ASYMPTOTICS OF THE FIRST NODAL LINE

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Introduction

In this note, we announce the result that the first nodal line of a convex planar domain tends to a straight line as the eccentricity tends to infinity.

Let Ω denote a bounded convex domain in $\mathbf{R}^2$. Denote by u the second Dirichlet eigenfunction. Then u satisfies

$$\begin{aligned} \Delta u &= -\lambda_2 u \quad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ represent the eigenvalues in increasing order. The first nodal line Λ is the zero set of u.

$$\Lambda = \{z \in \Omega : u(z) = 0\}$$

In order to state a precise theorem, let us normalize the region to lie within an $N \times 1$ rectangle. Let P_x and P_y denote the orthogonal projection on the x and y axes, respectively. First rotate so that the projection $P_y\Omega$ is *smallest*. Then dilate and translate so that $P_y\Omega = (0,1)$ and $P_x\Omega = (0,N)$. (The choice of orientation of the y axis is crucial for what follows, but the dilation and translation are merely for notational convenience.)

Theorem 1. *With the normalization above, there is an absolute constant C such that*

$$\textit{length}\, P_x\Lambda \le C/N$$

Furthermore, this estimate is sharp. The case of a long, thin, circular sector shows that $C \ge 1/2$.

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Outline of the Proof

Step 1 $O(1)$ estimate. With the normalization of Theorem 1,

$$\Omega = \{(x, y) : f_1(x) < y < f_2(x),\ 0 < x < N\}$$

where f_1 is a convex function and f_2 is a concave function. Define the height function $h(x) = f_2(x) - f_1(x)$. Consider the ordinary differential operator $\mathcal{L}$ defined by

$$\mathcal{L} = -\frac{d^2}{dx^2} + \frac{\pi^2}{h(x)^2}$$

Recall the following theorem, which implies a weaker version of Theorem 1, namely, the same estimate without the factor $1/N$.

Theorem 2 [J3]. *With the normalization of Theorem 1, let ϕ_2 be the second eigenfunction for $\mathcal{L}$ with Dirichlet boundary conditions on $[0, N]$. Let x_0 be the unique zero of ϕ_2 in $(0, N)$. There is an absolute constant A such that*

$$P_x\Lambda \subset [x_0 - A, x_0 + A]$$

This theorem says in a very crude sense that u resembles the function

$$\phi_2(x) \sin \ell_x(y)$$

where

$$\ell_x(y) = \frac{\pi(y - f_1(x))}{h(x)}.$$

The function $\ell_x(y)$ is chosen to be the linear function in y that has the value 0 on the bottom, $(x, f_1(x))$, of Ω and π on the top, $(x, f_2(x))$, of Ω. Thus, $\sin \ell_x(y)$ is the lowest Dirichlet eigenvalue for $-(d/dy)^2$ on the interval $f_1(x) \le y \le f_2(x)$ of length $h(x)$. (The fact that we have rotated so that h is as small as possible plays a crucial role.)

In addition to this estimate, we will need another consequence of [J3], expressed in terms of a parameter L defined as follows.

Definition. *The length scale L of Ω is the length of the rectangle R contained in Ω with the lowest (first) Dirichlet eigenvalue.*

Up to order of magnitude, L is the largest number such that $h(x) > 1 - 1/L$ on an interval of length L. When Ω is a rectangle, $R = \Omega$ and $L = N$. When Ω is a triangle of length N, then $L \approx N^{1/3}$. In general, $N^{1/3} \le L \le N$. The example of a trapezoid shows that all intermediate sizes for L are possible.

The heuristic principle behind L is that ϕ_2 resembles $\sin(2\pi(x - x_0)/L)$, the second eigenfunction of the interval $[x_0 - L/2, x_0 + L/2]$ and u resembles $\sin(2\pi(x - x_0)/L) \sin \pi y$, the second eigenfunction of the rectangle of length L and width 1 with nodal line at $x = x_0$. This is true to within order of magnitude near the "central" portion of Ω, with an exponential tail in the thin regions of Ω. More precisely we have,

Proposition. *Let* $u_+(x,y) = \max\{u(x,y),0\}$ *and* $u_-(x,y) = \max\{-u(x,y),0\}$. *Then*

$$u_+(x_0+A+1,1/2) \approx u_-(x_0-A-1,1/2) \approx \max|u|/L$$
$$u_+(x_0+L/20,1/2) \approx u_-(x_0-L/20,1/2) \approx \max|u|$$

The proposition follows from the methods of [J3]. (See especially Proposition A of [J3].)

Step 2. Denote by

$$e(x,y) = (h(x)/2)^{-1/2}\sin \ell_x(y)$$

the normalized first eigenfunction on $f_1(x) \le y \le f_2(x)$. Denote

$$\psi(x) = \int_{f_1(x)}^{f_2(x)} u(x,y)e(x,y)dy$$

Then

$$u(x,y) = \psi(x)e(x,y) + v(x,y)$$

where $\psi(x)$ is the "best" coefficient possible and $v(x,y)$ should be a small error term. Because of Theorem 2, there exists a number x_1, such that $|x_0 - x_1| < A$ and $\psi(x_1) = 0$.

Lemma 1. $\psi'(x) \approx 1/L$ *on* $|x - x_0| \le L/20$. *In particular,* ψ *is strictly increasing and* x_1 *is the only zero of* ψ *on that interval.*

Lemma 2. $|v(x,y)| \le S/L$ *where*

$$S = \max_{|x-x_1|\le L/20}(|f_1'(x)| + |f_2'(x)|)e^{-c|x-x_1|} + Ce^{-cL}$$

The number S represents the slope of the boundary near x_1 plus the slope at a further distance decreased by an exponential factor. In the range $|x - x_0| \le L/20$, $|f_1'(x)| + |f_2'(x)| \le C/L^3$. The ideas of the proofs of Lemmas 1 and 2 will be presented in the next section. For now let us complete the outline of the proof of Theorem 1.

Step 3. If $u(x,1/2) = 0$, then

$$\frac{1}{L}|x - x_1| \approx |\psi(x)| = |v(x,1/2)|/e(x,1/2) \le S/L$$

Therefore,

$$|x - x_1| \le S$$

Moreover,

$$S \le C/L^3 \le C/N$$

This is the end of the proof for points of the nodal line in the middle of Ω. Near the boundary $\partial\Omega$, the denominator $e(x,y)$ vanishes, so additional ideas are needed. One uses maximum principle and Hopf type estimates of [J1,J2,J3] and extra estimates on the rate of vanishing of $v(x,y)$ at the boundary.

Proofs of Lemmas 1 and 2

The idea of the proof of Lemma 1 is as follows. One calculates that

$$\mathcal{L}\psi - \lambda_2\psi = -\psi'' + \left(\frac{\pi^2}{h(x)^2} - \lambda_2\right)\psi = \sigma$$

where σ is small and

$$\left(\lambda_2 - \frac{\pi^2}{h(x)^2}\right) \approx \frac{1}{L^2}$$

in the range $|x - x_0| \leq L/20$. Next one deduces from the proposition above that with the normalization $\max |u| = 1$,

$$\psi(x_0 + L/20) - \psi(x_0 - L/20) \approx 1$$

Then comparison with constant coefficient ordinary differential equations gives $\psi'(x) \approx 1/L$ for $|x - x_0| \leq L/20$.

The idea of the proof of Lemma 2 is to follow the Carleman method of differential inequalities. In that method, one considers a *harmonic* function, say w, in a region, say Ω, which vanishes on a portion of the boundary. Then one considers the function

$$f(x) = \int_{f_1(x)}^{f_2(x)} w(x,y)^2 dy$$

Using the equation $\Delta w = 0$ the zero boundary values and integration by parts one can find a differential inequality for f of the form $f''(x) \geq a(x)f(x)$. This convexity property makes it possible to deduce rates of vanishing for w.

To prove Lemma 2, one considers

$$g(x) = \int_{f_1(x)}^{f_2(x)} v(x,y)^2 dy,$$

and deduce a differential inequality of the form

$$g'' \geq 2\left(\frac{(2\pi)^2}{h(x)^2} - \lambda_2\right) g - \beta\sqrt{g} \geq g - \beta\sqrt{g}$$

The crucial point is that because we have subtracted the first eigenfunction in the y direction ($v = u - \psi(x)e(x,y)$), the coefficient on g involves $(2\pi)^2$ rather than π^2. It follows that

$$g(x) \approx \frac{\cosh(x - x_1)}{\cosh(L/2)} + \beta \text{ dependence}$$

The first term is exponentially small and the second term is controlled by S, proving Lemma 2.

To illustrate the mechanism of the lemmas explicitly, we carry out a sine series computation in a special case. Note that the size and sign of $(k\pi)^2 - \lambda_2$ for $k = 1$ versus $k \geq 2$ is at issue. We consider the special case in which $f_1(x) = 0$ and $f_2(x) = 1$ for $0 \leq x \leq N - 1$.

Then $N-1 \le L \le N$, so N and L are comparable. By comparison with rectangles of length N and $N-1$ we find that

$$\pi^2\left(1+\frac{4}{N^2}\right) \le \lambda_2 \le \pi^2\left(1+\frac{4}{(N+1)^2}\right)$$

For $0 \le x \le N-1$,

$$u(x,y) = \sum_{k=1}^{\infty} u_k(x)\sin(k\pi y)$$

where

$$u_k(x) = 2\int_0^1 \sin(k\pi y)u(x,y)dy$$

Furthermore, the Fourier coefficient u_k satisfies

$$u_k''(x) + (\lambda_2 - (k\pi)^2)u_k = 0$$

The function $\psi(x) = u_1(x)$ and $\lambda_2 - \pi^2 \approx 1/N^2 \approx 1/L^2$. Thus

$$u_1(x) = -c_1 \sin\sqrt{\lambda_2 - \pi^2}x$$

The coefficient satisfies $c_1 > 0$ because u is negative on the left half and positive on the right half of Ω. Normalize so that $\max u = 1$. By Theorem 2, $u_\pm$ is large at $x_0 \pm L/20$, and hence c_1 is larger than a positive absolute constant. This yields Lemma 1, as well as the precise location of x_1 as a function of λ_2.

On the other hand, the remaining terms of the series are small. For all $k > 1$, $\lambda_2 - k^2\pi^2 < -1$. Therefore,

$$u_k(x) = c_k \sinh\sqrt{(k\pi)^2 - \lambda_2}x$$

The unit bound on u implies

$$\sum_{k=1}^{\infty} u_k(x)^2 \le 2$$

In particular,

$$\sum_{k=2}^{\infty} c_k^2 \sinh^2[\sqrt{(k\pi)^2 - \lambda_2}(N-1)] \le 2$$

This implies that for $k \ge 2$,

$$|u_k(x)| \le Ce^{-kN} quad \text{ for } \quad |x - x_1| \le N/4$$

Hence $v(x,y)$ is exponentially small, which proves Lemma 2 in the special case.

Where is Λ?

Recall that the nodal line may be in the exact middle ($x_1 = N/2$), as in the case where Ω is a rectangle, or it may be very near the fat end of the region, as in the case of a circular sector with vertex at the origin: $x_1 \approx N - cN^{1/3}$. Theorems 1 and 2 give numerical schemes for approximating the location of the nodal line as follows.

Recall that x_0 was defined above as the zero of the eigenfunction ϕ_2. Since the second eigenvalue for $\mathcal{L}$ on $[0,N]$ is the same as the first Dirichlet eigenvalue for the operator on the two intervals $[0,x_0]$ and $[x_0,N]$, Theorem 2 implies the following prescription.

ODE Eigenvalue Scheme. Choose x_0 to be the unique number such that the lowest Dirichlet eigenvalue for the operator $\mathcal{L}$ on the intervals $[0, x_0]$ and $[x_0, N]$ are equal. Then

$$P_x\Lambda \subset [x_0 - A, x_0 + A]$$

The min-max principle implies that any curve dividing the region Ω into two halves with equal eigenvalues must intersect the nodal line. Theorem 1 implies that Λ is particularly close to a vertical straight line. This leads to the following prescription.

PDE Eigenvalue Scheme. Choose x_2 so that the least eigenvalues for the Dirichlet problem for the Laplace operator on the two regions

$$\Omega \cap \{(x,y) : x < x_2\} \quad \text{and} \quad \Omega \cap \{(x,y) : x > x_2\}$$

are equal. Then Theorem 1 implies

$$P_x\Lambda \subset [x_2 - C/N, x_2 + C/N]$$

The first scheme requires knowledge of the lowest eigenvalue of an ode, which is in standard numerical packages. The second scheme requires knowledge of the lowest eigenvalue on a convex domain, which is not quite as standard. Toby Driscoll [D] has recently developed a very effective program for computing both eigenfunctions and eigenvalues on polygons. Preliminary experiments with triangles with $3 \leq N \leq 150$ indicate that A in the first scheme may be $1/100 + 1/N$. (This seems too good to be true, but perhaps $A = 1/10 + 1/N$ will work in general.) The bound C/N in Scheme 2 seems to be $1/N$ as predicted by the case of a sector. We must confess, however, that the rigorous proofs of these bounds give ridiculous values like $C = 10^{20}$.

Conjectures

The methods outlined here should also give information about the size of the first eigenfunction, improving by a factor of L the bounds given in [J3].

Conjecture 1. *With the normalizations on Ω of Theorem 1, let u_1 denote the first eigenfunction for Ω such that* $\max |u_1| = 1$. *Then there is an absolute constant C and a suitable multiple of the first eigenfunction ϕ_1 for the operator $\mathcal{L}$ on $[0, N]$ satisfies*

$$|u_1(x,y) - \phi_1(x)| \leq C/L$$

Conjecture 1 is motivated by the elementary inequality

$$|\sin(x/L) - \sin(x/(L+1))| \leq C/L \quad \text{on} \quad 0 \leq x \leq \pi L$$

Corollary to Theorem 1. *Let (x,y) be a point of Λ satisfying $1/4 \leq y \leq 3/4$, that is, far from $\partial\Omega$. Let η be a unit vector tangent to Λ at (x,y). Then*

$$|\eta \cdot e_1| \leq CS \leq C/L^3$$

Conjecture 2. *The corollary is valid up to the boundary.*

(Conjecture 2 implies Theorem 1.)

Finally let us speculate about the higher-dimensional case. We begin by explaining the significance of L in another way. Let e be a unit vector and define

$$\Omega(t, e) = \{x + se : x \in \Omega,\ 0 \le s < t\}$$

Thus $\Omega(t, e)$ is Ω stretched by t in the direction e. Define

$$P(e) = -\frac{d}{dt}\lambda_1(\Omega(t, e))|_{t=0}$$

This is the first variation of the lowest eigenvalue. It is analogous to the projection body function in the theory of convex bodies. (See [J4].) In a convex domain normalized as above,

$$\begin{aligned} P(e_1) &\approx \min_e P \approx 1/L^3 \quad \text{and} \\ P(e_2) &\approx \max_e P \approx 1 \end{aligned}$$

Moreover, the direction e_1 is necessarily within $1/L^3$ of the values of e for which the exact minimum is attained.

In $\mathbf{R}^n$, $n \ge 3$ one can define the same function P on the unit sphere. In the spirit of quadratic forms, choose v_1 so that

$$P(v_1) = \min P$$

Choose v_2 perpendicular to v_1 such that

$$P(v_2) = \min_{v \perp v_1} P(v)$$

(One can continue inductively to form an orthonormal basis $v_1, v_2, \dots, v_n$.)

Conjecture 3. *There is a dimensional constant C such that if v is a unit vector tangent to Λ, then*

$$|v \cdot v_1| \le CP(v_1)/P(v_2)$$

This conjecture is intended to give specific bounds on the way the nodal set tends to a plane as the eccentricity tends to infinity. It is an analogous conjecture concerning the shape of the second eigenfunction to conjectures in [J4] concerning the shape of the first eigenfunction. One could also formulate even more detailed and even more speculative conjectures relating all the numbers $P(v_k)$ to the location of Λ.

References

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