UNIFORM BOUNDS FOR EIGENFUNCTIONS OF THE LAPLACIAN ON MANIFOLDS WITH BOUNDARY

D. GRIESER

Abstract. Let u be an eigenfunction of the Laplacian on a compact manifold with boundary, with Dirichlet or Neumann boundary conditions, and let $-\lambda^2$ be the corresponding eigenvalue. We consider the problem of estimating $\max_M u$ in terms of λ, for large λ, assuming $\int_M u^2 = 1$. We prove that $\max_M u \leq C_M \lambda^{(n-1)/2}$, which is optimal for some M. Our proof simplifies some of the arguments used before for such problems. We review the 'wave equation method' and discuss some special cases which may be handled by more direct methods.

1. Introduction

Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 2$, with (smooth) boundary ∂M. Let Δ denote the Laplace-Beltrami operator on functions on M. Consider a solution of the eigenvalue problem, with Dirichlet or Neumann boundary conditions,

$$(1) \qquad \begin{aligned} &(\Delta + \lambda^2)u = 0, \\ &u_{|\partial M} = 0 \quad \text{or} \quad \partial_n u_{|\partial M} = 0, \end{aligned}$$

(∂_n denotes the normal derivate) normalized by the condition

$$(2) \qquad \|u\|_2 = 1.$$

The subscript p will always indicate the $L^p(M)$ norm. In this paper we consider the problem of bounding $\|u\|_\infty = \max_{x \in M} |u(x)|$ in terms of λ, for large λ. The size of $\|u\|_\infty$ may be considered as a rough measure for how unevenly the function u is distributed over M: If u is 'small' (say $o(1)$ as $\lambda \to \infty$) on a 'large' set S (say, of area $|S| = |M| - \delta$) then the condition (2) forces $\|u\|_\infty$ to be at least of order $\delta^{-1/2}$; therefore upper bounds on $\|u\|_\infty$ imply lower bounds on δ, i.e. the area of the set where u is 'concentrated'. It may happen that $\delta \to 0$ as $\lambda \to \infty$, for certain (sequences of) eigenfunctions, for example on the sphere or the disk. Also, upper bounds on $\|u\|_\infty$ yield upper bounds for the multiplicities of the eigenvalues of Δ.

In addition to single eigenfunctions we also consider sums of eigenfunctions of the form

$$u_I(x) = \sqrt{\sum_{j:\lambda_j \in I} |u_j(x)|^2}$$

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for finite intervals $I \subset \mathbb{R}$. Here $u_1, u_2, \dots$ is any orthonormal basis of eigenfunctions with eigenvalues $\lambda_1 < \lambda_2 \le \dots \to \infty$. (The sum is independent of the choice of basis.) A simple argument using a suitable version of Sobolev's embedding theorem (see, for example, [Hör85], Thm. 17.5.3) shows that

$$\sup_x u_{[0,\lambda]}(x) \le C\lambda^{n/2}. \tag{3}$$

(Here and everywhere C will denote some constant only depending on (M, g).) Together with Weyl's law

$$\#\{j : \lambda_j \le \lambda\} \sim \gamma_M \lambda^n \quad \text{as } \lambda \to \infty \tag{4}$$

(with $\gamma_M = (2\pi)^{-n}\mathrm{vol}_{\mathrm{eucl}}(B^n)\mathrm{vol}(M)$, B^n=unit ball in $\mathbb{R}^n$) this shows that for each x the average size of $u_1(x), u_2(x), \dots$ is of order $O(1)$. However, if M is a sphere or a ball, for example, then there is a subsequence $u_{j'}$ of eigenfunctions with $\|u_{j'}\|_\infty \ge c\lambda_{j'}^{(n-1)/2}$, for some constant $c > 0$ (see Section 2.3). Our main result shows that this is the worst possible rate of growth:

Theorem 1. *Let M^n be a compact Riemannian manifold with boundary. There is a constant $C = C(M)$ such that any solution of* (1) *satisfies*

$$\|u\|_\infty \le C\lambda^{(n-1)/2}. \tag{5}$$

Corollary 2. *Under the same assumptions as in the theorem, the multiplicity of the eigenvalue λ^2 of $-\Delta$ is at most $C\lambda^{n-1}$.*

The theorem is already non-trivial in the case of a domain in $\mathbb{R}^n$ with the Euclidean metric: 'Interior' estimates are simpler in this case due to the constant coefficients of Δ (see Section 2.4), but the boundary must still be curved, and this causes the main difficulty. While the bound (5) is optimal for balls and spheres it is far from optimal for a torus and a rectangle (see Section 2.2), and possibly for more general M under assumptions on the curvature. For example, in the case of negative curvature, the results of Bérard [Bér77] imply a bound of $C\lambda^{(n-1)/2}/\log\lambda$. The first time that the exponent in (5) was improved in any case of nonzero curvature was in the paper ([IS95]) by Iwaniec and Sarnak, for certain arithmetic surfaces M. We do not analyze the dependence of the constant C in (5) on the metric. The methods used imply that it is bounded in terms of a finite number (depending on n) of derivatives of the metric and of the geodesic curvature of ∂M.

Outline of the proof of Theorem 1. The idea is to use the standard wave kernel method outside a boundary layer of width λ^{-1} and a maximum principle argument inside that layer.

Let us first recall the wave kernel method (cf. [Hör68, Sog93]). Certain weighted sums over many eigenfunctions turn out to be easier to estimate than single eigenfunctions, since they have a local character. More precisely, given $\epsilon > 0$, choose a Schwartz function ρ such that

$$\rho \ge 0, \quad \rho_{|[0,1]} \ge 1, \quad \mathrm{supp}\,(\hat\rho) \subset (-\epsilon, \epsilon).$$

Here $\hat\rho(t) = \int_{-\infty}^{\infty} e^{-it\lambda}\rho(\lambda)\,d\lambda$ denotes the Fourier transform. Proving the existence of such a ρ is an easy exercise.

If $u_1, u_2, \ldots$ is an orthonormal basis of real eigenfunctions of the Dirichlet Laplacian, with eigenvalues $\lambda_1 < \lambda_2 \leq \ldots \to \infty$, then we consider the sum, convergent by (3),

$$\sum_j \rho(\lambda - \lambda_j) u_j(x) u_j(y). \tag{6}$$

Actually, this is the integral kernel of the operator $\rho(\lambda - \sqrt{-\Delta})$, but we won't use this fact. If we write, using Fourier's inversion formula,

$$\begin{aligned} \rho(\lambda - \lambda_j) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\rho}(t) e^{it(\lambda - \lambda_j)}\, dt \\ &= [\hat{\rho}(t) e^{-it\lambda_j}]\check{}\,(\lambda) \\ &= 2[\hat{\rho}(t)\cos(t\lambda_j)]\check{}\,(\lambda) - \rho(\lambda + \lambda_j), \end{aligned}$$

(the superscript $\check{}$ denotes inverse Fourier transform $t \to \lambda$) then we get the important identity

$$\sum_j \rho(\lambda - \lambda_j) u_j(x) u_j(y) = 2[\hat{\rho}(t) K(t, x, y)]\check{}\,(\lambda) + O(\lambda^{-\infty}) \tag{7}$$

where the error term is small by (3) and rapid decay of ρ, and

$$K(t, x, y) = K_M(t, x, y) = \sum_j \cos(t\lambda_j) u_j(x) u_j(y) \tag{8}$$

is the *wave kernel*, i.e. for each fixed $y \in \dot{M}$ (the interior of M) it is the solution of

$$\begin{aligned} (\frac{\partial^2}{\partial t^2} - \Delta_x) K(t, x, y) &= 0 \quad \text{in } \mathbb{R}_t \times \dot{M}_x \\ K(0, x, y) &= \delta_y(x) \\ \frac{\partial}{\partial t} K(t, x, y) &= 0 \\ K_{|x \in \partial M} &= 0. \end{aligned} \tag{9}$$

The convergence of (8) is in the sense of distributions (i.e. weakly) in t, for each fixed x, y, as the argument leading to (7) shows, for example.

The point of (7) is that K may be analyzed directly from (9). In particular, solutions of the wave equation have *finite propagation speed* (see [Hör85], Lemma 17.5.12, for example, for the easy proof using energy estimates); for K this means

$$\operatorname{supp} K \subset \{(t, x, y) : \operatorname{dist}(x, y) \leq |t|\}$$

and that $K(t, x, y)$ depends only on the data (i.e. M and the metric g) in $B_t(x, y) := \{z : \operatorname{dist}(x, z) + \operatorname{dist}(y, z) \leq |t|\}$. Therefore, (7) shows that the sum (6) depends (up to an error $O(\lambda^{-\infty})$) only on the data in $B_\epsilon(x, y)$, and this is the *local character* mentioned before.

We will use (7) only on the diagonal, i.e. for $x = y$. From the assumptions on ρ we get

$$u_{[\lambda-1,\lambda]}(x)^2 \leq 2[\hat{\rho}(t) K(t, x, x)]\check{}\,(\lambda) + O(\lambda^{-\infty}), \tag{10}$$

so the theorem would follow from an estimate

$$|\int e^{it\lambda}\hat\rho(t)K(t,x,x)\,dt| \le C\lambda^{n-1}. \tag{11}$$

Since $\hat\rho$ is smooth, this is a statement about the singularities of K as a distribution in t, uniformly in the parameter x.

As a first step one may now obtain the *interior estimate*

$$u_{[\lambda-1,\lambda]}(x) \le C_\epsilon \lambda^{(n-1)/2} \quad \text{if } \operatorname{dist}(x,\partial M) > \epsilon. \tag{12}$$

In the case of a domain $M \subset \mathbb{R}^n$ one has $K_M = K_{\mathbb{R}^n}$ for $|t| < 2\operatorname{dist}(x,\partial M)$ by locality (finite propagation speed), so (12) follows easily from (10) by using the explicit expression for the Euclidean wave kernel in terms of an x-space Fourier transform, see Section 2.4. In the case of a nonflat metric, K can be analyzed near $t = 0$ by the geometric optics approximation (see [Hör68],[Hör85]) or the scaling technique introduced by Melrose in [Mel84], and this gives the interior estimate in that case. Another method to obtain (12) uses the Hadamard parametrix for the resolvent, see [Ava56], [Sog88], for example.

Since $|u(x)| \le u_{[\lambda-1,\lambda]}(x)$ for solutions u of (1), (12) gives in particular

$$|u(x)| \le C_\epsilon \lambda^{(n-1)/2} \quad \text{if } x \in M_\epsilon. \tag{12'}$$

The claim of Theorem 1 is that C_ϵ in (12') may be chosen independent of ϵ. When x approaches the boundary in the argument above, one has to either shrink the support of $\hat\rho$ – which means increasing its maximum since $\int \hat\rho = \rho(0) = 1$ (and this gives only $C_\epsilon \le C\epsilon^{-1/2}$ in (12)) – or analyze the wave kernel in the presence of boundary conditions. The difficulty with the latter is that the geometric optics construction becomes much more complicated near the boundary because of diffraction and multiple reflection of geodesics, and a satisfactory parametrix for the wave equation has so far only been constructed near points where the boundary is strictly convex or concave (see [MT], [Mel80]). This parametrix was used in [Gri92] to prove Theorem 1 near concave boundary points (e.g. near the inner circle of an annular domain). On the diagonal $x = y$ the singularities of K have been analyzed in sufficient detail near an arbitrary smooth boundary by Ivrii, Melrose and Hörmander ([Ivr80, Mel84, Hör85]) to allow us to obtain the desired estimates at points x outside a boundary layer of width λ^{-1}. In the boundary layer, a simple maximum principle argument then completes the proof, because there u can have at most one oscillation in the direction perpendicular to the boundary.

Related results. Estimates of $\|u\|_\infty$ for large λ are closely related to asymptotic improvements over the bound (3) on $u_{[0,\lambda]}$ (which is often referred to as the 'spectral function of Δ on the diagonal'). That improvements might be possible is suggested by the observation that $\int_M u_I^2 = \#\{j : \lambda_j \in I\}$, and Weyl's law (4). Carleman [Car35] was the first to prove the interior pointwise asymptotics corresponding to (4) (as $\lambda \to \infty$)

$$u_{[0,\lambda]}(x) = \gamma'\lambda^{n/2} + o_\epsilon(\lambda^{n/2}), \quad \text{for } \operatorname{dist}(x,\partial M) > \epsilon \tag{13}$$

(for domains in $\mathbb{R}^n$; for manifolds see [MP49], [Går53]). Here $\gamma' = (2\pi)^{-n/2}\sqrt{\mathrm{vol}_{\mathrm{eucl}}(B^n)}$. The error term was improved to

$$u_{[0,\lambda]}(x) = \gamma'\lambda^{n/2} + O_\epsilon(\lambda^{(n-1)/2}), \quad \text{for dist}\,(x, \partial M) > \epsilon \tag{14}$$

([Ava52, Lev53] in $\mathbb{R}^n$, [Ava56] for manifolds, [Hör68] for higher order operators on manifolds). The connection to Theorem 1 is established by writing $u^2_{[\lambda-1,\lambda]} = u^2_{[0,\lambda]} - u^2_{[0,\lambda-1]}$. Then (14) gives immediately the interior estimate (12), and also shows the optimality of the power $(n-1)/2$ in (12), for *any* M (as opposed to (12') which is optimal only for some M).

It follows from the very precise and general results in Ivrii's book ([Ivr98]) that the ϵ-dependence in (14) may be removed, and hence that our theorem even holds with u replaced by $u_{[\lambda-1,\lambda]}$. Our main point here is the simplification of the arguments near the boundary. However, not all difficulties can be avoided: We still have to refer to results on the boundary parametrix in the treatment of the layer $\{x : \lambda^{-1} \leq \mathrm{dist}\,(x, \partial M) \leq 1\}$.

If one considers $\|u\|_p$ for $p \in (0, \infty)$ instead of $p = \infty$ then one obtains interesting phenomena related to the 'restriction theorem for the Fourier transform' of Stein-Tomas [Tom79]. The optimal interior estimates were obtained by Sogge [Sog88, Sog93]. The same estimate extends uniformly to concave portions of the boundary, as shown in [Gri92] for $n = 2$ and in [SS95] for all n, but not in general (e.g. for the 'whispering gallery' eigenfunctions on the disk, see [Gri92] and the remark at the end of Section 2.3). The problem of finding optimal L^p-bounds for general boundary geometry is still open.

Corollary 2 also follows from the 'sharp Weyl formula' improving (4)

$$\#\{j : \lambda_j \leq \lambda\} = \gamma_M \lambda^n + O(\lambda^{n-1}).$$

While this follows from Ivrii's results again, there are simpler proofs, see [See80, Pha81].

Contents of the paper. In Section 2 we collect some basic facts about our problem which are well-known to the experts but scattered or not present in the literature. In particular, we prove Corollary 2 from Theorem 1 and give two simple proofs of the interior estimate in the case of flat domains. Also, we discuss the torus and the disk. In Section 3 we prove estimate (11) outside a boundary layer of width λ^{-1}, and in Section 4 we derive from this the estimate on u inside this layer, for Dirichlet boundary condition. Finally, in Section 5 we describe the modifications needed for the Neumann problem.

2. Basic facts about $\|u\|_\infty$

2.1. **Multiplicities.** The following proposition shows that Corollary 2 is a consequence of Theorem 1.

Proposition 3. *If $V \subset L^2(M)$ is a subspace of dimension m, then*

$$\sup_{\substack{u \in V \\ \|u\|_2 = 1}} \|u\|_\infty \geq |M|^{-1/2} m^{1/2}. \tag{15}$$

Proof. Let $v_1, \dots, v_m$ be an orthonormal basis of V. For simplicity (and sufficient for our purpose), we assume that the v_i are continuous, the general case is only slightly harder. Define for $x, y \in M$

$$u_y(x) = \sum_{i=1}^m \overline{v_i(y)} v_i(x), \quad a(y) = u_y(y) = \sum_{i=1}^m |v_i(y)|^2.$$

We have $\int_M a(y) = \sum_{i=1}^m \|v_i\|_2^2 = m$, so

$$a(\tilde{y}) \geq m/|M|$$

for some $\tilde{y}$. Now by orthonormality of the v_i

$$\|u_{\tilde{y}}\|_2^2 = \sum_{i=1}^m |\overline{v_i(\tilde{y})}|^2 = a(\tilde{y})$$

and $\|u_{\tilde{y}}\|_\infty \geq u_{\tilde{y}}(\tilde{y}) = a(\tilde{y})$, so $\|u_{\tilde{y}}\|_\infty^2 / \|u_{\tilde{y}}\|_2^2 \geq a(\tilde{y}) \geq m/|M|$. Therefore, $u = u_{\tilde{y}}/\|u_{\tilde{y}}\|_2$ satisfies the desired bound. □

2.2. **The torus.** Besides proving Corollary 2 from Theorem 1, Proposition 3 has another interesting consequence: Let $M = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ be the 'square' torus. The 'standard' normalized eigenfunctions are

$$u_a(x) = (2\pi)^{-n/2} \exp(ia \cdot x), \quad a \in \mathbb{Z}^n, \tag{16}$$

with eigenvalue $\lambda^2 = |a|^2 = a_1^2 + \ldots + a_n^2$. These are uniformly bounded, which is much better than the bound (5). However, since the multiplicity of λ^2 is not uniformly bounded as $\lambda \to \infty$, one may construct another sequence of eigenfunctions (as in the proof of Proposition 3) with non-uniformly bounded maxima. In fact, the multiplicity of λ^2 equals the number of representations of λ^2 as the sum of n squares of integers. From standard results on these numbers (see [Gro85], for example) one obtains:

> If M is the square n-torus, then for any N there is a sequence of L^2-normalized eigenfunctions u with eigenvalues λ^2 tending to infinity and satisfying
>
> $$\|u\|_\infty \geq c_N \lambda^{(n-2)/2} (\log \lambda)^N$$
>
> for some $c_N > 0$.

As a simple example we take $\lambda^2 = 5^l$, $l = 1, 2, 3, \ldots$. This has $4(l+1)$ representations as sum of two squares, for example 5 arises from $(\pm 1, \pm 2)$ and $(\pm 2, \pm 1)$. This gives a sequence as desired for $n = 2$ and $N = 1$.

Note also that in the case of the torus one has equality in (15), as follows immediately from (16). The number-theoretic results referred to above therefore also yield the upper bound

$$\|u\|_\infty \leq C_\epsilon \lambda^{(n-2)/2+\epsilon}$$

for any $\epsilon > 0$. For more on the case of the torus see Bourgain's article [Bou93]. Of course, the same results hold for a square (or cube) in $\mathbb{R}^n$.

See [Jak97] for more about eigenfunctions on tori and [TZ00] for interesting recent work on manifolds with uniformly bounded eigenfunctions.

2.3. **The ball.** For completeness and to show that the bound in Theorem 1 cannot be improved in general, we shortly discuss the case $M = \{x \in \mathbb{R}^n : |x| \leq 1\}$. In polar coordinates, the Euclidean Laplacian is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S \tag{17}$$

where Δ_S is the Laplacian on the unit sphere $S = \{|x| = 1\}$. By separation of variables one obtains that there is a basis of eigenfunctions of the form

$$u(r\omega) = r^{-(n-2)/2} J_m(\lambda r)\Phi(\omega), \quad \omega \in S, \tag{18}$$

where Φ is an eigenfunction of $-\Delta_S$ (with eigenvalue μ^2), J_m is the Bessel function (see [Wat48]) of order $m = \sqrt{\mu^2 + (n-2)^2/4}$ (which is always an integer), and λ is a positive zero of J_m (for Dirichlet boundary conditions).

Consider a radial eigenfunction, i.e. $\mu = 0$ and $\Phi \equiv 1$. This is not normalized as in (2), so instead of $\|u\|_\infty$ we need to estimate $\|u\|_\infty/\|u\|_2$. We have[1] $m = (n-2)/2$, $\|u\|_\infty = u(0) = C_n\lambda^{(n-2)/2}$, and the asymptotics

$$J_m(s) \sim cs^{-1/2}\cos(s - m\pi/2 - \pi/4) + O(s^{-5/2}), \quad s \to \infty$$

easily imply $\|u\|_2^2 \approx \int_0^1 (r^{-(n-2)/2}(\lambda r)^{-1/2})^2 r^{n-1}\, dr = \lambda^{-1}$, so $\|u\|_\infty/\|u\|_2 \approx \lambda^{(n-1)/2}$, and this shows that the bound (5) is saturated by u.

Similar (but more involved) Bessel function estimates may be used to prove Theorem 1 directly for the ball, and even the stronger estimate for $u_{[\lambda-1,\lambda]}$. See [Gri92].

Remark: The radial functions are one extreme case of (18). The other extreme case is obtained by taking $\lambda = \lambda_{m1}$, the first positive zero of J_m. For $m = 1, 2, 3, \ldots$ this yields the sequence of 'whispering gallery eigenfunctions' (say $n = 2$ for simplicity). They concentrate on a strip of width $\approx \lambda^{-2/3}$ at the boundary, as follows from (and is made precise by) the estimates

$$\begin{aligned} \lambda_{m1} &= m + am^{1/3} + O(m^{-2/3}), \quad \text{as } m \to \infty, \text{ with } a > 0 \\ J_m(m + tm^{1/3}) &\geq m^{-1/3} \quad \text{for } t \in [-a/2, a/2] \\ J_m(m + tm^{1/3}) &\leq Cm^{-1/3}e^{-c|t|^{3/2}} \quad \text{for } -m^{2/3} \leq t \leq 2a \end{aligned}$$

for positive constants c, C. (These are easy consequences of well-known asymptotic formulas for J_m, see [Olv54] for example, or [Wat48], Sec.8.4 for weaker but sufficient bounds.) Sogge showed that away from the boundary concentration can happen only on sets of area $\geq \lambda^{-1/2}$. This follows from his estimate $\|u\|_6 \leq C\lambda^{1/6}\|u\|_2$. In contrast, the whispering gallery eigenfunctions have $\|u\|_6 \approx \lambda^{1/3}\|u\|_2$. Note that the L^∞ estimate (11) only implies area $\geq \lambda^{-1}$ for concentration. This shows two things:

[1] That $r^{-m}J_m(r)$ attains its maximum at $r = 0$ follows immediately from Poisson's integral

$$J_m(r) = C_m r^m \int_{-1}^{1} e^{irt}(1-t^2)^{m-\frac{1}{2}}\, dt,$$

see [Wat48], Section 3.3. See also [Wat48], Section 15.31, for a different proof.

(1) Optimal bounds on concentration phenomena are obtained from certain L^p, $p < \infty$, rather than L^∞ bounds on eigenfunctions.
(2) As opposed to L^∞ bounds, these L^p-bounds are sensitive to the presence (and geometry) of a boundary.

In general, such optimal bounds are still unknown.

2.4. **General domains in $\mathbb{R}^n$.** Here we prove the interior estimate (12) for Euclidean domains $M \subset \mathbb{R}^n$. We give two proofs: First, we finish the wave equation proof outlined in the Introduction, and second, we give a more direct proof using averaging and Bessel functions.

First proof: As argued in the introduction, it is sufficient to prove (11), with K replaced by the wave kernel in $\mathbb{R}^n$. One way to represent $K_{\mathbb{R}^n}$ is as an oscillatory integral, obtained from using the x-space Fourier transform and solving the ordinary differential equation that results from (9),

$$K_{\mathbb{R}^n}(t,x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \cos(t|\xi|)\, e^{i(x-y)\xi}\, d\xi. \tag{19}$$

From this it is not hard to get (11) directly. Let us describe a more conceptual approach which yields the power $n-1$ by pure homogeneity arguments (this is the scaling technique in [Mel84] for this special case): From the homogeneity of the equation satisfied by $K_{\mathbb{R}^n}$ (or from (19) directly) one has the homogeneity

$$K_{\mathbb{R}^n}(t,x,y) = \epsilon^{-n} K_{\mathbb{R}^n}(t/\epsilon, x/\epsilon, y/\epsilon)$$

for $\epsilon > 0$. Since Δ has constant coefficients, one has furthermore translation invariance $K_{\mathbb{R}^n}(t,x,y) = K_{\mathbb{R}^n}(t, x-y, 0)$. Therefore, $K_{\mathbb{R}^n}(t,x,x) = K_{\mathbb{R}^n}(t,0,0)$ is a distribution in t, homogeneous of degree $-n$. Then its singular support must be contained in $\{0\}$, and its inverse Fourier transform is also smooth outside zero and homogeneous of degree $n-1$ (see [Hör83], Theorem 7.1.18), so it satisfies the desired bound. Then it is straightforward to see that $\rho * K_{\mathbb{R}^n}\check{}$ satisfies the same bound, i.e. (11).

Second proof: Let $x_0 \in M$, and assume that the ball B of radius R around x_0 is contained in M. We show that

$$|u(x_0)| \leq C\lambda^{(n-1)/2} R^{-1/2} \|u\|_{L^2(B)} \tag{20}$$

which clearly implies (12'). To simplify notation, assume $x_0 = 0$. In this proof, $\| \|_p$ denotes the L^p norm on B, not M.

Define the spherical average

$$h(r) = \frac{1}{|S|} \int_S u(r\omega)\, d\omega,$$

considered as function on B. Since h is the average over the functions g^*u over all rotations $g \in SO(n)$ and since these rotations induce isometries on $L^2(B)$, Minkowski's inequality gives

$$\|h\|_2 \leq \|u\|_2.$$

Furthermore, h solves $(\Delta + \lambda^2)h = 0$ (since Δ is rotation invariant) and is radial, so it is of the form (18) with $\Phi = const$. Therefore, the same calculation as in Section 2.3 shows that

$$\frac{\|h\|_\infty}{\|h\|_2} \approx R^{-1/2}\lambda^{(n-1)/2}.$$

Finally, $\|h\|_\infty = |h(0)| = |u(0)|$, so (20) follows. We remark that this proof may be adapted to prove (12) instead of (12'), see [Gri92].

3. Estimates outside the boundary layer

In this section we prove Theorem 1 for points x with

$$\operatorname{dist}(x, \partial M) \geq \lambda^{-1}.$$

As explained in the Introduction, we only have to prove (11) for these x. Propagation of singularities (see [Hör85]) implies that the singular support of the distribution $t \to K(t, x, x)$ is contained in the set of lengths of geodesics, possibly reflected at the boundary, which start and end at x. Clearly, for small ϵ the only singularities in $|t| < \epsilon$ are therefore at $t = 0$ and possibly at $t = \pm 2\operatorname{dist}(x, \partial M)$. Therefore, K may be expected to be and indeed is representable, for small $|t|$, as the sum of two distributions, a 'direct' term which is only singular at $t = 0$, and a 'reflected' term. To describe the direct term, choose a closed manifold $\tilde{M}$ extending M, and let

$$K^{\mathrm{dir}}(t, x, y)$$

be the solution of the problem (9) on $\tilde{M}$.

To describe the reflected term, it is convenient to introduce geodesic coordinates with respect to the boundary; that is, we identify points x of M close to the boundary with pairs $(x', x_n) \in \partial M \times [0, c)$, $c > 0$, via the map

$$\partial M \times [0, c) \to M \tag{21}$$

sending (x', x_n) to the endpoint of the geodesic of length x_n which starts at $x' \in \partial M$ perpendicular to ∂M. This map is a diffeomorphism onto its image for sufficiently small c. We have

$$x_n = \operatorname{dist}(x, \partial M).$$

Also, let χ_+^α, $\alpha \in \mathbb{C}$, be the distribution on $\mathbb{R}$ obtained by analytic continuation in α from $\{\mathrm{Re}\ \alpha > -1\}$, where it is defined by

$$\chi_+^\alpha(s) = \begin{cases} s^\alpha/\Gamma(\alpha+1) & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$

and satisfies $(\chi_+^{\alpha+1})' = \chi_+^\alpha$, see [Hör83], Section 3.2. Alternatively, it is the inverse Fourier transform of

$$\hat{\chi}_+^\alpha(\sigma) = e^{-i(\alpha+1)\pi/2}\frac{1}{(\sigma - i0)^{\alpha+1}}. \tag{22}$$

Theorem 17.5.9 in [Hör85] then gives the following description of the singularity of K at $t = \pm 2\operatorname{dist}(x, \partial M)$:

Proposition 4. *For sufficiently small ϵ, there is a distribution*

$$I(x', \theta, t) \in \mathcal{D}'(\partial M \times \mathbb{R} \times (-\epsilon, \epsilon))$$

so that for $|t| < \epsilon$ and $x_n > 0$ we have

$$K(t, x, x) = K^{dir}(t, x, x) - t^{-n} I(x', \frac{2x_n}{t}, t).$$

Furthermore, I has support in $|\theta| \le 1$ and singular support in $|\theta| = 1$, and near $\theta = 1$ we have

$$I(x', \theta, t) = \sum_{j=0}^{N-1} a_j(x', \theta, t) \chi_+^{j-(n+1)/2}(1 - \theta) + R_N(x', \theta, t)$$

with smooth functions a_j and a continuous remainder R_N, for any $N > (n-1)/2$. A similar expansion exists near $\theta = -1$.

Note: In the statement of Theorem 17.5.9 in [Hör85] the direct term (called $t^{-n}I_1$ there) is not identified like in the statement above. But in the proof (middle of page 59, loc. cit.) it is chosen as K^{dir} like above.

The proposition is obtained by analyzing the regions $t < 3x_n$ and $t \ge 3x_n$ separately: In $t < 3x_n$ one uses a suitable scaling (cf. Section 2.4) and the Hadamard parametrix, and in $t \ge 3x_n$ (where no singularities should occur) one uses propagation of singularities estimates (this is the harder part).

Since the estimates for K^{dir} are already known by the interior estimate, (11) will follow from:

Lemma 5. *For $x_n \ge \lambda^{-1}$, we have*

$$\left| \int_0^\infty e^{it\lambda} \hat{\rho}(t) t^{-n} I(x', \frac{2x_n}{t}, t)\, dt \right| \le C\lambda^{n-1}.$$

For convenience, we restrict to positive t. Negative t are handled in the same way.

Proof. The integrand is zero for $t < 2x_n$ and for $t > \epsilon$. We split the integral up into a part where $t > 3x_n$ and a part where $t < 4x_n$, using a cutoff function smooth in t/x_n. By Proposition 4, I is smooth on the first part and therefore bounded, so this part is dominated by a constant times

$$\int_{x_n}^\infty t^{-n}\, dt \le x_n^{-n+1} \le \lambda^{n-1}.$$

In the second part, where $t < 4x_n$, we split up I as in the proposition, with a fixed $N > (n-1)/2$. Then the term R_N can be handled in the same way as the first part.

It remains to analyze the singular terms. If we denote the cutoff function by $\psi(t/x_n)$, $\psi \in C_0^\infty(1, 4)$, they take the form (assuming for simplicity that $\hat{\rho}$ was chosen to be constant near zero)

$$\int e^{it\lambda} a(x', \frac{2x_n}{t}, t) \chi_+^\alpha (1 - \frac{2x_n}{t}) \psi(\frac{t}{x_n}) t^{-n}\, dt,$$

$\alpha = j - (n+1)/2$. If we change variables $\tau = t/2x_n$ and use the homogeneity of χ_+^α, this becomes

$$x_n^{-n+1} \int e^{2i\tau\lambda x_n} b(x', \tau, t) \chi_+^\alpha(\tau - 1)\, d\tau$$

with b smooth and supported in $\tau \in (1/2, 2)$. Using (22) one sees by a short standard calculation that this is bounded by a constant times

$$x_n^{-n+1}(\lambda x_n)^{(n-1)/2-j} = \lambda^{n-1}(\lambda x_n)^{-(n-1)/2-j} \leq \lambda^{n-1}.$$

□

4. Estimates in the boundary layer

So far, we have proved $|u(x)| \leq C\lambda^{(n-1)/2}$ for $\operatorname{dist}(x, \partial M) \geq \lambda^{-1}$. The proof of Theorem 1 in the Dirichlet case will therefore be completed by the following lemma:

Lemma 6. *If u is a solution of*

$$(\Delta + \lambda^2)u = 0$$

vanishing at the boundary of M, then

$$\max_{x: dist(x,\partial M) < \lambda^{-1}} |u(x)| \leq \max_{x: dist(x,\partial M) = \lambda^{-1}} |u(x)|. \tag{23}$$

Proof. Without loss of generality, assume u is real valued. Under the identification near the boundary given by (21), the metric takes the form

$$g(x) = dx_n^2 + g'(x', x_n)$$

where $g'(\cdot, x_n)$ is a Riemannian metric on ∂M for each x_n. Therefore, the Laplacian has the form

$$\Delta = \frac{\partial^2}{\partial x_n^2} + a(x)\frac{\partial}{\partial x_n} + P(x', x_n, D_{x'}) \tag{24}$$

with a Laplacian P on the boundary (depending on the parameter x_n) and a smooth function a. The generalized maximum principle says that if v is a positive function on the strip

$$S = \{0 \leq x_n \leq \lambda^{-1}\} \subset M$$

with

$$(\Delta + \lambda^2)v \leq 0$$

then

$$\max_S \frac{|u|}{v} \leq \max_{\partial S} \frac{|u|}{v}.$$

(Apply [PW84], Theorem 10, to u and then to $-u$.) We apply this with the function

$$v(x', x_n) = \sin(\frac{\pi}{2} + \frac{3}{2}\lambda x_n).$$

We have $v > 0.07$ on S and, from (24),

$$(\Delta + \lambda^2)v = -\frac{5}{4}\lambda^2 v + O(\lambda) < 0 \text{ on } S \text{ for large } \lambda,$$

so the generalized maximum principle applies, and the lemma follows. □

5. The Neumann problem

Theorem 1 and Corollary 2 remain true if the Dirichlet boundary condition $u_{|\partial M} = 0$ is replaced by the Neumann boundary condition

$$\partial_n u_{|\partial M} = 0$$

in (1), where ∂_n denotes the outward normal derivative.

Let us sketch the proof: The proof of Proposition 3 (and thus the proof of Corollary 2 from Theorem 1) as well as the reduction of Theorem 1 to the wave kernel estimate (11) carry over literally, except that K is replaced by the Neumann wave kernel (i.e. $\partial_n K_{|x\in\partial M} = 0$ instead of $K_{|\partial M} = 0$ in (9), where ∂_n refers to the x-coordinates). (Also, Section 2.4 applies literally to the Neumann problem.)

Proposition 4 holds for the Neumann wave kernel as well (with different a_j), by straightforward modification of the arguments in [Hör85].

Finally, Lemma 6 remains true for Neumann eigenfunctions. To see this, just note that, in the comparison of the functions u and v, the maximum of u/v over S not only must occur at a boundary point x_0 of S, but also the outward normal derivative at x_0 must be strictly positive. See Theorem 10 of [PW84] (this is sometimes called Zaremba's principle). But v was chosen such that

$$\partial_n v = -\partial v/\partial x_n = 0 \quad \text{for } x_n = 0$$

and so $\partial_n(u/v) = 0$ for $x_n = 0$. This means that the maximum of u/v over S must occur on the 'inner' part of ∂S, i.e. for $x_n = \lambda^{-1}$. Applying the same argument to $-u$ finally yields (23).

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HUMBOLDT-UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMATIK, SITZ: RUDOWER CHAUSSEE 25, 10099 BERLIN, GERMANY

E-mail address: grieser@mathematik.hu-berlin.de