we defined $P_\alpha(x, \ln\zeta) = \int_0^{\ln\zeta} p_\alpha(x,s)\, ds$; and, for $p \in \mathcal{S}(\mathbb{R})[\ln\zeta]$, we write $p(x, \ln\zeta)$ where $\ln\zeta$ denotes the polynomial variable.

The value of SAL's statement clearly depends on the function f_p appearing in the estimates (2.9a)–(2.9b'). In the original work [**BS85**] f_p was constant. In some situations, even on cones (see [**Les97**]) one needs to relax this, by using the following criterion:

PROPOSITION 2.9 (supplement to SAL). *Assume that we have for some $\varphi \in C([0,1])$, $\varepsilon > 0$, $T > 0$, the estimates*

$$
|\sigma(x,\zeta)| \le \varphi(x)|\zeta|^{-1}, \qquad 0 < |\zeta| \le 1, 0 < x \le 1, \tag{2.11a}
$$

$$
|\sigma(\theta s, \xi s)| \le c\theta^{-T} s^{\varepsilon-1}, \qquad 0 < \theta \le 1, 0 < s \le 1 \tag{2.11b}
$$

uniformly in $|\xi| = 1$. Then

$$
\int_0^1 |\sigma(\theta s, \xi s)|\, ds = \mathcal{O}(\ln\theta) \text{ as } \theta \to 0. \tag{2.12}
$$

This represents the case "$p = 0$" in SAL and allows to identify terms in the asymptotics up to order $\ln t$.

3. The Push-Forward Theorem

Since the b-calculus is covered already in another contribution in this volume [**Gri00**] we will keep this section short compared to the previous one. The main sources for this section are [**Mel90, Mel92**].

Let X be a manifold with corners. An index set is a non-empty discrete set $K \subset \mathbb{C} \times \mathbb{N}_0$ with

$$
\begin{gathered}
(\alpha_j, k_j) \in K,\ |(\alpha_j, k_j)| \to \infty \Rightarrow \Re\alpha_j \to \infty, \\
(\alpha, k) \in K, p \in \mathbb{N} \Rightarrow (\alpha + p, k) \in K, \\
(\alpha, k) \in K, p \in \mathbb{N}_0,\ 0 \le p < k \Rightarrow (\alpha, p) \in K.
\end{gathered} \tag{3.1}
$$

An index family $\mathcal{K}$ is an assignment of an index set $\mathcal{K}(H)$ to every boundary hypersurface H. For a given index family one defines the space of polyhomogeneous conormal functions $\mathcal{A}^{\mathcal{K}}_{\mathrm{phg}}(X)$, and these are the functions u which have an asymptotic expansion

$$u \sim \sum_{(\alpha,p)\in\mathcal{K}(H)} \rho^{\alpha} \ln^{p} \rho \, v_{\alpha,p} \tag{3.2}$$

near every boundary hypersurface H, with ρ a defining function of H; the coefficients $v_{\alpha,p}$ are polyhomogeneous conormal on H, with index family $\mathcal{K}$ restricted to H. Some care needs to be taken in the definition of $\sim$, see (5.8) for the precise definition in the case needed here. If $X = \mathbb{R}_+$ then these u clearly correspond to the functions in $L_{\infty,\infty}$ with complete asymptotics from Definition 2.1, with two boundary points 0 and ∞. The index set at 0 (resp. ∞) corresponds to the α_j (resp. $-\beta_j$) and the exponents of the polynomials p_{α_j} (resp. q_{β_j}), and a set of such exponents generates an index set by "completion" according to (3.1).

Now we define several types of maps between manifolds with corners: If H is a boundary face of codimension k we can choose local coordinates $x_1, \dots, x_k, y_1, \dots, y_{n-k}$ of X such that $x_j \geq 0$, and $H = \{x_j = 0\}$ near a point $\bar{x}$. We do the same 'in prime' near $\bar{x}' = f(\overline{x})$, if f is a map $X \to Y$ between manifolds with corners. Then, f is called b-normal if these coordinates can be chosen so that

$$f^*(x'_j) = \prod_{r\in I_j} x_r^{e(r,j)} \text{ for } j = 1, \dots, k'. \tag{3.3a}$$

for some disjoint $I_j \subset \{1, \dots, k\}$. Here, the nonnegative integers $e(r,j)$ define the exponent matrix of f. f is called b-submersion if the coordinates can be chosen so that

$$f^*(y'_p) = y_p \text{ for } p = 1, \dots, n' - k'. \tag{3.3b}$$

f is called b-fibration if both (3.3a) and (3.3b) can be achieved simultanuously. See [**Gri00**] for a more geometric characterization and motivation of this definition.

The push-forward of an index family is defined as follows:

Definition 3.1. If $f : X \to Y$ is a b-fibration and $\mathcal{K}$ is an index family for X then the index family $f_{\#}\mathcal{K}$ for Y is defined by

$$f_{\#}\mathcal{K}(H) = \overline{\bigcup_{\substack{G\in M_1(X)\\ e_f(G,H)\neq 0}}} \left\{ \left(\frac{z}{e_f(G,H)}, p \right) \mid (z,p) \in \mathcal{K}(G) \right\} \tag{3.4}$$

for any $H \in M_1(Y)$.

The extended union used in the previous definition is the following: Let K, I be index sets, then

$$K \,\overline{\cup}\, I = K \cup I \cup \{(z, p' + p'' + 1) \mid (z,p') \in K, (z,p'') \in I\}. \tag{3.5}$$

Finally, we define the nullset $\text{null}(e_f)$ of f:

$$\text{null}(e_f) = \{G \in M_1(X) \mid \forall_{H \in M_1(Y)} : e_f(G,H) = 0\} \tag{3.6}$$

THEOREM 3.2 (Push-Forward Theorem). *If $f : X \to Y$ is a b-fibration between compact manifolds with corners, and if $\mathcal{K}$ is any index family for X with*

$$G \in \text{null}(e_f) \Rightarrow \Re\mathcal{K}(G) > 0 \tag{3.7}$$

then push-forward under f gives a mapping

$$f_* : \mathcal{A}^{\mathcal{K}}_{phg}(X; {}^b\Omega) \to \mathcal{A}^{f_\#\mathcal{K}}_{phg}(Y; {}^b\Omega) \tag{3.8}$$

(See Section 5 for the meaning of ${}^b\Omega$, i.e. b-densities.) There is also a version with finite asymptotics, and for partially polyhomogeneous functions. Clearly, the theorem extends to non-compact X when pushing forward only compactly supported densities.

The Push-Forward Theorem shows that additional logarithmic terms at a hypersurface $H \subset Y$ may be produced in $f_*\omega$ if there are two different hypersurfaces $G_1, G_2 \subset X$ with $e_f(G_j, H) \neq 0, j = 1, 2$, (i.e. both G_1 and G_2 are mapped into H by f), and if the same power occurs in the expansions of ω at G_1 and G_2 (at least for generic such ω) [4]. This explains the occurence of logarithmic terms when pushing forward densities with purely polynomial asymptotics (e.g. smooth densities).

4. The Example Revisited

Let us now apply the theorems from the previous sections to our introductory example (1.1), if we push forward smooth densities. Remember $\sigma(x,y) = u(x, \frac{1}{y})\frac{1}{y}$. Therefore, if u is smooth we have the asymptotic expansion

$$\sigma(x,\zeta) \underset{\zeta\to\infty}{\sim} \sum_{j\geq 0} \frac{1}{j!}\partial_2^j u(x,0)\zeta^{-j-1} \tag{4.1}$$

from the Taylor series. Recast in the language of SAL's assumption (2.8) this means

$$p_\alpha(x, \ln\zeta) = \frac{1}{(-1-\alpha)!}\partial_2^{(-1-\alpha)} u(x,0), \quad \alpha \in -\mathbb{N}. \tag{4.2}$$

[4] Actually, a simple partition of unity argument shows that additional log-terms are created only if G_1, G_2 intersect, as the x- and y-axis in Figure 1 do. It is easy to see that then the level sets of f look roughly like (a higher-dimensional version of) those in Figure 1. Thus: 'log's are produced by level sets pushing into a corner.'

Furthermore, $\partial_1^j \sigma(x,\zeta) = \partial_1^j u(x, \frac{1}{\zeta})\frac{1}{\zeta}$, and $P_\alpha(x, \ln\zeta) = p_\alpha(x,0)\ln\zeta$ from the definitions. Thus, the expansion given by SAL is:

$$\begin{aligned} f_* u(t)\cdot t &= \int_0^\infty \sigma(x, xz)\,dx \\ &\underset{t\to 0}{\sim} \sum_{j\geq 0} \fint_0^\infty \frac{\zeta^{j-1}}{j!}\partial_1^j u(0, \tfrac{1}{\zeta})\,d\zeta \cdot t^{j+1} \\ &\quad + \sum_{j\geq 0} \fint_0^\infty \frac{x^{-1-j}}{j!}\partial_2^j u(x,0)\,dx \cdot t^{j+1} \\ &\quad - \sum_{j\geq 0} \frac{t^{j+1}}{(j!)^2}\partial_1^j \partial_2^j u(0,0)\ln t. \end{aligned} \tag{4.3}$$

In particular, we reproduce the result for $u \equiv u_0$, the characteristic function of $[0,1]^2$, from the introductory section. [5]

In the language of the b-calculus, $X = \mathbb{R}_+^2$ has two boundary hypersurfaces (x- and y-halfaxis), $Y = \mathbb{R}_+$ has one (the point 0). The map f is a b-fibration: Near the corner $(0,0)$ we can use cartesian coordinates, near the hypersurface $\{x=0\}$ (resp. $\{y=0\}$) we can use coordinates xy and y (resp. xy and x) to see that it is a b-fibration, and that the exponent map is 1. If u is smooth the corresponding index set is $K = \{(n,0) \mid n \in \mathbb{N}_0\}$ for both hypersurfaces. The union is $K \cup K = K$ of course, but the extended union is

$$K \,\overline{\cup}\, K = \{(n,0) \mid n \in \mathbb{N}_0\} \cup \{(n,1) \mid n \in \mathbb{N}_0\} \tag{4.4}$$

from (3.5). This is the b-explanation for the logarithmic terms.

5. Equivalences

In the previous section we compared SAL and PFT in the following special situation: The function u on $\mathbb{R}_+^2$ to which PFT is applied is smooth and has compact support. Correspondingly, the function $\sigma(x,\zeta)$ to which SAL is applied has the special asymptotics (4.1) as $\zeta \to \infty$ and vanishes whenever $x > C$ or $\zeta < c$, for some positive numbers c, C. The assumptions of SAL allow more general σ: There is no restriction on the support, and the asymptotics may be more general.

In this section we construct a setting for PFT which corresponds to this. In particular, we will see that the integrability conditions (2.9) of SAL correspond to condition (3.7) of PFT. For simplicity, we only consider the case of complete asymptotic expansions, i.e. $p = \infty$ in SAL. Also, we assume that σ has compact support in x, the extension to non-compact support in x (but Schwartz function behavior as expressed in (2.8)) being straightforward.

We need to find a manifold with corners X and a b-fibration $F : X \to \mathbb{R}_+$ such that integration $\int \sigma(x, xz)\,dx$ corresponds to a pushforward by F and the asymptotics $z \to \infty$ corresponds to the conormal expansion at $0 \in \mathbb{R}_+$. The latter

[5] Notice the symmetry of (4.3) in x and y after the substitution $y = 1/\zeta$ in the first integral.

suggests introducing $t = 1/z$ as before. Integration $\int h(x,t)\,dx$ of a function h on $\mathbb{R}^2_+$ is just the push-forward under the map $\pi_2 : \mathbb{R}^2_+ \to \mathbb{R}_+, (x,t) \mapsto t$. In our case, $h(x,t) = \sigma(x, x/t)$ behaves badly at $(0,0)$, and the occurence of x/t here suggests blowing up this point. Thus, set

$$X = [\mathbb{R}^2_+, (0,0)], \quad F = \pi_2 \circ \beta : X \to \mathbb{R}_+ \tag{5.1}$$

where $\beta : X \to \mathbb{R}^2_+$ is the blow-down map[6]. See Figure 2 for the level sets of F.

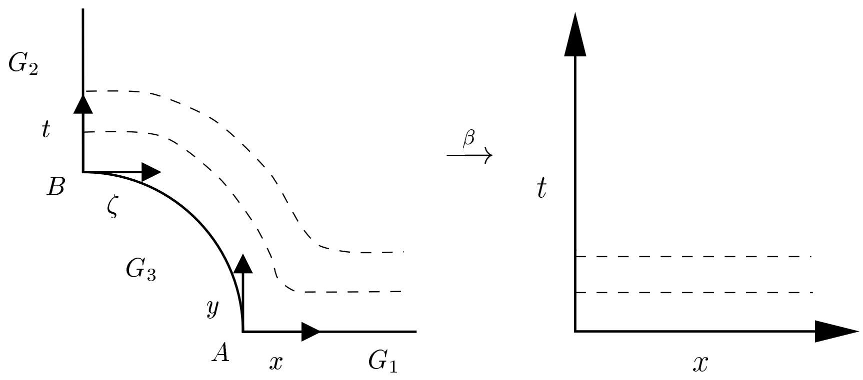

FIGURE 2. The blow up $[\mathbb{R}^2_+, (0,0)]$ of $\mathbb{R}^2_+$

The coordinate functions x, t can be considered (via pullback by β) as functions on X, and on the interior X^o we also have the functions

$$y = t/x, \quad \zeta = x/t = 1/y. \tag{5.2}$$

Referring to Figure 2 for notations (where G_1, G_2, G_3 denote the boundary lines and $A = G_1 \cap G_3, B = G_2 \cap G_3$ the corners), we see that (x, y) provide a local coordinate system for X near A (i.e. that they extend smoothly and with independent differentials to $X \setminus G_2$ and that x, y are boundary defining functions for G_3, G_1 respectively), and that (ζ, t) are coordinates near B (i.e. extend to $X \setminus G_1$ and define G_2, G_3 respectively).

We now use b-densities on X since there is no 'canonical' density to identify functions with densities and since PFT is formulated in terms of b-densities (rather than densities). By definition, a b-density $u \in \mathcal{F}(X, {}^b\Omega)$ (for some function space $\mathcal{F}$) on X can be written

$$u = u_A(x,y) \frac{dx}{x} \frac{dy}{y}$$

[6] $[\mathbb{R}^2_+, (0,0)]$ may be defined simply as $[0, \pi/2] \times \mathbb{R}_+$, with blow-down map polar coordinates: $\beta(\theta, r) = (r\cos\theta, r\sin\theta)$. While r, θ have the advantage of providing global coordinates on X, we prefer to avoid transcendental functions and rather use local coordinates naturally suggested by the problem, following R. Melrose's tradition.

near A and

$$u = u_B(\zeta, t)\frac{d\zeta}{\zeta}\frac{dt}{t}$$

near B (omitting absolute value signs for simplicity), with $u_A, u_B \in \mathcal{F}$. Note that on X^o we have $\frac{dx}{x}\frac{dy}{y} = \frac{d\zeta}{\zeta}\frac{dt}{t}$ from (5.2) so that u_A and u_B simply transform like functions, i.e.

$$u_B(\zeta, t) = u_A(\zeta t, 1/\zeta). \tag{5.3}$$

(If one would use just densities, there would be additional factors occuring here.)

To finish the description of the PFT-setting, we define index sets for the three boundary faces of X: Let K_1 and K_2 be any index sets and $K_3 = \{(n, 0) : n = 1, 2, 3, \ldots\}$ be the index set of smooth functions vanishing to first order at G_3. Note that for the map F above we have $(e_F(G_i, H))_{i=1,2,3} = (1, 0, 1)$ where $H = \{0\}$ is the only boundary face of $\mathbb{R}_+$. Therefore, F is a b-fibration and $\mathrm{null}(e_F) = \{G_2\}$, so we have

$$(3.7) \text{ is satisfied } \Leftrightarrow \Re K_2 > 0.$$

The correspondence between SAL and PFT can then be expressed as follows:

Proposition 5.1. *Define X and F as in* (5.1). *Let u be a compactly supported b-density on X, smooth in the interior X^o. Define $\sigma : (\mathbb{R}_+^\times)^2 \to \mathbb{R}_+$ by*

$$\sigma(x, \zeta) = \frac{1}{x} u_A\left(x, \frac{1}{\zeta}\right). \tag{5.4}$$

Then

$$F_* u = \left(\int_0^\infty \sigma(x, \frac{x}{t})\, dx\right) \frac{dt}{t}, \tag{5.5}$$

and we have the following equivalences:

(A) *σ is smooth in $\{x \geq 0, \zeta > 0\}$ $\iff$ u is polyhomogeneous at G_3^o, with index set K_3 (i.e. $u = xv$ with v smooth up to G_3^o),*

(B) *σ has complete asymptotics, i.e.*

$$\left|\partial_1^K \left(\sigma(x, \zeta) - \sum_{(\alpha,k)\in K_1, \Re\alpha < N} \zeta^{-\alpha} \ln^k \zeta\, \sigma_{\alpha k}(x)\right)\right| \leq C_{NK}\zeta^{-N} \tag{5.6}$$

for all N, K, $x > 0, \zeta > 1$, with $\sigma_{\alpha k} \in C_0^\infty(\mathbb{R}_+)$

$$\iff$$

u is polyhomogeneous, with index sets K_1, K_3, on $X \cap \{y < 1\}$ (i.e. 'near A'),

(C) *If $u \in \mathcal{A}^{(K_1,K_2,K_3)}(X)$ then*

$$\int_0^1 \zeta^p |\partial_1^p \sigma(\zeta t, \zeta)|\, d\zeta \leq C_p \text{ for all } p \text{ and } t \in [0, 1] \tag{5.7}$$

$$\iff$$

condition (3.7) *of the PFT is satisfied for u.*

REMARK 5.2. Roughly speaking, (B) and (C) say that conditions 3. and 4. of SAL correspond to conditions on the behavior of u near A and B. More precisely, the first halves of (A), (B) and (C) are analogues of 1.+2., 3. and 4. in SAL for the case $p=\infty$, when σ has compact x-support and f_p =const, for the case of real ζ (but SAL 4 is a little weaker than (5.7) plus polyhomogeneity, see below).

In (C) we assume that K_2 is the 'minimal' index set for u, i.e. that the coefficients in the corresponding expansion are non-trivial.

PROOF OF 5.1. (5.5) is just (1.3) in the present notation.

(A) This is immediate from (5.4) since x defines G_3 outside B. Note that the boundary points A, B of G_3 correspond to $\zeta=\infty$ and $\zeta=0$, respectively.

(B) Polyhomogeneity of u near A means asymptotic expansions at the interiors of G_1, G_3 (i.e. for $y \to 0$ with x bounded away from 0 and ∞, and vice versa), plus uniformity ('joint asymptotics') as both x, y approach zero. By definition, this means that, for all N,

$$u_A(x,y) - \sum_{(\alpha,k)\in K_1, \Re\alpha<N} y^\alpha \ln^k y \, u_{\alpha k}(x) \in \dot{C}^N_{G_1}(X), \tag{5.8}$$

the space of N times differentiable functions (in $x \geq 0, y \geq 0$) vanishing to order N at $G_1=\{y=0\}$, with $u_{\alpha k} \in \mathcal{A}^{K_3}(G_3)$. Since $y=1/\zeta$, it is easily seen that this is equivalent to (5.6).

(C) From (5.3) and (5.4) we have

$$u_B(\zeta,t)=\zeta t \, \sigma(\zeta t, \zeta).$$

Setting $v_B(\zeta,t)=u_B(\zeta,t)/t$ and taking t-derivatives, we get

$$\partial_2^p v_B(\zeta,t)=\zeta^{p+1}\partial_1^p\sigma(\zeta t,\zeta),$$

so (5.7) is equivalent to

$$\int_0^1 |\partial_2^p v_B(\zeta,t)| \frac{d\zeta}{\zeta} \leq C_p. \tag{5.7'}$$

Now polyhomogeneity of u near B with the index sets K_2, K_3 implies that $\partial_2^p v_B \in \mathcal{A}^{K_2}(G_2)$ for each p and each $t \geq 0$. (5.7') holds precisely when the leading term in the asymptotics $\zeta \to 0$ of $\partial_2^p v_B$ has a positive ζ-power, i.e. when $\Re K_2 > 0$. $\square$

REMARK 5.3. In the proof of (C) very little of the polyhomogeneity at G_2 was used. Condition 4 of SAL indicates the precise sense in which this condition in PFT could be weakened.

REMARK 5.4. The setup of Sections 1 and 4 corresponds to the special case where u is supported in $X \setminus G_2$ and smooth.

6. Conclusion

The basic problem behind both the Singular Asymptotics Lemma and the Push-Forward Theorem is to determine the asymptotics of integrals like (1.3) as $t \to 0$ (or equivalently (1.6) as $z \to \infty$), under certain assumptions on the asymptotics and support of u (resp. σ).

In the PFT these integrals are interpreted as the push-forward of u under the map f in (1.1) (resp. F in (5.1) when the y-support of u is not compact). The singular nature of this push-forward operation – evident from the creation of additional log-terms in the asymptotics of $f_* u$ – stems from the fact that f (resp. F) is not a fibration (df is not surjective at $(0,0)$, dF at A). The PFT shows that for a certain general class of maps f (b-fibrations) between manifolds with corners, which fail to be fibrations in a controlled way, the asymptotic type of $f_* u$ can be predicted (or rather 'estimated from above') from that of u. The additional log-terms are predicted from the geometry of the map f.

The SAL deals with the integral $\int_0^\infty \sigma(x, zx)\, dx$ directly and therefore gives a more explicit result; the singular nature of this integral is reflected in the existence of two 'regimes' ($x > Cz^{-1}$, $x < Cz^{-1}$) in which the integrand behaves in different ways. [7]

We saw in Sections 4 and 5 that the qualitative statements of SAL may be derived by applying PFT to a special case. On the other hand, SAL makes somewhat weaker regularity assumptions and gives explicit formulas for the coefficients of the resulting expansion; in particular, this allows to check whether the log-terms (whose *possible* existence is predicted by PFT) actually appear, i.e. have non-zero coefficients.

References

[BS85] Jochen Brüning and Robert Seeley, *Regular singular asymptotics*, Adv. in Math. **58** (1985), 133–148.

[BS87] Jochen Brüning and Robert Seeley, *The resolvent expansion for second order regular singular operators*, J. Funct. Anal. **73** (1987), 369–429.

[Gri00] Daniel Grieser, *Basics of the b-calculus*, in this volume.

[Les97] Matthias Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner–Texte zur Mathematik, vol. 136, 1997.

[Mel90] Richard B. Melrose, *Pseudodifferential operators, corners and singular limits*, (Kyoto), August 1990, Plenary address at the International Congress of Mathematicians.

[Mel92] Richard B. Melrose, *Calculus of conormal distributions on manifolds with corners*, Internat. Math. Res. Notices **3** (1992), 51–61.

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[7] As pointed out in [**Gri00**], 'regimes' correspond to minimal faces of X, i.e. to the points A, B. Under the more restrictive support conditions of Sections 1 and 4 the regime $x < Cz^{-1}$ is empty, so only one corner is needed.