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On the L^2–Stokes theorem and Hodge theory for singular algebraic varieties

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Abstract. For a projective algebraic variety V with isolated singularities, endowed with a metric induced from an embedding, we consider the analysis of the natural partial differential operators on the regular part of V. We show that, in the complex case, the Laplacians of the de Rham and Dolbeault complexes are discrete operators except possibly in degrees $n, n \pm 1$, where n is the complex dimension of V. We also prove a Hodge theorem on the operator level and the L^2–Stokes theorem outside the degrees $n-1, n$. We show that the L^2-Stokes theorem may fail to hold in the case of real algebraic varieties, and also discuss the L^2-Stokes theorem on more general non-compact spaces.

1. Introduction

The interplay between geometric differential operators on a Riemannian manifold and the geometry of the underlying manifold has been the focus of many efforts; one of the early highlights is the Atiyah-Singer Index Theorem.

Since the work of Atiyah and Singer one has become more and more interested in various types of manifolds with singularities. While the case of a smooth compact manifold is fairly well understood, the picture is far from complete for singular manifolds. It is impossible to give a complete account of the existing literature here. We only mention Cheeger's work on manifolds with cone-like singularities [7], Melrose's b-calculus [14] and Schulze's calculus on singular manifolds [26].

However, singularities occuring in "the real world" are often much more complicated than just conical. A very natural class of singular spaces is the class of (real or complex) projective algebraic varieties. These are special cases of stratified spaces. Topologically, stratified spaces are of iterated cone-type, and probably this was Cheeger's main motivation to develop an analysis of elliptic operators on such manifolds. However, it seems that the inductive step, i.e. the generalization of Cheeger's theory to stratified spaces, has still not been done. A more serious problem is that the natural metrics on algebraic varieties, i.e. those induced from a metric on projective space are not

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of iterated cone–type. A great deal of efforts have been made to find local models of such metrics [11], [12], [23].

Nevertheless, there exist partial results about the interplay between L^2–cohomology and intersection cohomology [18], [19], mixed Hodge structure [20], and the so-called L^2-Kähler package [6], [2], [21]. Having the L^2–Kähler package on a complex algebraic variety in general would be extremely nice, because it implies many of the fundamental operator identities which one has in the compact case.

The problems we discuss in this note are the L^2–Stokes theorem and discreteness of the Laplace-Beltrami operators:

Let V be a real or complex projective variety and let $M := V \setminus \operatorname{sing} V$ be its regular part. We equip M with the Riemannian metric g induced by a smooth metric on projective space (in the complex case we assume g to be Kähler). Furthermore, let $(\Omega_0(M), d)$ be the de Rham complex of differential forms acting on smooth forms with compact support. A priori, the operator d has several closed extensions in the Hilbert space of square integrable forms. These lie between the "minimal" and the "maximal" one. The latter are defined by

$$\begin{aligned} d_{\min} &:= \overline{d} = \text{closure of } d, \\ d_{\max} &:= (d^t)^* = \text{adjoint of the formal adjoint } d^t \text{ of } d. \end{aligned}$$

Some authors address the maximal as the Neumann and the minimal as the Dirichlet extension. We do not adopt this terminology since it may be misleading: On a compact manifold with boundary the Laplacians corresponding to the maximal/minimal extensions both are of mixed Dirichlet/Neumann type. The maximal/minimal extensions of d produce so-called Hilbert complexes. A detailed account of the functional analysis of Hilbert complexes was given in [1]. We note that the cohomology of the $d_{\max}$ complex is the celebrated L^2–cohomology

$$H^i_{(2)}(M) := \ker d_{i,\max} / \operatorname{im} d_{i-1,\max},$$

which has been the subject of intensive studies.

Having to make the distinction between $d_{\max}$ and $d_{\min}$ can be tedious. If $d_{\max} \neq d_{\min}$ then even simple facts known from compact manifolds might not be true. Therefore, it is desirable to have *uniqueness*, i.e. $d_{\max} = d_{\min}$, which is better known as the *L^2–Stokes theorem (L^2ST)* because of its equivalent formulation (2.5). We would like to emphasize that validity of L^2ST does *not* imply essential selfadjointness of the Laplace-Beltrami operator Δ (defined on compactly supported smooth forms), as can be seen in the case of cones already. Instead, L^2ST is equivalent to the selfadjointness of the specific extension $d^t_{\min} d_{\min} + d_{\min} d^t_{\min}$ of Δ.

On a compact manifold without boundary the L^2ST follows by a simple mollifier argument. It is known to hold for several types of non-compact manifolds. We give a short account of known results and their proofs in Section 2. The conjecture that the L^2ST holds for complex projective varieties has been around implicitly for quite a while, e.g. in [8], although the authors cannot give a reference where it is stated explicitly, except for complex surfaces [16].

The L^2ST is quite plausible for complex projective varieties because all strata are of even codimension, in particular there is no boundary. So, no "boundary terms" should occur in the integration by parts implicit in the L^2ST. However, this picture cannot be complete since we will prove the following (Proposition 2.10).

Theorem 1.1. *There exists an even dimensional real projective variety whose singular set consists of a single point, such that the L^2–Stokes theorem does not hold.*

Thus, if L^2ST holds for complex projective varieties the reason must lie in the complex structure.

Call two Riemannian metrics g, $\tilde{g}$ on M *quasi-isometric* if there is a constant C such that for each $x \in M, v \in T_xM$ one has

$$C^{-1} g_x(v,v) \le \tilde{g}_x(v,v) \le C g_x(v,v).$$

The domains of $d_{\min}$ and $d_{\max}$ and therefore validity of L^2ST are quasi-isometry invariants. Therefore, if L^2ST holds for one metric induced from projective space, it holds for all such metrics. Thus its validity is independent of the Kähler structure. But surprisingly enough the Kähler structure can be very useful as a tool. For example the estimate [20, Prop. 2.13] (see also Proposition 3.1 below) proved by Pardon and Stern on complex projective varieties with isolated singularities makes essential use of the Kähler structure. They apply their estimate to derive a Hodge structure on the L^2–cohomology. In particular, they conclude that in various degrees the cohomologies of the $d_{\min}$ and $d_{\max}$ de Rham complexes coincide. Together with results in [2] the estimates of Pardon and Stern can be used to prove more:

Theorem 1.2. *Let $V \subset \mathbb{C}P^N$ be an algebraic variety with isolated singularities, of complex dimension n, and let $M = V \setminus \operatorname{sing} V$, equipped with a Kähler metric induced by a Kähler metric on $\mathbb{C}P^N$. Then, for $k \neq n-1, n$ resp. $p+q \neq n-1, n$, the L^2 Stokes theorem holds and we have uniqueness for the Dolbeault operators, i.e.*

$$d_{k,\max} = d_{k,\min}, \qquad k \neq n-1, n, \tag{1.1}$$

$$\partial_{p,q,\max} = \partial_{p,q,\min}, \qquad p+q \neq n-1, n. \tag{1.2}$$

Furthermore, for $k \neq n, n \pm 1$ we have

$$\begin{aligned} & d_{k-1,\min} d^t_{k-1,\min} + d^t_{k,\min} d_{k,\min} \\ & \quad = \Delta^{\mathcal{F}}_k = 2 \bigoplus_{p+q=k} \Delta^{\mathcal{F}}_{p,q,\partial} \\ & \quad = 2 \bigoplus_{p+q=k} \partial_{p-1,q,\min} \partial^t_{p-1,q,\min} + \partial^t_{p,q,\min} \partial_{p,q,\min}, \end{aligned} \tag{1.3}$$

i.e. the Hodge decomposition holds in the operator sense.

Here, $\Delta^{\mathcal{F}}_k$ is the Friedrichs extension, see Section 3. We expect that L^2ST is true in fact for all degrees and that the first equality in (1.3) holds in all degrees except n (in degree n it will usually not hold). This would be a very interesting result since it would imply the Kähler package, as shown in [2, Th. 5.8].

As already mentioned, essential self-adjointness cannot be expected for the Laplacian. It is therefore quite surprising to obtain its Friedrichs extension from the de

Rham complex. It was shown in [2] that this case is exceptional and that it has some nice consequences.

It is interesting to know more about the structure of the spectrum of the Friedrichs extension of the Laplacian. We will prove the following.

Theorem 1.3. *Under the assumptions of Theorem* 1.2 *the operator on k-forms*

$$\Delta_k^{\mathcal{F}} = d_{k-1,\min} d_{k-1,\min}^t + d_{k,\min}^t d_{k,\min}$$

is discrete for $k \neq n, n \pm 1$.

For $k = 0$ on an algebraic surface or threefold with isolated singularities, this follows from [15] and [22], where an estimate for the heat kernel is proved. For algebraic curves, the full asymptotic expansion of the heat trace was proved by Brüning and Lesch [3]. The existence of such an asymptotic expansion in general remains a challenging open problem.

This note is organized as follows: In Section 2 we discuss various aspects of the L^2 Stokes theorem and its history. Furthermore, we prove Theorem 1.1. Section 3 is devoted to complex projective varieties and the proof of Theorems 1.2 and 1.3.

2. The L^2–Stokes theorem

We start with some general remarks about elliptic complexes and their ideal boundary conditions on non-compact manifolds. In particular we recall some results from [1, 2].

Let (M, g) be a Riemannian manifold and

$$D : C_0^\infty(E) \longrightarrow C_0^\infty(F)$$

a first order differential operator between sections of the hermitian vector bundles E, F. We consider D as an unbounded operator $L^2(E) \to L^2(F)$ and define two closed extensions of D by

$$\begin{aligned} D_{\min} &:= \overline{D} = \text{closure of } D, \\ D_{\max} &:= (D^t)^* = \text{adjoint of the formal adjoint } D^t \text{ of } D. \end{aligned}$$

Note that $D_{\min} \subset D_{\max}$ and $D_{\min} = (D^*)^* = (D_{\max}^t)^*$ where we write $D_{\max}^t = (D^t)_{\max}$. The domains of $D_{\max/\min}$ can be described as follows:

$$\begin{aligned} \mathcal{D}(D_{\min}) &= \left\{ s \in L^2(E) \,\middle|\, \begin{array}{l} \text{There exists a sequence } (s_n) \subset C_0^\infty(E) \\ \text{with } s_n \to s, Ds_n \to Ds \text{ in } L^2(E) \end{array} \right\}, \\ \mathcal{D}(D_{\max}) &= \{ s \in L^2(E) \mid Ds \in L^2(E) \}. \end{aligned} \tag{2.1}$$

$D_{\max}$ is maximal in the sense that it does not have a proper closed extension that has $C_0^\infty(F)$ in the domain of its adjoint. Of course, $D_{\max}$ does have (abstractly defined) proper closed extensions without this property.

The following well-known fact shows that the 'difference' between $D_{\max}$ and $D_{\min}$ only depends on the behavior of (M, g) at 'infinity', i.e. when leaving any compact subset of M. We include a proof for completeness.

Lemma 2.1. *If $s \in \mathcal{D}(D_{\max})$ and $\phi \in C_0^\infty(M)$ then $\phi s \in \mathcal{D}(D_{\min})$.*

Proof. We use a Friedrichs mollifier, i.e. a family of operators $J_\varepsilon : \mathcal{E}'(M) \to C_0^\infty(M), \varepsilon \in (0,1]$, such that $J_\varepsilon f \to f$ in $L^2(M)$ for any $f \in L^2_{\rm comp}(M)$ and such that J_ε and the commutator $[D, J_\varepsilon]$ are bounded operators on $L^2(M)$, uniformly in ε. For the existence of such mollifiers see [29, Ch. II.7].

Now we have $J_\varepsilon(\phi s) \in \mathcal{D}(D_{\min})$, $J_\varepsilon(\phi s) \to \phi s$, and $D(J_\varepsilon(\phi s)) = [D, J_\varepsilon](\phi s) + J_\varepsilon(D(\phi s))$ is uniformly bounded in $L^2(M)$, as $\varepsilon \to 0$. Therefore, there is a constant C such that for all $t \in \mathcal{D}(D^*)$ we have

$$|(\phi s, D^* t)| = \lim_{\varepsilon\to 0} |(J_\varepsilon(\phi s), D^* t)| = \lim_{\varepsilon\to 0} |(D(J_\varepsilon(\phi s)), t)| \le C\|t\|. \tag{2.2}$$

This means $\phi s \in \mathcal{D}((D^*)^*) = \mathcal{D}(D_{\min})$. □

We now turn to elliptic complexes. Elliptic complexes on manifolds with singularities have been studied systematically e.g. in [24], [25]. For a general discussion of Fredholm complexes and Hilbert complexes we also refer to [27] and [1]. Let

$$(C_0^\infty(E), d) : 0 \longrightarrow C_0^\infty(E_0) \xrightarrow{d_0} C_0^\infty(E_1) \xrightarrow{d_1} \dots \xrightarrow{d_{N-1}} C_0^\infty(E_N) \longrightarrow 0 \tag{2.3}$$

be an elliptic complex. The main examples are the de Rham complex $(\Omega_0^*(M), d)$ and, for a Kähler manifold, the Dolbeault complexes $(\Omega_0^{*,q}(M), \partial_{\cdot,q})$ and $(\Omega_0^{p,*}(M), \bar\partial_{p,\cdot})$.

Next we recall the notion of *Hilbert complex* (cf. [1], for example). This is a complex

$$(\mathcal{D}, D) : 0 \longrightarrow \mathcal{D}_0 \xrightarrow{D_0} \mathcal{D}_1 \xrightarrow{D_1} \dots \xrightarrow{D_{N-1}} \mathcal{D}_N \longrightarrow 0 \tag{2.4}$$

of closed operators D_k with domains $\mathcal{D}_k$ lying in Hilbert spaces H_k. An *ideal boundary condition (ibc)* of an elliptic complex $(C_0^\infty(E), d)$ is a choice of extensions D_k of d_k which form a Hilbert complex (with $H_k = L^2(E_k)$). That is, an ibc is a choice of closed extensions D_k of $d_k, 0 \le k < N$, with the additional property that $D_k(\mathcal{D}_k) \subset \mathcal{D}_{k+1}$. $d_{k,\min}$ and $d_{k,\max}$ are examples of ibc's for any elliptic complex.

The main question that we address here is whether there is only one ibc, i.e. whether $d_{k,\min} = d_{k,\max}$ for all k. This is called the case of *uniqueness*. Uniqueness is equivalent to

$$(d_{\max}\omega, \eta) = (\omega, d^t_{\max}\eta) \quad \text{for all } \omega \in \mathcal{D}(d_{\max}), \eta \in \mathcal{D}(d^t_{\max}). \tag{2.5}$$

Definition 2.2. We say that the *L^2–Stokes theorem* (L^2ST) holds for (M, g) if 2.5 is true for the de Rham complex on M.

This means that no boundary terms appear in the integration by parts that is implicit in 2.5. In particular, L^2ST holds for closed M.

We now return to general elliptic complexes 2.3 and their ibc's 2.4. Given an ibc $(\mathcal{D}, D)$, define the associated Laplacian by

$$\Delta(\mathcal{D}, D) = (D + D^*)^2.$$

This is a self-adjoint operator. The main instances are $\Delta^{a/r}$, which are associated with $D = d_{\max/\min}$, respectively. Note that Δ^a and Δ^r are not comparable unless they are equal. Lemma 3.1 from [2] says (as a special case) that this happens if and only if $d_{\max} = d_{\min}$. Explicitly, the restriction of $\Delta^{a/r}$ to sections of E_k (e.g. k-forms) is given by

$$\Delta_k^{a/r} = d_{k-1,\max/\min} d^t_{k-1,\min/\max} + d^t_{k,\min/\max} d_{k,\max/\min}.$$

Also, we denote

$$\Delta = (d + d^t)^2 \quad \text{on } C_0^\infty(E)$$

and Δ_k its restriction to sections of E_k.

Proposition 2.3. *Let $(C_0^\infty(E), d)$ be an elliptic complex.*

1. $C^\infty(E) \cap \mathcal{D}(d_{\max/\min}) \cap \mathcal{D}(d^t_{\min/\max})$ *is (graph) dense in* $\mathcal{D}(d_{\max/\min})$.
2. *Fix k and set*

$$\tilde{\Delta}_k := d_{k-1,\min} d^t_{k-1,\min} + d^t_{k,\min} d_{k,\min}.$$

$\tilde{\Delta}_k$ is self-adjoint if and only if $d_{k,\max} = d_{k,\min}$ and $d_{k-1,\max} = d_{k-1,\min}$.

In particular, if Δ_k is essentially self-adjoint for all k then uniqueness holds.

Proof. 1. By ellipticity, we have

$$\bigcap_{n \geq 0} \mathcal{D}((\Delta_k^{a/r})^n) \subset C^\infty(E_k). \tag{2.6}$$

By Lemma 2.11 in [1] the left hand side in 2.6 is a core for $d_{k,\max/\min}$ (i.e. dense in the graph topology), so the claim follows.

2. 'If' is obvious. To prove the converse, observe first $\tilde{\Delta}_k \subset \Delta_k^{a/r}$. Since all of these operators are self-adjoint, this implies $\Delta_k^r = \tilde{\Delta}_k = \Delta_k^a$. Since $\mathcal{D}(\Delta_k^{a/r})$ is a core for $d_{k,\max/\min}$ and for $d^t_{k-1,\min/\max}$ we are done. □

Now we turn to the de Rham complex and the L^2ST.

First we note that the validity of L^2ST is a quasi-isometry invariant since the domains $\mathcal{D}(d_{k,\max/\min})$ are quasi-isometry invariants in view of their characterization 2.1. (Note that d_k itself is independent of the metric.)

We will be mainly interested in the case of projective varieties with induced metrics. That is, we start with a variety V in real or complex projective space and let $M = V \setminus \operatorname{sing} V$ be its regular part. The metric on M is obtained by restriction of a Riemannian metric on projective space. Since any two Riemannian metrics on projective space are quasi-isometric, all metrics on M obtained in this way are mutually quasi-isometric, so if L^2ST holds for one such metric then it holds for all.

The main open problem about L^2ST is the following:

Conjecture 2.4. *The L^2–Stokes theorem holds for complex projective varieties.*

Conjectures closely related to Conjecture 2.4 were stated in the basic paper [8], but Conjecture 2.4 was not stated explicitly. For complex surfaces, Conjecture 2.4 was formulated in [16].

We now review some results about the validity of L^2ST.

1. *Complete manifolds:* Gaffney [9], [10] showed that Δ is essentially self-adjoint, in particular L^2ST holds.

2. *Cones, horns and pseudomanifolds:* Let (N^n, g_N) be a Riemannian manifold satisfying L^2ST, and such that Range $d_{n/2-1}$ is closed (e.g. N compact). Define the cone ($\gamma = 1$) or horn ($\gamma > 1$) over N by
$$M = (0, \infty) \times N, \quad g = dx^2 + x^{2\gamma} g_N.$$
Cheeger [5, Thm. 2.2] proved that L^2ST holds for (M, g) if and only if there are no square-integrable $n/2$-forms α on N satisfying $d\alpha = d^t\alpha = 0$ (if N is compact, this is the cohomological condition $H^{n/2}(N, \mathbb{R}) = 0$). Using this inductively, he showed L^2ST for 'admissible Riemannian pseudomanifolds' (Proof of Thm. 5.1 in loc. cit.). Youssin [31] generalized these results to the L^p Stokes theorem for any $p \in (1, \infty)$ and carried out a detailed study of their relation to L^p-cohomology.

3. *(Conformal) cones, complex projective algebraic curves:* A different proof for cones over compact N and a generalization to conformal cones was given by Brüning and Lesch [2]. In particular, this implies L^2ST for complex projective algebraic curves. The last result was obtained before by Nagase [17].

4. *Functions on real or complex projective algebraic varieties with singularities of real codimension at least two:* Li and Tian [13] proved that $d^t_{0,\min} d_{0,\min}$ is self-adjoint for such M. So, in view of Proposition 2.3 uniqueness holds for $k = 0$. This was proved before by Nagase [15] and Pati [22] for complex surfaces and threefolds, respectively.

5. *Complex projective algebraic varieties with isolated singularities:* Pardon and Stern [20] proved that the cohomology of the $d_{k,\min}$ and $d_{k,\max}$–complexes coincide in degrees k with $|k - n| \geq 2$, $n = \dim_{\mathbb{C}} M$. We will prove below (Propositions 3.1 and 3.2) that their estimates actually imply the stronger $d_{k,\max} = d_{k,\min}$ for $k \neq n-1, n$.

6. *Real analytic surfaces with isolated singularities:* In this case, uniqueness was proved in all degrees by Grieser [11].

7. *Orbit spaces:* If G is a compact Lie group acting isometrically on a smooth compact Riemannian manifold X then the quotient X/G is a stratified space with a Riemannian metric on its smooth part M (see [28]). Then L^2ST holds on M if and only if X/G is a Witt space, i.e. the links of all the odd-codimensional strata have vanishing middle L^2-cohomology. A study of when this happens was made for S^1-actions by Sjamaar [28].

The methods used to prove L^2ST all proceed essentially in the following way: First, one derives certain boundedness or decay properties for forms $\omega \in \mathcal{D}(d_{\max})$, then one uses these to show that, for a suitable sequence of cutoff-functions $\{\phi_n\} \in C_0^\infty(M)$, the forms $\phi_n\omega$ converge to ω in graph norm. By Lemma 2.1, this implies $\omega \in \mathcal{D}(d_{\min})$. In the first step, it may be useful to restrict to certain dense (in graph norm) subspaces of $\mathcal{D}(d_{\max})$ with better (or easier to derive) properties. For example, from the proof of Proposition 2.3 (1) we see in particular that

$$\mathcal{D}(d_{k,\max}) \cap \mathcal{D}(d^t_{k-1,\max}) \cap C^\infty(\Lambda^k M)$$

is a core for $d_{k,\max}$. Thus, we may assume

$$\omega \in L^2, \quad (d + d^t)\omega \in L^2. \tag{2.7}$$

This allows to use elliptic techniques to estimate ω.

On functions, one can do slightly better:

Lemma 2.5.(cf. [13, Sec. 4]) *Let (M,g) be a Riemannian manifold. Then*

$$\{f \in \mathcal{D}(d_{0,\max}) \mid \sup_{x\in M} |f(x)| < \infty\}$$

is a core for $d_{0,\max}$.

Proof. For $f \in \mathcal{D}(d_{0,\max}) \cap C^\infty(M)$ choose a sequence (a_n) of regular values of f with $\lim_{n\to\infty} a_n = \infty$ and put $f_n := \max(-a_n, \min(f(x), a_n))$. Then f_n is bounded and it is straightforward to check that $f_n \to f$ in the graph norm of d. □

In general, if $\omega \in \mathcal{D}(d_{k,\max})$ and $(\phi_n) \subset C_0^\infty(M)$, in order to prove $\omega \in \mathcal{D}(d_{k,\min})$ it suffices to show

$$\phi_n\omega \longrightarrow \omega \quad \text{in } L^2(\Lambda^k M), \tag{2.8}$$

$$\|d(\phi_n\omega)\|_{L^2(\Lambda^k(M))} \leq C. \tag{2.9}$$

This follows from Lemma 2.1 and the analogue of estimate (2.2).

By Lebesgue's dominated convergence theorem, (2.8) is fulfilled if

$$\text{Condition 1: } \phi_n \longrightarrow 1 \text{ pointwise and } \sup_{x,n} |\phi_n(x)| < \infty.$$

(The second condition could be weakened to $\sup_n |\phi_n(x)| \leq \phi(x)$ for some ϕ with $\phi\omega \in L^2$.) Then, (2.9) will be satisfied if

$$\text{Condition 2: } \|d\phi_n \wedge \omega\|_{L^2} \leq C$$

(plus $\phi\, d\omega \in L^2$ in case of the weaker Condition 1).

Specifically, the proofs of the results cited above proceed in the following way:

1. Gaffney finds ϕ_n satisfying Condition 1 and with $d\phi_n$ uniformly bounded as $n \to \infty$.

2. Cheeger proves (2.5) directly, using the weak Hodge decomposition of $\omega \in \mathcal{D}(d_{\max})$ and estimating its parts separately. Since validity of L^2ST is a local property (see [5, Lemma 4.1]) the result on pseudomanifolds follows by localization and iteration, using a partition of unity with uniformly bounded differentials.

3. Brüning and Lesch show that (2.7) implies $\omega \in \mathcal{D}(d_{\min})$ modulo the pull-back of a harmonic $n/2$-form on N, by writing the equation $D\omega = f \in L^2$ as an operator-valued singular ordinary differential equation in the axis variable x, whose solutions can be analyzed fairly explicitly.

4. Li and Tian first restrict to bounded functions using Lemma 2.5 and then use a sequence ϕ_n satisfying Condition 1 and $\|d\phi_n\|_{L^2} \to 0$, which clearly implies Condition 2.

5. Pardon and Stern show, using the Kähler identities, that $d\omega, d^t\omega \in L^2$ implies $\omega/r \in L^2$ if $|k-n| \geq 2$, where r is the distance from the singularity [20, Prop. 2.27]. Then they use a cutoff sequence satisfying Condition 1 and with $rd\phi_n$ tending to zero uniformly as $n \to \infty$. This again implies Condition 2.

6. Grieser shows that a neighborhood of a singular point is quasi-isometric to a union of cones and horns, so the result follows from (2).

7. This follows from the results on Riemannian pseudomanifolds of (2) since the metric on M is locally quasi-isometric to a metric of iterated cone type.

We now show that on real algebraic varieties L^2ST may very well fail, even near isolated singularities. For this, we construct examples which locally near the singularity are quasi-isometric to

$$\begin{aligned} M &= \mathbb{R}_+ \times N_1 \times N_2 \\ g &= dr^2 + r^{2\alpha_1} g_1 + r^{2\alpha_2} g_2 \end{aligned} \tag{2.10}$$

with compact oriented Riemannian manifolds (N_i, g_i) and real numbers $\alpha_i \geq 1$, $i = 1, 2$. Here, r denotes the first variable.

Lemma 2.6. *Let (M, g) be as in 2.10. If $H^k(N_2) \neq 0$ then L^2ST fails for k-forms supported near $r = 0$, where*

$$k = \frac{\alpha_1 n_1 + \alpha_2 n_2}{2\alpha_2}, \quad n_i = \dim N_i. \tag{2.11}$$

Proof. Let ω be a non-zero harmonic k-form on N_2, i.e. $d_2\omega = d_2^t\omega = 0$ (in this proof, the subscripts 1,2 refer to N_1, N_2 respectively). Choose cut-off functions $\phi, \psi \in C_0^\infty([0,\infty))$ such that $\phi = 1$ near zero, $\operatorname{supp}\psi \subset \{\phi = 1\}$, and $\psi(0) \neq 0$. For simplicity, denote the pullbacks of ω, ϕ, ψ to M by the same letters.

Set $\mu = \phi\omega \in \Omega^k(M)$, $\nu = \psi dr \wedge \omega \in \Omega^{k+1}(M)$. We claim that

$$(d\mu, \nu) \neq (\mu, d^t\nu).$$

To prove this, recall $(d\mu, \nu) - (\mu, d^t\nu) = \int_M d(m \wedge *\nu)$. A simple calculation shows $*\nu = \pm\psi(r) r^\beta dvol_1 \wedge *_2\omega$ (the sign is unimportant), where $\beta = \alpha_1 n_1 + \alpha_2(n_2 - 2k)$. So $\mu \wedge *\nu = \pm\psi(r) r^\beta dvol_1 \wedge (\omega \wedge *_2\omega)$. Since ω is closed and coclosed, we get $d(\mu \wedge *\nu) = \pm d(\psi r^\beta) \wedge dvol_1 \wedge (\omega \wedge *_2\omega)$. Therefore, if $\beta \geq 0$ we have $\int_M d(\mu \wedge *\nu) = \pm\|\omega\|^2_{L^2(N_2)} \psi(r) r^\beta_{|r=0}$. This is non-zero for $\beta = 0$, i.e. k as in (2.11). □

For example, if

$$\alpha_1 n_1 = \alpha_2 n_2$$

then $k = n_2$, so L^2ST fails since always $H^{n_2}(N_2) \neq 0$. By symmetry, L^2ST also fails for $k = n_1$.

Lemma 2.7. *Let n, m, p, q be positive integers, $p > q$. For $x \in \mathbb{R}^n, y \in \mathbb{R}^m, z \in \mathbb{R}$ define the homogeneous polynomial f by*

$$f(x,y,z) = |x|^{2p} - |y|^{2q} z^{2(p-q)}$$

and the variety V by

$$V = \{[x,y,z] \mid f(x,y,z) = 0\} \subset \mathbb{R}P^{n+m}.$$

Then, the singularities of V are given by

$$\operatorname{sing} V = \{[0,0,1]\} \cup \{[0,y,0] \mid y \in \mathbb{R}^m \setminus \{0\}\}. \tag{2.12}$$

Thus, V has an isolated singularity at $P = [0,0,1]$ and one singular stratum isomorphic to $\mathbb{R}P^{m-1}$. Given any Riemannian metric on $\mathbb{R}P^{n+m}$, a (pointed) neighborhood of P in V is quasi-isometric to a neighborhood of $r = 0$ in

$$\begin{aligned} M &= \mathbb{R}_+ \times S^{n-1} \times S^{m-1} \\ g &= dr^2 + r^2 g_{S^{n-1}} + r^{2\alpha} g_{S^{m-1}} \end{aligned} \tag{2.13}$$

where $\alpha = p/q$.

Proof. That $\operatorname{sing} V$ is given by the right hand side of 2.12 can be checked by a straightforward calculation.

For $z \neq 0$, we use the standard chart $(x,y) \in \mathbb{R}^{n+m} \to [x,y,1] \in \mathbb{R}P^{n+m}$. In this chart, V is given by $\{|x|^{2p} = |y|^{2q}\}$, which clearly has an isolated singularity at $(0,0)$. Parametrize V near but outside $(0,0)$ by

$$\begin{aligned} \phi : &\mathbb{R}_+ \times S^{n-1} \times S^{m-1} \longrightarrow V \\ &(r,v,w) \longmapsto (rv, r^\alpha w), \quad \alpha = p/q. \end{aligned}$$

Any Riemannian metric on $\mathbb{R}P^{n+m}$ is quasi-isometric to the standard Euclidean metric $|dx|^2 + |dy|^2$ near P. By the usual formula for polar coordinates, we have

$$\begin{aligned} \phi^*(|dx|^2) &= dr^2 + r^2 g_{S^{n-1}} \\ \phi^*(|dy|^2) &= (d(r^\alpha))^2 + r^{2\alpha} g_{S^{m-1}}, \end{aligned}$$

so $\phi^*(|dx|^2 + |dy|^2) = (1 + \alpha^2 r^{2(\alpha-1)}) dr^2 + r^2 g_{S^{n-1}} + r^{2\alpha} g_{S^{m-1}}$, which is quasi-isometric to (2.13) for bounded r. □

Proposition 2.8. *Let V be as above and assume $p(m-1) = q(n-1)$. Then L^2ST fails for V for $(n-1)$-forms and for $(m-1)$-forms.*

Proof. This follows from Lemma 2.7 and Lemma 2.6 in the special case mentioned after its proof. □

In the previous proposition we may choose m odd and n even (e.g. $m = 2k+1, n = 2k+2, p = 2k+1, q = 2k,\ k > 0$). Then V is even–dimensional and all strata are of even codimension.

We can even slightly modifiy V to produce a variety with one isolated singularity for which L^2ST does not hold:

Lemma 2.9. *Let $f, g \in C^\infty(\mathbb{R})$ with $f^{(j)}(0) = 0, 0 \le j < k$, $f^{(k)}(0) \neq 0$, $g^{(j)}(0) = 0$, $0 \le j < l$, $g^{(l)}(0) \neq 0$, $k \ge l$. Then, locally near 0, the quasi–isometry class of*

$$V_{f,g} := \{(x, y) \in \mathbb{R}^{n+m} \mid f(|x|^2) = g(|y|^2)\}$$

depends only on k and l.

Proof. W.l.o.g. we may assume that f and g are non-negative for small positive argument. Let $\tilde f(r) = r^k, \tilde g(r) = r^l$. Then, there exist $\varphi, \psi \in C^\infty(\mathbb{R})$, $\varphi(0) = \psi(0) = 0$, $\varphi'(t), \psi'(t) > 0$ for $|t| < \varepsilon$, such that

$$\tilde f = f \circ \varphi, \quad \tilde g = g \circ \psi.$$

We put $\varphi_1(t) := \varphi(t)/t, \psi_1(t) := \psi(t)/t$ and

$$\Phi(x, y) := (\varphi_1(|x|^2)^{1/2}x, \psi_1(|y|^2)^{1/2}y), \quad (x, y) \in \mathbb{R}^{n+m}, \quad |x|, |y| < \varepsilon.$$

Then Φ is a local diffeomorphism at $(0, 0)$ mapping $V_{\tilde f,\tilde g} \cap U$ onto $V_{f,g} \cap U$ for some neighborhood U of $(0, 0)$. Hence $V_{\tilde f,\tilde g} \cap U$ and $V_{f,g} \cap U$ are quasi–isometric. □

Summing up, we have:

Proposition 2.10. *Under the assumptions of Lemma 2.7, consider the variety*

$$W = \{[x, y, z] \mid |x|^{2p} = |y|^{2q}z^{2(p-q)} + |y|^{2p}\} \subset \mathbb{R}P^{n+m}.$$

Then $\operatorname{sing} W = \{[0, 0, 1]\}$, i.e. W has exactly one singular point. Moreover, given any Riemannian metric on $\mathbb{R}P^{n+m}$, a (pointed) neighborhood of $[0, 0, 1]$ in W is quasi-isometric to a pointed neighborhood of $[0, 0, 1]$ in the variety V of Lemma 2.7.

In particular, if $p(m-1) = q(n-1)$, then L^2ST fails to hold for W.

Proof. The determination of $\operatorname{sing} W$ is straightforward. The other statements follow from the previous discussion. □

3. Complex algebraic varieties

In this section we discuss the L^2–Stokes theorem and the discreteness of the spectrum of the Laplace-Beltrami operator on complex projective varieties with isolated singularities. The main ingredients are an estimate due to Pardon and Stern [20] and results by Brüning and Lesch [2].

Throughout this section, $V \subset \mathbb{C}P^N$ denotes an algebraic variety with isolated singularities and $M = V \setminus \operatorname{sing} V$ its smooth part, $n := \dim_{\mathbb{C}} M$. Let g be a Kähler metric on M which is induced by a Kähler metric on $\mathbb{C}P^N$. Fix a smooth function $r : M \to (0, \infty)$ which, near any singularity $p \in \operatorname{sing} V$, is comparable to the distance from p.

Denote by $\partial_{p,q}, \bar\partial_{p,q}$ the Dolbeault operators on forms of type (p, q). Pardon and Stern show in [20]:

Proposition 3.1.

1. *Assume* $n - p - q \geq 2$. *If* $\omega \in \mathcal{D}(\partial_{p,q,\max}) \cap \mathcal{D}(\partial^t_{p-1,q,\max})$ *then*
$$\omega/r \in L^2(\Lambda^{p,q} M)$$
and $\omega \in \mathcal{D}((\bar\partial + \bar\partial^t)_{\min}) \cap \mathcal{D}((\partial + \partial^t)_{\min})$.
2. *Assume* $n - k \geq 2$. *If* $\omega \in \mathcal{D}(d_{k,\max}) \cap \mathcal{D}(d^t_{k-1,\max})$ *then*
$$\omega/r \in L^2(\Lambda^k M)$$
and $\omega \in \mathcal{D}((d + d^t)_{\min})$.

For the proof see [20], Prop. 2.27 and Lemma 2.18.
We combine this with a result in [2].

Proposition 3.2. *Let* $(C_0^\infty(E), d)$ *be an elliptic complex. If for some* $k \geq 0$
$$\mathcal{D}(d_{k,\max}) \cap \mathcal{D}(d^t_{k-1,\max}) \subset \mathcal{D}((d + d^t)_{\min}) \tag{3.1}$$
then we have
$$d_{k,\max} = d_{k,\min}, \tag{3.2}$$
$$d^t_{k-1,\max} = d^t_{k-1,\min}, \tag{3.3}$$
and
$$\Delta_k^{r/a} = d_{k-1,\min} d^t_{k-1,\min} + d^t_{k,\min} d_{k,\min} = \Delta_k^{\mathcal{F}},$$
where $\Delta_k^{\mathcal{F}}$ *denotes the Friedrichs extension of the Laplacian* Δ_k.

For the proof see [2], Lemma 3.3. Note that (3.1) is equivalent to
$$D_{k,\max} = D_{k,\min}$$
for $D_k := d_k + d^t_{k-1}$. However, one needs to be careful when considering the 'rolled-up' complex, i.e. the operator $D = \bigoplus_{k=0}^N D_k$: Even if (3.1) holds for all k, this does *not* imply $D_{\max} = D_{\min}$! The reason for this is that, in general,
$$\mathcal{D}(D_{\max}) \supsetneq \bigoplus_k \mathcal{D}(D_{\max}) \cap L^2(\Lambda^k M)$$
since the ranges of $d_{\max}$ and $d^t_{\max}$ might not be orthogonal. An example can be easily constructed using the de Rham complex on the unit disk in $\mathbb{R}^2$.

Proof of Theorem 1.2. First, consider the de Rham complex. Propositions 3.1 and 3.2 imply that 3.2 and 3.3 hold for $k = 0, \ldots, n-2$. From $d_k = \pm * d^t_{2n-k-1} *$ we get the same relations for $k = n+2, \ldots, 2n$. Finally, 3.3, applied with $k = n+2$, gives 3.2 for $k = n+1$ by taking adjoints. This proves 1.1. The same argument applied to the Dolbeault complex proves 1.2.

The first and third equality in 1.3 follow from the last identity in Proposition 3.2, applied to the de Rham and Dolbeault complex, respectively. It remains to prove
$$\Delta_k^{\mathcal{F}} = 2 \bigoplus_{p+q=k} \Delta^{\mathcal{F}}_{p,q,\partial}. \tag{3.4}$$

Now the Kähler identities (see [30], Ch. V, Thm. 4.7) show that, for a compactly supported smooth k-form $\phi = \bigoplus_{p+q=k} \phi_{p,q}$, one has

$$\Delta_k \phi = 2 \bigoplus_{p+q=k} \Delta_{p,q,\partial}\, \phi_{p,q}. \tag{3.5}$$

The domain of the Friedrichs extension of Δ_k is defined as the completion of $C_0^\infty(\Lambda^k M)$ with respect to the scalar product

$$(\phi,\psi)_{\Delta_k^{\mathcal{F}}} = (\Delta_k \phi, \psi) + (\phi,\psi).$$

By (3.5) this scalar product is the direct sum of the analogous scalar products $(\ ,\ )_{\Delta_{p,q}^{\mathcal{F}}}$ on $C_0^\infty(\Lambda^{p,q}M)$, $p+q=k$, so we obtain (3.4). □

Remark 3.3. For the operators $\partial, \bar\partial$ this result is sharp. This can be seen already in the case of algebraic curves: In [4] it was shown that for $n=1$ in general $\partial_{0,\max} \neq \partial_{0,\min}, \partial_{0,1,\max} \neq \partial_{0,1,\min}$.

We now turn to the question of discreteness. Recall that a self–adjoint operator T in some Hilbert space H is called *discrete* if its spectrum consists only of eigenvalues of finite multiplicity (with ∞ as the only accumulation point). Note that T is discrete if and only if it has a compact resolvent or equivalently if the embedding $\mathcal{D}(T) \hookrightarrow H$ is compact. Here, $\mathcal{D}(T)$ carries the graph topology.

Proposition 3.4. *Let (M,g) be a Riemannian manifold such that there is a function $r \in C^\infty(M)$, $r > 0$, with the following properties:*

(i) *For $\varepsilon > 0$ the set $r^{-1}([\varepsilon,\infty))$ is compact,*

(ii) *$f \in \mathcal{D}(\Delta_k^{\mathcal{F}})$ implies $f/r \in L^2(\Lambda^k(M))$.*

Then $\Delta_k^{\mathcal{F}}$ is discrete.
In particular, if M is as in Theorem 1.2 then $\Delta_k^{\mathcal{F}}$ is discrete for $k \neq n, n\pm 1$.

Proof. We first note that (ii) implies

$$\|f/r\| \leq C\|(I+\Delta_k^{\mathcal{F}})f\|$$

for $f \in \mathcal{D}(\Delta_k^{\mathcal{F}})$. ($\|\cdot\|$ denotes the L^2 norm.) This follows from the closed graph theorem since the operator of multiplication by $1/r$ is closable with domain containing $\mathcal{D}(\Delta_k^{\mathcal{F}})$. In order to prove discreteness, it is enough to show that the embedding $\mathcal{D}(\Delta_k^{\mathcal{F}}) \hookrightarrow L^2$ is compact, where $\mathcal{D}(\Delta_k^{\mathcal{F}})$ carries the graph norm. Thus, let $(\phi_n) \subset \mathcal{D}(\Delta_k^{\mathcal{F}})$ be a bounded sequence, i.e.

$$\|\phi_n\|, \|\Delta_k^{\mathcal{F}}\phi_n\| \leq C_1.$$

We need to show that (ϕ_n) has a subsequence that is convergent in $L^2(\Lambda^k(M))$.

Since Δ_k is an elliptic differential operator and since $\{r \geq 1\}$ is compact there exists a subsequence $(\phi_n^{(1)})$ which converges in $L^2(\Lambda^k(\{r \geq 1\}))$. Choose a subsequence

$(\phi_n^{(2)})$ of $(\phi_n^{(1)})$ which converges in $L^2(\Lambda^k(\{r \geq 1/2\}))$. Continuing in this way and then using a diagonal argument, we find a subsequence $(\phi_n^{(\infty)})$ such that

$$(\phi_n^{(\infty)}|\{r \geq 1/m\}) \text{ converges in } L^2(\Lambda^k(\{r \geq 1/m\})), \qquad \text{for all } m.$$

We show that $(\phi_n^{(\infty)})$ is a Cauchy sequence in $L^2(\Lambda^k(M))$: First, we estimate

$$\|\phi_n^{(\infty)}\|_{L^2(\{r\leq 1/m\})} \leq \frac{1}{m}\|\frac{1}{r}\phi_n^{(\infty)}\| \leq \frac{C}{m}\|(I+\Delta_k^{\mathcal{F}})\phi_n^{(\infty)}\| \leq C'/m.$$

Given $\varepsilon > 0$ choose m such that $C'/m < \varepsilon/3$. Then

$$\|\phi_k^{(\infty)} - \phi_l^{(\infty)}\|_{L^2(\Lambda^k(M))} \leq \frac{2}{3}\varepsilon + \|\phi_k^{(\infty)} - \phi_l^{(\infty)}\|_{L^2(\{r\geq 1/m\})}.$$

Since the last term tends to zero as $k, l \to \infty$, we are done. □

Proof of Theorem 1.3. This follows by applying Proposition 3.4 to the situation of Theorem 1.2 in view of Proposition 3.1. □

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