Contents

List of included papers

[GJ1] D.Grieser, D.Jerison, *Asymptotics of the first nodal line of a convex domain*, Inventiones Math. **125**(1996), 197-219.

[GJ2] D.Grieser, D.Jerison, *The size of the first eigenfunction of a convex planar domain*, Journal of the AMS **11**(1998), 41-72.

[G1] D.Grieser, *Local geometry of singular real analytic surfaces*, Preprint, 1999.

[G2] D.Grieser, *Variations on quasiisometry*, Preprint, 1999.

[G3] D.Grieser, *Basics of the b-calculus*, to appear in J. Gil et al. [eds.], *Approaches to Singular Analysis,* Advances in Partial Differential Equations, Birkhäuser, Basel, 2000.

[GG] D.Grieser, M.Gruber, *Singular asymptotics lemma and Push-forward theorem*, to appear in J. Gil et al. [eds.], *Approaches to Singular Analysis,* Advances in Partial Differential Equations, Birkhäuser, Basel, 2000.

[CGIKO] S.Chanillo, D.Grieser, M.Imai, K.Kurata, I.Ohnishi, *Symmetry breaking and other phenomena in the optimization of eigenvalues for composite membranes*, to appear in Comm. Math. Physics.

[CGK] S.Chanillo, D.Grieser, K.Kurata, *The free boundary problem in the optimization of composite membranes*, to appear in Contemp. Math., AMS.

Introduction

The common theme of the papers which form the core of this Habilitationsschrift is the singular analysis. Here, 'analysis' means the study of solutions of partial differential equations, especially of the spectral problem for elliptic operators; 'singular' means the emphasis on phenomena which stem from non-smoothness or non-uniform ellipticity of the coefficient matrix. In most cases the operators have a geometric origin, i.e. they are Laplace or Dirac operators; then their singular character comes from the singular nature of the underlying space.

I now give an overview over the mathematical context surrounding this Habilitation. On any Riemannian manifold (M, g) several elliptic partial differential operators are defined naturally: The Laplace (or Laplace-Beltrami) operator Δ, acting on functions or, more generally, on differential forms on M; and the Dirac operators associated to various bundles.

One of the great mathematical problems of the 20th century is to understand the relations between the analytic properties of these operators and the geometric and topological properties of M. Assume for the moment that M is compact. Then some of the interesting analytic properties are:

(a) The eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$ and eigenfunctions $u_1, u_2, \ldots$ of Δ; their investigation splits into two quite different directions:

 (a1) 'small' eigenvalues (for example λ_1, u_1),

 (a2) 'large' eigenvalues (asymptotic behavior of λ_i, u_i for $i \to \infty$).

 Some important properties of the eigenfunctions are: The size and location of the nodal set $u_i^{-1}(0)$ and the critical set $|du_i|^{-1}(0)$, the size of the maximum of $|u_i|$ relative to its L^2 norm, and concentration phenomena on small sets.

(b) The Fredholm index of the Dirac operators.

Other important analytic properties, which we don't consider here, are: The determinant of Δ, the η invariant and the analytic torsion; also the scattering theory of Δ which replaces (a) in the case of non-compact M.

Part of the motivation to study these properties comes from physics: The eigenvalues of Δ are the resonant frequencies of a vibrating body of 'shape' M, and the eigenfunctions describe the form of the vibration; at the same time the eigenvalues are the possible sharp values of the energy of a quantum mechanical particle which moves freely on M, and the eigenfunctions are the corresponding states. The Fredholm index is important in string theory.

A prerequisite for dealing with the problems above is an understanding of the basic properties of elliptic operators. Among these are the following.

(c) Existence and uniqueness, more generally Fredholm theory; regularity; selfadjointness of symmetric operators; discreteness of the spectrum.

For compact manifolds with sufficiently smooth metric these things have been understood for a long time, see [25], for example; this is also true for compact manifolds with piecewise smooth boundary; among the latter an important special case are domains in $\mathbb{R}^n$ with piecewise smooth boundary, endowed with the Euclidean metric.

There is an extensive literature dealing with the problems above. The investigation of the eigenvalues of Δ was often motivated by the attempt to solve the *inverse spectral problem* (to reconstruct M from $\{\lambda_1, \lambda_2, \dots\}$, [52]) . I now list a few representative results; in parantheses I put the relevant geometric or topological properties of M whenever this makes sense:

(a1) Inequalities of Faber-Krahn [31, 54] (volume), Hayman [44, 75] (inradius) and Cheeger [19] ('girth') for λ_1, proof of the nodal line conjecture for u_2 on convex plane domains [61] and counterexample in the multiply connected case [45, 33],

(a2) Weyl asymptotics for the eigenvalues [84, 57, 3, 47], trace formula and from this sharpening of the Weyl asymptotics (dynamics of the geodesic flow) [18, 30], L^p bounds for the eigenfunctions [83, 38, 82, 39], improvement of the general bound on eigenfunctions in the arithmetic case [49], bounds for the area of the nodal set [6, 28, 29] and the dimension of the critical set [43];

on the inverse spectral problem: construction of the heat invariants (curvature integrals over M) [68, 36] and the wave invariants (curvature integrals along geodesics) [24, 42], compactness of isospectral sets [7, 64, 74], spectral determination of generic convex axisymmetric plane domains with analytic boundary [86], counterexamples to the spectral determination of plane domains [37],

(b) the Atiyah-Singer index theorem (characteristic classes) [2] and (in the case with boundary) the Atiyah-Patodi-Singer index theorem [1], and their 'heat equation proofs' [34, 67].

There are still many open problems, especially in the areas (a1) and (a2), for example the spectral determination of convex domains, the nodal line conjecture for simply connected domains, and the qualitative properties of eigenfunctions of high energy in the case of chaotic geodesic flow ('quantum chaos').

So far I have only talked about the case of compact smooth manifolds, possibly with boundary. Natural generalizations may be obtained in various ways: By weakening one of the requirements of compactness or smoothness of the space, or by considering parameter dependent families of smooth compact (or more general) spaces which 'degenerate' when one approaches certain parameter values.

Important classes of *non-smooth spaces* are the following:

(A) Algebraic varieties (or more generally subanalytic sets)

(B) Orbit spaces of proper group actions on smooth manifolds.

The question arises how the notions of Riemannian metric and differential operator should be generalized to such spaces. In this work we take the position that these are only defined on the smooth part of the space. This is an open dense subset and a manifold itself, which is not compact in general. The idea is that the singular set (i.e. the complement of the smooth part) should become 'visible' for the analysis through the form of the metric near it. (A different point of view is taken in [79].) Natural classes of Riemannian metrics are obtained in case (A) by embedding the variety in a manifold N and pulling back a smooth metric on N, and in case (B) by taking the quotient of an invariant metric on the total space. These metrics are not complete if the space under consideration is actually singular. Some authors take the position that one should only consider complete metrics instead since then certain analytic difficulties disappear (see for example [67]; note that any metric is conformal to a complete metric, therefore these problems are closely related).

Important examples of *degenerating families* of smooth compact manifolds are the following:

(C) ('Adiabatic limit') A fibration $\pi : M \to N$ with smooth base and fiber with metrics of the form $g_\varepsilon = \pi^* h + \varepsilon g_0$, $\varepsilon > 0$, where h, g_0 are metrics on N and M, respectively; for $\varepsilon \to 0$ the fibers are shrunk to points.

(D) The set of bounded convex domains in $\mathbb{R}^2$. (Here the parameter space is infinite dimensional; the most important degeneration is: eccentricity := diameter/inradius $\to \infty$.)

A typical feature of the geometric differential operators in all of these examples of non-smooth and degenerating spaces is that they are not uniformly elliptic or that certain coefficients tend to infinity when one approaches the singular set of the space (or the parameter space). Therefore, I put their investigation under the common header 'singular problems'.

An important common feature of examples (A)-(C) is their strong *structure*: They are closely related to the smooth compact case in that they may be related to it by certain locally finite processes (e.g. resolution of singularities). In particular these spaces are stratified. However, stratifyability is a much weaker condition. It is beyond the scope of this introduction to make this vague notion of strong structure precise. One way to do this was proposed by R. Melrose with his notion of 'boundary fibration structures' [65, 63] (see also [G3]). A typical characteristic of such problems is the existence of complete asymptotic expansions of the given and desired quantities (for example in terms of the distance to the singular set). Accordingly it is natural to adapt the theory of pseudodifferential operators – which is also highly structured – for such problems.

In contrast, example (D) is only weakly structured, as well as certain other classes of singular spaces, e.g. those characterized by weak regularity (for example Lipschitz manifolds). In such problems one may only expect estimates in orders of magnitude instead of complete asymptotics; this is also reflected in the techniques that may be used (e.g. maximum principle, convexity).

Singular problems were increasingly investigated since the 1970s; their theory is much less developed than the theory in the smooth compact case. Even basic analytic questions, for example those of selfadjointness and discreteness, are still open for many singular spaces (see [40, 11]). Cheeger, Goresky, and MacPherson conjectured [23] that for complex projective varieties the L^2 cohomology (defined analytically, with respect to an induced Kähler metric) coincides with the intersection homology, which is defined topologically. There are many partial results on this conjecture, see [48, 71, 73, 78, 85]; [81] proves the same claim for orbit spaces. The conjecture is still open in general. Closely related is the problem of generalizing the Hodge theorem to the singular case [20, 21, 72, 12, 76, 40]; this is also open for general varieties. Let me mention a little gem in this context: The analysis of the Hodge theory of a fibration under the adiabatic limit degeneration ((C) above) leads to the Leray spectral sequence of the fibration [60].

The simplest singularities are cones and horns; they have locally the form $[0,1) \times N/\{0\} \times N$, with metric $dx^2 + x^{2\alpha} g_N$ on the smooth part $(0,1) \times N$, where (N, g_N) is a closed Riemannian manifold and $\alpha \geq 1$. For these cases many of the properties mentioned above have been investigated extensively (see [20, 21, 14, 16, 12, 55, 35, 69] and [22, 15, 32, 8, 56, 17] for index theorems). Some of these papers also admit spaces which are locally iterated cones or products $\mathbb{R}^k \times$ cone.

As I mentioned above, it is natural to try to develop pseudodifferential calculi for strongly structured problems. This is done systematically by the Melrose [63] (see also [G3]) and Schulze [80] and their schools. For example, there are pseudodifferential calculi for cones and for the adiabatic limit.

Few authors have considered the eigenvalues and eigenfunctions of the Laplacian on singular spaces which are not locally cone- or horn-like. See [41] for semialgebraic sets and [9, 10] for orbit spaces, and for example [27, 4] for certain weakly structured situations. Also, there are almost no generalizations of the index theorem for more complex singularities (see [53] and [58] for certain classes of orbit spaces). One reason for this is that even the local geometry of induced metrics is poorly understood (but see the important papers [70, 77])

The behavior of eigenvalues and eigenfunctions has been investigated for numerous forms of degeneration, since such problems often appear in applications (e.g. vibrations of thin sheet metal). See for example [59]. A method which is used often in this context is that of 'matched asymptotic expansions'. This is closely related to the construction of adapted pseudodifferential calculi (see [G3]), but usually simpler (and with weaker results, which, however, often suffice for the application at hand).

Now I explain how the papers contained in this Habilitationsschrift fit into the context sketched above. More detailed accounts of each paper are given in the next chapter and in the introduction to the paper. In the papers [GJ1], [GJ2] we investigate low eigenvalues and their eigenfunctions, for the Dirichlet Laplacian on plane convex domains; the goal is to get quantitative results which are optimal in order of magnitude, uniformly for domains of arbitrarily large eccentricity. This is a weakly structured problem of degeneration.

In the papers [G1], [G2] I investigate the local geometry of real subanalytic surfaces with isolated singularities and draw conclusions about the basic analytic properties of the geometric operators. These papers are mainly geometric.

In [G3], [GG] we describe and discuss general techniques of singular analysis which are used for strongly structured problems.

The papers [CGIKO], [CGK] deal with an optimization problem for eigenvalues: The first eigenvalue is to be minimized over a family of configurations, and the problem is to understand qualitative properties of the solutions. At first glance, the problem does not seem to be singular in the sense explained above. However, it contains a free boundary problem whose solutions are generally non-smooth; also, in the proofs of the central results on symmetry breaking we analyze certain degenerating families of domains (thin annuli and dumbbells with thin handle).

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Summary of the papers

1 The papers [GJ1],[GJ2]

1.1 Problem and previous results

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Let $0 < \lambda_1 < \lambda_2$ be the two lowest eigenvalues of the Laplacian on Ω, with Dirichlet boundary conditions, and let u_1, u_2 be associated eigenfunctions. Our goal is to describe $\lambda_{1/2}$ and $u_{1/2}$ as precisely as possible by geometric data of Ω. The estimates should be optimal in order of magnitude, uniformly for domains of arbitrarily large eccentricity. (The eccentricity is defined as the ratio of diameter of Ω and diameter of the largest inscribed circle.)

Before we make this more precise we normalize Ω. The problem is invariant under dilations and Euclidean motions. Therefore, we may assume:

Normalization:

1. The projection of Ω on the y-axis is the interval $(0, 1)$, and the projection of Ω on any other line has length ≥ 1.
2. The projection of Ω on the x-axis is the interval $(0, N)$, for some $N = N(\Omega) \geq 1$.

See Figure 1 in [GJ2] (with $a = 0, b = N$). It is easy to see that

$$\text{inradius}(\Omega) = \frac{1}{2} + O(N^{-2}) \tag{1}$$

$$\text{diam}(\Omega) = N + O(N^{-1}). \tag{2}$$

Therefore, the eccentricity roughly equals N. We normalize the eigenfunctions as follows:

$$u_1 > 0, \quad \max_{\Omega} u_1 = 1,$$

$$\max_{\Omega} |u_2| = 1.$$

The following facts are well-known:

1. u_1, u_2 are real analytic functions in the interior of Ω.

2. The level sets $\{u_1 \geq c\}$ are convex for any c. In particular (using 1.), u_1 attains its maximum at precisely one point:
$$u_1^{-1}(1) =: \{(x_{\max}, y_{\max})\}.$$
3. The *nodal line*
$$\mathcal{N} := u_2^{-1}(0)$$
is real analytic, and it divides Ω in two connected components. (This is true for any second eigenfunction, in case the multiplicity of λ_2 is greater than one. In the case of interest here, where N is large, the multiplicity equals one, see Theorem 1.3 below.)

Let $\mathcal{C}$ be the set of domains Ω which are normalized as above. For $\Omega \in \mathcal{C}$ and $x \in (0, N)$ set
$$J_x = \{y : \ (x, y) \in \Omega\}$$
and define functions $f_1, f_2, h : (0, N) \to (0, 1]$ by
$$J_x = (f_1(x), f_2(x)), \quad h = f_2 - f_1.$$
$h(x)$ is the 'thickness' of Ω at x. h is a concave function. Furthermore, Ω defines an ordinary differential operator
$$\mathcal{L} := -\frac{d^2}{dx^2} + \frac{\pi^2}{h(x)^2}$$
on $(0, N)$, with Dirichlet boundary conditions. Let
$$\mu_1, \mu_2$$
be the lowest eigenvalues of $\mathcal{L}$ and ϕ_1, ϕ_2 associated eigenfunctions, normalized by
$$\phi_1 > 0, \quad \max \phi_1 = 1,$$
$$\max |\phi_2| = 1,$$
and define $\xi_{\max}, \xi_{\mathrm{nod}} \in (0, N)$ by
$$\phi_1^{-1}(1) = \{\xi_{\max}\},$$
$$\phi_2^{-1}(0) = \{\xi_{\mathrm{nod}}\}.$$
Convexity and positivity of the potential π^2/h^2 imply that both of these sets have only one element.

Definition 1.1 *Let $l \in \{0,1,2\}$. A* quantity of order l *is a function $F : \mathcal{C} \to \mathbb{R}$ which can be evaluated in an elementary way from the solution of a differential equation in l variables; the data of the equation (coefficients, domain of definition) that determines $F(\Omega)$ must be directly determined by Ω.*

A quantity of order zero is also called a geometric quantity. Since this definition is used only to formulate the problem, we do not give precise definitions of the expressions 'in an elementary way' and 'directly'. Instead, we just give the relevant examples:

Examples

- Quantities of *second order* are: $\lambda_{1/2}$, $x_{\max}, y_{\max}$, $\|u_{1/2}\|_2$ (the L^2 norm), the length of the nodal line of u_2.
- Quantities of *first order* are: $\mu_{1/2}$, $\xi_{\max}$, ξ_{nod}.
- Quantities of *order zero* are: $\operatorname{vol}(\Omega)$, N.

(Above we assumed that λ_2 is simple.) Now we can state our problem more precisely and more generally:

Main Problem: Determine how well the second order quantities mentioned above may be approximated by quantites of order one or zero.

As an illustration we consider λ_1: Since Ω is contained in an $N \times 1$ rectangle we have

$$\lambda_1(\Omega) \geq \lambda_1([0,N] \times [0,1]) = \pi^2(1 + N^{-2}). \tag{3}$$

This raises the question whether there may be an estimate of the form $\lambda_1 \leq \pi^2(1 + CN^{-2})$, for some constant C. However, this is not the case: For a circular sector

$$\Omega = K_N := \{(r\cos\theta, r\sin\theta) : \ r \in (0,N), \ \sin\theta \in (0, N^{-1})\}$$

one can express the eigenfunctions in terms of Bessel functions, and their asymptotic behavior shows that

$$\lambda_1(K_N) = \pi^2(1 + c_0 N^{-2/3} + O(N^{-4/3}))$$

for $N \to \infty$ (see [51]).

In [51] D. Jerison defined a geometric quantity that may be used to give a better approximation for λ_1 (and λ_2) than (3):

For $\Omega \in \mathcal{C}$ let L be the length of the longest interval $I \subset (0, N)$ satisfying

$$h \geq 1 - \frac{1}{L^2} \quad \text{on } I. \tag{4}$$

It is easy to see (see [50]) that

$$\frac{1}{2}N^{1/3} \leq L \leq N;$$

in particular, large L corresponds to large eccentricity. For the circular sector K_N we have $L \sim N^{1/3}$ and for the $N \times 1$ rectangle $L = N$.

The solution for the Main Problem in the case of $\lambda_{1/2}$ was given by D. Jerison:

Theorem 1.2 ([50],[51]) *There are positive constants c, C such that for all $\Omega \in \mathcal{C}$ and $i = 1, 2$ we have*

$$\begin{aligned} \pi^2 + cL^{-2} \leq \lambda_i &\leq \pi^2 + CL^{-2} \\ |\lambda_i - \mu_i| &\leq CL^{-3}. \end{aligned} \tag{5}$$

In other words, λ_i may be approximated by quantities of order zero with an error L^{-2} and by quantities of order one with an error L^{-3}.

The second main result of [51], together with results from [50], concerns the nodal line:

Theorem 1.3 ([51]) *There is a constant C such that for all $\Omega \in \mathcal{C}$ with $L > C$ the second eigenvalue is simple and the nodal line $\mathcal{N}$ has the following properties:*

(a) $\overline{\mathcal{N}} \cap \partial\Omega \neq \emptyset$.

(b) Let $\mathcal{N}_x$ be the projection of $\mathcal{N}$ on the x axis. Then we have

$$\mathcal{N}_x \subset [\alpha - CL, \beta + CL]$$

where $I = [\alpha, \beta]$ is the interval in (4).

(c) $\mathcal{N}$ may be localized more precisely by

$$\mathcal{N}_x \subset [\xi_{\text{nod}} - C, \xi_{\text{nod}} + C].$$

(a) solves the 'nodal line problem' for large eccentricity. This problem is the conjecture that the nodal line of any second eigenfunction touches the boundary, for any convex (or even any simply connected) domain. This was proved in the general convex case by Melas [62], shortly after publication of [51]. M. and T. Hoffmann-Ostenhof and N. Nadirashvili [46] gave an example showing that the corresponding statement for multiply connected domains is false in general.

(b) and (c) show how well the position of $\mathcal{N}$ may be approximated by quantities of order zero and one, respectively. The error in (c) is optimal in order of magnitude, i.e. C cannot be replaced by $CF(L)$ for a function F tending to zero for $L \to \infty$. This follows from the analysis of the following example, which we sketch:

For fixed $\varepsilon \in (0, 1]$ and arbitrary $N > 1$ let

$$\Omega_1(N) = \{(x, y) : 0 < y < \min\{x/\varepsilon, 1\}, 0 < x < N\}.$$

This is a right triangle (above $0 < x < \varepsilon$) with an $(N - \varepsilon) \times 1$ rectangle attached on the right. Define $\Omega_2(N)$ as the symmetrization of $\Omega_1(N)$ with respect to the line $y = 1/2$. The domains $\Omega_1(N)$ and $\Omega_2(N)$ have the same function h and therefore the same function ϕ_1 and the same ξ_{nod}. We have $h(x) = 1$ for $x \geq \varepsilon$. Now for any domain having this last property one can show the following: For $x \geq N/10$ the second eigenfunction u_2 must be very close (exponential in $-N$) to the second eigenfunction

$$\sin 2\pi \frac{x - \gamma}{N - \gamma} \sin \pi y$$

of the rectangle $(\gamma, N) \times (0, 1)$, for a certain $\gamma \in [0, 1)$ which is a spectral invariant of the 'end piece' $\Omega \cap \{0 < x < \varepsilon\}$; therefore, $\mathcal{N}_x$ is very close (uniformly in ε) to the point $(N + \gamma)/2$. Furthermore one can show that the value of γ is different for the two domains $\Omega_1(N)$, $\Omega_2(N)$, at least for small ϵ. From this it follows immediately that there is a constant $c > 0$ such that the nodal lines of $\Omega_1(N)$ and $\Omega_2(N)$ have distance at least c, for large N. This proves the optimality of (c).

1.2 Results

The main results of [GJ1] are:

Theorem 1.4 ([GJ1]) *There is a constant C such that for all $\Omega \in \mathcal{C}$ the projection $\mathcal{N}_x$ of the nodal line onto the x axis is an interval of length at most CL^{-3}.*

In the case where the derivative of h near $\mathcal{N}_x$ is much smaller than L^{-3} we even get a more precise estimate, see Theorem 3 in [GJ1].

Note that this estimate for the width of $\mathcal{N}$ is better than the estimate which follows from Theorem 1.3(c). On the other hand, here the position of $\mathcal{N}_x$ cannot be given in terms of a quantity of lower order (as it was the case in Theorem 1.3(c)).

We also show a corresponding pointwise estimate for the slope of $\mathcal{N}$:

Theorem 1.5 ([GJ1]) *Let (η_1, η_2) be a unit vector tangent to $\mathcal{N}$ in the point (x, y). For any $\varepsilon > 0$ there is $C = C(\varepsilon)$ such that for all $(x, y) \in \mathcal{N}$ having distance greater than ε from $\partial\Omega$ we have*

$$|\eta_1| \le CL^{-3}.$$

Both results are optimal: In the case of the circular sector $\mathcal{N}$ is a circular arc whose radius is of order N, therefore its projection on the x axis has width of order $N^{-1} = L^{-3}$. We conjecture that the estimate in Theorem 1.5 holds up to the boundary, i.e. that C may be chosen independent of ε.

The paper [GJ2] is about the first eigenfunction. First, we consider the position of its maximum:

Theorem 1.6 ([GJ2]) *There is a constant C such that for all $\Omega \in \mathcal{C}$ we have*

$$|x_{\max} - \xi_{\max}| \le C.$$

This corresponds to Theorem 1.3(c), but it is harder to prove, see below. The same example as above shows that this estimate is sharp. There is also an analogue to Theorem 1.3(b), but it is comparatively easy to prove, see below.

The second main result of [GJ2] shows how well u_1 may be approximated uniformly using the first order quantity ϕ_1: Define

$$\alpha(x, y) = \pi \frac{y - f_1(x)}{h(x)}.$$

Theorem 1.7 ([GJ2]) *There is a constant C such that for all $\Omega \in \mathcal{C}$ we have*

$$|u_1(x, y) - \phi_1(x) \sin \alpha(x, y)| \le CL^{-1} \quad \text{for } x \in I'.$$

Here I' is the interval concentric with I, of half the length.

Note that, for any fixed x, the sine factor occuring here is simply the first Dirichlet eigenfunction of d^2/dy^2 on the y-interval J_x.

1.3 Methods, idea of proof

As already in [51], the basic idea is that for large eccentricity one should have an approximate separation of variables. This means that the eigenfunction u (i.e. u_1 or u_2) should be well approximated by a function of the form

$$\tilde{u}(x, y) = e(x, y)\psi(x)$$

where $e(x,\cdot)$ is the L^2-normalized first eigenfunction of d^2/dy^2 on the y-interval J_x. Since the first eigenvalue of this operator is $-\pi^2/h(x)^2$ and the spectral gap $\mu_2-\mu_1$ tends to zero for $N\to\infty$ (see (5)) one expects that a good candidate for ψ is ϕ_1 or ϕ_2, respectively. This ansatz was used by Jerison in [51].

The essential new idea in [GJ1] and [GJ2] is to use the following function instead:

$$\psi(x)=\int_{f_1(x)}^{f_2(x)} e(x,y)u(x,y)\,dy.$$

This means that $\psi(x)$ is the coefficient of the term of lowest frequency in the Fourier decomposition of $u(x,\cdot)$ on the interval J_x. In particular, $\psi(x)$ is no longer a quantity of first order but computed from the eigenfunction u itself. Therefore one may expect ψ to yield a good approximation $\tilde{u}$ of u, in any case a better approximation than ϕ_1 resp. ϕ_2.

At the same time it is more difficult to analyze ψ than to analyze ϕ_1 and ϕ_2. This analysis becomes possible since one can derive an equation

$$\mathcal{L}\psi=\lambda\psi+\sigma$$

($\lambda=\lambda_1$ resp. λ_2), where the term σ stems from the x-dependence of e (so that $\sigma(x)=0$ if $\partial_x e\equiv 0$ near x). Near the central interval I the derivative $\partial_x e$ is bounded by the slope of the boundary curves $y=f_1(x)$, $y=f_2(x)$ of Ω, which is small for large eccentricity (e.g. of order L^{-3} near $\mathcal{N}$). From this and weak a priori-bounds for u we obtain optimal bounds for σ which then enable us to analyze ψ.

In the case of the second eigenfunction we show in this way that ψ has a unique zero; therefore, the zero set of $\tilde{u}$ is just a vertical line. Theorem 1.4 then follows from an upper bound for the 'error' $u-\tilde{u}$ and a lower bound for the derivative of ψ near its zero.

In the case of the first eigenfunction we also need to estimate the difference of ψ and ϕ_1 (suitably scaled). The arguments in the proof of Theorem 1.6 are necessarily more complicated than those in the proof of Theorem 1.3(c) since the derivative of the first eigenfunction (of $\mathcal{L}$) vanishes at its maximum (which makes it harder to localize the maximum) while the derivative of the second eigenfunction at its zero is of order L^{-1}.

General techniques used throughout these papers are the generalized maximum principle, the generalized Harnack inequality ('generalized' means the theorems for eigenfunctions instead of harmonic functions) and Carleson's Lemma which permits estimates uniformly up to the boundary. See [51].

For more details see the introductions and bodies of the papers [GJ1], [GJ2]. As an illustration we sketch the proof of the following result (see Proposition A' in [51], where the proof is left to the reader):

Proposition 1.8 *There is a constant C such that for all $\Omega \in \mathcal{C}$ we have*

$$x_{\max} \in [\alpha - CL, \beta + CL];$$

here, $I = [\alpha, \beta]$ is the interval in (4).

Proof. (sketch) From the normalization of Ω one sees by an elementary geometric argument (see Lemma 3.1 in [GJ2]) that $\max h = 1$. Let $x_0 \in I$ be any point with $h(x_0) = 1$. By definition of I the inequality $h(x) < 1 - L^{-2}$ holds for $x \notin I$; therefore, concavity of h and $|I| = L$ imply $|h'(x)| > L^{-3}$ for $x \notin I$, and so $h(x) \leq 1 - |x - x_0| L^{-3}$ for $x \notin I$. Then the second inequality in (5) implies that there is a constant C_0 such that for all $\Omega \in \mathcal{C}$ one has

$$\frac{\pi^2}{h(x)^2} - \lambda_1 > \frac{1}{L^2} \quad \text{for } |x - x_0| > C_0 L. \tag{6}$$

In order to localize $x_{\max}$ we want to show that $u = u_1$ is small far away from I. For this purpose we first consider the function

$$\rho(x) = \int_{I_x} u(x,y)^2 \, dy, \quad x \in (0, N).$$

An easy calculation shows (all integrals are over I_x with the measure dy, and $u_x = \partial_x u$ etc.)

$$\begin{aligned}
\rho'' &= 2\left(\int u u_{xx} + \int u_x^2\right) && (7)\\
&\geq 2 \int u u_{xx} = 2 \int u(-u_{yy} - \lambda_1 u) && (8)\\
&= 2 \int u_y^2 - 2\lambda_1 \int u^2 && (9)\\
&\geq 2\left(\frac{\pi^2}{h^2} - \lambda_1\right)\rho. && (10)
\end{aligned}$$

In the third line we integrated by parts in y, and in the fourth line we used that $\pi^2/h(x)^2$ is the first eigenvalue on I_x and that this eigenvalue can be characterized by the Rayleigh quotient.

From (6) we now get

$$\rho''(x) \geq 2L^{-2}\rho(x) \quad \text{for } |x - x_0| > C_0 L.$$

Using $\max \rho \leq 1$ and $\rho(0) = \rho(N) = 0$ we then conclude, using an elementary comparison argument (the one-dimensional version of the generalized maximum principle), that

$$\rho(x) \leq e^{-\sqrt{2}\,\mathrm{dist}\,(x,I_0)/L} \tag{11}$$

where $I_0 = [x_0 - C_0 L, x_0 + C_0 L]$.

Finally, we want to estimate u pointwise by its mean ρ. Consider first a point x with $h(x) \geq 1/2$; let $I = \{x\} \times I_x$ and let $I' \subset I$ be the interval concentric with I of half the length. The Carleson Lemma shows that $\max_I u \leq C \max_{I'} u$, and Harnack's inequality yields $\max_{I'} u \leq C \min_{I'} u$ for some constant C. Using $\min_{I'} u^2 \leq |I'|^{-1} \rho(x)$ we then get

$$\max_y u(x,y)^2 \leq C\rho(x) \quad \text{for } h(x) \geq 1/2.$$

An elementary barrier argument (construction of a comparison function and use of the generalized maximum principle) yields

$$\max_y u(x,y) \leq CL^{-1} \quad \text{for } h(x) \leq 1/2.$$

(See [GJ2], Lemmas 3.2 und 3.12 for more details on the last two estimates.) Together with (11) these two estimates give

$$u(x,y) < 1 \quad \text{for } x \notin [\alpha - C_1 L, \beta + C_1 L]$$

for a suitable constant $C_1 > C_0$. From this it follows that u must take its maximum inside this interval. □

2 The papers [G1], [G2]

In these papers I analyze the local geometry of sub-analytic surfaces with induced metric and give some applications to the analysis of the geometric differential operators on these surfaces.

Let V be a compact sub-analytic subset of a real analytic manifold M. Assume that V has dimension two everywhere and only isolated singularities. Thus

$$V = V_{\mathrm{reg}} \cup \{p_1, \ldots, p_k\},$$

where $p_1, \ldots, p_k$ are non-isolated points of V and V_{reg} is a smooth manifold.

Furthermore, assume that a real analytic Riemannian metric is given on M, and let g be the induced Riemannian metric on V_{reg}.

The first two main results of [G1] give a description of the local geometry of V. The Riemannian metric on $V_{\rm reg}$ induces a distance function d on V, the *induced metric.* Denote by dist the distance function on M induced by the Riemannian metric on M. The restriction of dist to V is called *extrinsic metric.*

Important special cases are given by cones and horns: Let $\gamma \geq 1$ be a rational number. Define

$$V_\gamma = \{(x,y,z) : 0 \leq z < 1,\, z^\gamma = \sqrt{x^2+y^2}\} \subset \mathbb{R}^3.$$

On $\mathbb{R}^3$ use the Euclidean metric. Then (V_γ, d) is called a *cone* if $\gamma = 1$ and a *horn* if $\gamma > 1$. Next, let $\gamma_1, \dots, \gamma_m \geq 1$ be rational and $\varepsilon > 0$, and consider the set

$$V_{\gamma_1,\dots,\gamma_m} = \bigcup_{i=1}^m e_i \otimes V_{\gamma_i} \subset \mathbb{R}^{3m},$$

where $e_1, \dots, e_m$ are the standard basis vectors of $\mathbb{R}^m$. When the Euclidean metric is used on $\mathbb{R}^{3m}$ then $(V_{\gamma_1,\dots,\gamma_m}, d)$ is simply the union of $V_{\gamma_1}, \dots, V_{\gamma_m}$, glued together at their tips. All these spaces are semi-algebraic, in particular they are sub-analytic.

Two metric spaces $(X,d), (X',d')$ are called *quasiisometric* if there is a homeomorphism $\phi : X \to X'$ such that both ϕ and ϕ^{-1} are Lipschitz maps.

In the first main theorem I classify the spaces introduced above locally, up to quasiisometry:

Theorem 2.1 (Theorem 1.1 in [G1]) *Let (V,d) be as above and $p \in V$. There is a neighborhood of p in V which is quasi-isometric to one of the spaces $V_{\gamma_1,\dots,\gamma_m}$.*

In order to render this a classification one needs to show, in addition, that the spaces $V_{\gamma_1,\dots,\gamma_m}$, $V_{\delta_1,\dots,\delta_n}$ are quasiisometric if and only if the multisets $\{\gamma_1, \dots, \gamma_m\}$, $\{\delta_1, \dots, \delta_n\}$ are equal. This follows immediately from a result in [G2]:

Theorem 2.2 (Corollary 3.10 in [G2]) *If $\gamma_1, \gamma_2 \geq 1$ and V_{γ_1}, V_{γ_2} are quasiisometric then $\gamma_1 = \gamma_2$.*

Clearly, $m = 1, \gamma_1 = 1$ for $p \in V_{\rm reg}$. The spaces V_γ with $0 < \gamma < 1$ do not appear in the classification since they are quasiisometric to the cone V_1 (see Corollary 3.7 in [G2]).

In Theorem 1.2 in [G1] the quasiisometry in Theorem 2.1 is described more precisely. In particular I show that arc length parametrization along the curves

$$S_r(p) = \{q \in V : \mathrm{dist}\,(q,p) = r\}$$

naturally defines such a quasiisometry.

The second geometric theorem in [G1] concerns the length of the curves $S_r(p)$:

Theorem 2.3 (Theorem 1.3 in [G1]) *Let (V, d) be as above and $p \in V$. For $r \to 0$ there is a complete asymptotic expansion*

$$length(S_r(p)) \sim \sum_{i,j} C_{i,j} r^i (\log r)^j, \tag{12}$$

where i varies over a set of positive rationals with bounded denominator and j over $\{0, 1\}$.

Furthermore, the smallest exponent i that actually appears in the asymptotics equals $\gamma = \min\{\gamma_1, \dots, \gamma_m\}$ where $\gamma_1, \dots, \gamma_m$ are as in Theorem 2.1, and $C_{\gamma,1} = 0$; in other words, the leading term in the asymptotics contains no logarithm.

The asymptotics may me differentiated term by term.

The same is true for any connected component of a pointed neighborhood of p since connected components of sub-analytic sets are sub-analytic.

Using these geometric results I then derive a version of the Gauss-Bonnet Theorem:

Theorem 2.4 (Theorem 1.4 in [G1]) *Let (V, d) be as above. Let $l_1, \dots, l_k$ be the coefficients of r in the expansions (12) for all singular points $p_1, \dots, p_k$. Let K be the Gauss curvature of the Riemannian metric on V_{reg}.*

Then K is integrable over V_{reg}, and the Euler characteristic satisfies

$$\chi(V) = \frac{1}{2\pi} \left(\int_{V_{\mathrm{reg}}} K + \sum_{i=1}^{k} (2\pi - l_i) \right).$$

Finally I draw some conclusions of an analytic nature. Denote by $L^2(\bigwedge V_{\mathrm{reg}})$ the square integrable differential forms on V_{reg} and by d the exterior derivative. d and its transpose d^t are defined on $L^2(\bigwedge V_{\mathrm{reg}})$ in the sense of distributions.

Theorem 2.5 (Theorem 1.5 in [G1]) *(i) V_{reg} has the L^2 Stokes property; i.e. if $\omega, \eta, d\omega, d^t\eta \in L^2(\bigwedge V_{\mathrm{reg}})$ then*

$$(d\omega, \eta) = (\omega, d\eta).$$

(ii) The Gauss-Bonnet operator $D_{GB} = d + d^t$ and the Laplace-Beltrami operator $\Delta = D_{GB}^2$ are selfadjoint as operators on $L^2(\bigwedge V_{\mathrm{reg}})$ with domains $\mathcal{D}(D_{GB}) = H^1(\bigwedge V_{\mathrm{reg}})$ and $\mathcal{D}(\Delta) = H^2(\bigwedge V_{\mathrm{reg}})$.

Their spectra are discrete.

(iii) The L^2 Euler charakteristik of V, i.e. the index of D_{GB} considered as operator from even to odd forms, equals

$$\chi_2(V) = N + \frac{1}{2\pi}\left(\int_{V_{\mathrm{reg}}} K - \sum_{i=1}^{k} l_i\right),$$

where N is the total number of cones and horns at all singular points of V.

Concerning the proofs let me just mention that I use the resolution of singularities for Theorems 2.1 and 2.3. Then the problem is to determine the form of the pull-back metric on the desingularized space. However, the precise analysis shows that for a classification modulo quasiisometry a crude knowledge of this metric is sufficient.

For further details and a discussion of related literature see the introduction of [G1].

The paper [G2] deals with the question when two Riemannian metrics on pointed neighborhoods of 0 in $\mathbb{R}^n$ are quasiisometric, with respect to a diffeomorphism ϕ which becomes a homeomorphism when extended by setting $\phi(0) = 0$. In particular I prove (a generalization of) Theorem 2.2 (see Theorem 3.8 in [G2]). I also prove that the induced metric on the real Whitney umbrella $W = \{x^2 = y^2 z\} \subset \mathbb{R}^3$ is quasiisometric to a conic metric (when considering the normalization of W).

3 The papers [G3], [GG]

These papers are about general techniques of singular analysis. [G3] gives an extensive introduction to the basic ideas and concepts of R. Melroses b-calculus. In [GG] we investigate the precise relation between the Push-Forward Theorem, a central theorem in the b-calculus, and Brüning and Seeley's Singular Asymptotics Lemma.

The b-calculus is a theory which generalizes the classical pseudodifferential calculus to certain strongly structured singular situations. It was developed by R. Melrose and many of his coauthors since the early 1980s and is still being extended. Thus, it is not a complete theory but rather a collection of ideas and concepts which can be and was used for the solution of many singular problems.

It would exceed the scope of this synopsis to describe the general class of problems to which the b-calculus is supposed to be applied (the ‘boundary fibration structures’). Instead, we give three typical examples:

Typical problems for the b-calculus:

1. Resolvent expansion: Let X be a compact manifold and P a positive elliptic differential operator on X.[1] Determine the structure of the resolvent

$$(P+z)^{-1}, \quad z>0,$$

uniformly for $z \to \infty$.

2. b-calculus in the narrow sense: Let X be a compact manifold with boundary. Consider an elliptic operator P of Fuchs type (or 'totally characteristic' operator) on X; this means that in local coordinates $(x, y=(y_1, \ldots, y_{n-1})) : U \subset X \to \mathbb{R}_+ \times \mathbb{R}^{n-1}$ (where $\mathbb{R}_+ = [0,\infty)$) near an arbitrary boundary point P takes the form

$$\sum_{j+|\alpha|\leq m} a_{j\alpha}(x,y)(x\partial_x)^j \partial_y^\alpha, \quad a_{j\alpha} \text{ smooth},$$

and $\sum_{j+|\alpha|=m} a_{j\alpha}(x,y)\lambda^j\eta^\alpha \neq 0$ for $(\lambda,\eta) \neq 0$. Then the problem is to construct a parametrix for P and to understand its structure, in order to investigate mapping (in particular Fredholm) properties of P.

3. Adiabatic Limit: Let $\pi : M \to Y$ be a fibration of compact manifolds; let g and h be Riemannian metrics on M and Y, respectively. For $\varepsilon > 0$ consider the metric $g_\varepsilon = \pi^* h + \varepsilon^2 g$ on M. Then the problem is to investigate the behavior of certain invariants which are defined by the Laplace-Beltrami operator associated with g_ε, as $\varepsilon \to 0$. See [60] for the Hodge cohomology and [26] for the analytic torsion.

The first two examples were also treated by many other authors. However, the b-calculus provides a common frame for such problems.

Some essential characteristics of the b-calculus are:

1. Operators are described by their Schwartz kernels.

2. Singular behavior is characterized by geometric constructions ('blowing up' of the underlying space), as far as possible.

An operator which operates on (functions on) a space X and depends on parameters in a space Z has a Schwartz kernel which is a distribution K on $X \times X \times Z$. To understand the structure of the operator then means to find a blow-up $\beta : Y \to X \times X \times Z$ such that the pull-back of K under β has a simple structure which can be described through the geometry of Y. Technically speaking, Y should be a

[1] To simplify the notation we assume here that all operators are scalar, but this is not essential.

compact manifold with corners and $\beta^* K$ should be polyhomogeneous conormal on Y, in particular have conormal singularities along a submanifold of Y which has normal crossings with ∂Y. Compactness is imposed to reflect finiteness of structure. In order to achieve it one may have to compactify $X \times X \times Z$ first, for example by adding $z = \infty$ in the first example.

Let us consider the second example above: Schwartz kernels of operators on X may be expressed locally as distributions $K(x, y, x', y')$ with $x, x' \in \mathbb{R}_+$, $y, y' \in \mathbb{R}^{n-1}$; an essential step in constructing the parametrix is the insight that Schwartz kernels of Fuchs type operators have a very simple structure if one writes them using polar coordinates with respect to x, x'. Geometrically this simply means blowing up the submanifold $\partial X \times \partial X$ in $X \times X$ and looking at the kernels on the blown-up space. Then the parametrix may be constructed in a way largely parallel to the classical construction (however one needs to introduce a 'second symbol' if one wants to make the error terms Fredholm).

In [G3] these ideas and methods are introduced and motivated by many examples. Such an introductory exposition of the b-calculus did not exist in the literature before. We emphasize a few important aspects:

1. Extensive discussion of basic questions connected with asymptotic developments in several variables: Simultaneous asymptotics vs. asymptotics separately in each variable. (Section 2.1)
2. Detailed motivation and discussion of the role of blow-ups in the description of non-simultaneous asymptotics; notion of asymptotic type. (Sections 2.2–2.4)
3. Discussion of the connection of these ideas with the notions of 'regimes' and 'matching conditions' which are used mainly in the applied literature. (Section 2.5)
4. Introduction to the classical pseudodifferential calculus. (Sections 3.3, 4.1)
5. Extensive explanation of the roles of the Push-Forward Theorem and of the Pull-Back Theorem, and sketches of proof. (Sections 3.1, 3.2)
6. Characterization of the central notion of b-fibration (alternative to Melrose's original definition in [66]). (Definition 3.9)
7. Detailed exposition of the b-calculus in the narrow sense (example 2 above). (Chapter 4)

The paper [GG]: We investigate the precise relation between Melrose's Push-Forward-Theorem and Brüning and Seeley's Singular Asymptotics Lemma; we show that both deal with the same basic problem and solve it in different partial aspects.

The Push-Forward-Theorem is a far-reaching generalization of the following well-known theorem: Let μ be a smooth density on a manifold X and $f : X \to Y$ a smooth proper map to a manifold Y; if f is a submersion (and hence a fibration) then the push-forward measure $f_*\mu$ is a smooth density again. This is generalized in the following ways:

- X and Y are manifolds with corners, i.e. everywhere locally of the form
$$\mathbb{R}^k_+ \times \mathbb{R}^{n-k} \tag{13}$$
for some n, k.

- f need not be a fibration but only a *b-fibration*. A b-fibration is a fibration in the interior (i.e. f maps $\mathrm{int}(X)$ onto $\mathrm{int}(Y)$ as fibration), but not necessarily at the boundary. For the precise definition of a b-fibration see [66] or [G3], Definition 3.9. The simplest non-trivial example is
$$f : \mathbb{R}^2_+ \to \mathbb{R}_+, \quad (x, y) \mapsto xy. \tag{14}$$

- μ is not required to be smooth up to the boundary; instead, the weaker condition of 'polyhomogeneous conormality' is imposed; i.e. μ has complete asymptotic developments near all corners (13), simultaneously in all boundary defining variables $x_1, \dots x_k$, in terms of the form $\prod_{i=1}^k x_i^{\alpha_i} (\log x_i)^{\beta_i}$, with coefficients depending smoothly on $x_{k+1}, \dots, x_n$.

The Push-Forward Theorem (PFT) then says that $f_*\mu$ is also polyhomogeneous conormal, and calculates the exponents (α_i and β_i) that appear in $f_*\mu$ from those of μ and from combinatorial data on the boundary behavior of f. For example, for the map f in (14) and for μ smooth up to the boundary of $\mathbb{R}^2_+$ one obtains that the asymptotics of $f_*\mu$ at zero contains logarithmic terms in general.

The Singular Asymptotics Lemma (SAL) ([13]) calculates the asymptotics of the integral
$$\int_0^\infty \sigma(x, xz)\, dx$$
for $z \to \infty$ from the asymptotics of $\sigma(x, \zeta)$ for $x \to 0$ and $\zeta \to \infty$, under a certain integrability condition (near $x = \zeta = 0$). (The 'classical' SAL from [13] was generalized recently in [5].)

In [GG] we show that the problem posed by the SAL may be viewed as special case of the problem posed by the PFT. Therefore, part of the conclusion of the SAL may be derived from the PFT: the exponents that appear in the asymptotics. However, the SAL gives in addition explicit formulas for the coefficients. These formulas were used in the treatment of the index problem on spaces with conical singularities (see [15]). Also, the SAL imposes slightly weaker regularity conditions.

The map f which connects SAL and PFT is defined as follows: Let $\beta : [\mathbb{R}^2_+, 0] \to \mathbb{R}^2_+$ be the blow-up of $0 \in \mathbb{R}^2_+$ and $\pi : \mathbb{R}^2_+ \to \mathbb{R}_+$ the projection onto the first factor. Then $f = \pi \circ \beta : [\mathbb{R}^2_+, 0] \to \mathbb{R}_+$. For the special case where σ vanishes for ζ near zero one may use the simpler map (14) instead.

A common generalization of PFT and SAL would be obtained by solving the following problem:

Problem: Let $f : X \to Y$ be a proper b-fibration and μ a polyhomogeneous conormal density. Determine the coefficients in the boundary asymptotics of $f_*\mu$ in terms of regularized integrals over the coefficients of the boundary asymptotics of μ.

4 The papers [CGIKO], [CGK]

In these papers we investigate the following optimization problem:

Problem: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. For $\alpha > 0$ and measurable $D \subset \Omega$ consider the operator

$$P_{\alpha,D} = -\Delta + \alpha\chi_D,$$

where Δ is the Laplace operator and χ_D the characteristic function of D. Let $\lambda_\Omega(\alpha, D)$ be the first eigenvalue of the Dirichlet problem for $P_{\alpha,D}$.

Denote by $|D|$ the volume of D. Let $0 \le A \le |\Omega|$.

For which domains D with $|D| = A$ does the eigenvalue $\lambda_\Omega(\alpha, D)$ attain its smallest value?

We call such optimal domains D *optimal configurations* or solutions; (u, D) is called an *optimal pair* if D is an optimal configuration and u is a first eigenfunction of $P_{\alpha,D}$. u is determined up to scalar multiples by D.

We obtain results on the following problems:

- Existence, regularity and uniqueness,
- dependence on the parameters α, A,

- symmetry preservation and symmetry breaking,
- regularity of the boundary of an optimal configuration,
- convexity,
- connectivity,
- relation with a problem of optimal mass distribution in a membrane.

We cannot explain all results in this synopsis. Rather, we make a few basic remarks and then focus on the central themes of symmetry and regularity of the boundary.

We first prove some basics:

Theorem 4.1 (Theorem 1 in [CGIKO]) *Fix Ω, α, A as above. An optimal configuration exists. Any optimal pair (u, D) has the following properties:*

- *Regularity: $u \in C^{1,\delta}(\Omega) \cap H^2(\Omega)$ for any $\delta < 1$.*
- *Level set property: There is $t \geq 0$ such that*

$$D = \{u \leq t\}. \tag{15}$$

Here we always identify sets which only differ by a null set. In particular we may always assume (by (15)) that D is closed.

The question of uniqueness is far more complicated: We show that for $\Omega = \{x : |x| < 1\}$ there is a unique optimal configuration (for arbitrary values of the parameters α, A), while for certain other domains and certain parameter values there are several solutions. Both of these claims follow from symmetry considerations: The uniqueness for the ball follows from the following theorem on *symmetry preservation in convex situations*:

Theorem 4.2 (Theorem 4 in [CGIKO]) *Assume that Ω is symmetric and convex with respect to a hyperplane, say $\{x_1 = 0\}$; that is, for any $x' = (x_2, \ldots, x_n)$ the set*

$$\{x_1 : (x_1, x') \in \Omega\}$$

is either empty or an interval of the form $(-c, c)$.

Then the following holds for any optimal pair (u, D): u and D are symmetric with respect to $\{x_1 = 0\}$, the complement D^c is convex with respect to $\{x_1 = 0\}$, and u is decreasing in x_1 for $x_1 \geq 0$.

On the other hand, one may deduce non-uniqueness from *symmetry breaking*: For example, if Ω has a reflection symmetry but a solution D doesn't then the reflection of D is also a solution, and it is different from D. We prove symmetry breaking in two situations: For annular domains and for dumbbells. Annular domains are rotationally symmetric, dumbbells are symmetric with respect to reflection in the coordinate axes:

Theorem 4.3 (Theorem 6 in [CGIKO]) *Let $\alpha > 0$ and $\delta \in (0,1)$. For $b > 0$ let Ω_b be the annulus of width b,*

$$\Omega_b = \{x \in \mathbb{R}^2 : 1 < |x| < 1 + b\}.$$

There is $b_0 = b_0(\alpha, \delta)$ such that for $b < b_0$ any optimal configuration with parameters α and $A = \delta|\Omega_b|$ is not rotationally symmetric.

See Figure 2(b) in [CGIKO].

Theorem 4.4 (Theorem 7 in [CGIKO]) *For $h \in (0,1)$ define the dumbbell with handle of width $2h$*

$$\Omega_h = B_1(-2,0) \cup ((-2,2) \times (-h,h)) \cup B_1(2,0),$$

where $B_r(p) = \{x \in \mathbb{R}^2 : |x - p| < r\}$. For any $\alpha > 0$ and $A \in (0, 2\pi)$ there is $h_0 = h_0(\alpha, A)$ such that for all $h < h_0$ we have:

(i) Any optimal configuration D is not symmetric with respect to reflection in the x_2-axis.

(ii) If $A > \pi$ then the complement of any optimal configuration D is contained in one of the lobes $B_1(\pm 2, 0)$.

See Figure 3 in [CGIKO]. For the proofs of both theorems we use that an optimal pair (u, D) minimizes the Rayleigh quotient

$$R(u, D) = \frac{\int_\Omega |\nabla u|^2 + \alpha \int_D u^2}{\int_\Omega u^2}$$

over all $u \not\equiv 0$ and D with $|D| = A$. One needs to show that certain unsymmetric 'test pairs' have a smaller value of $R(u, D)$ than any symmetric optimal pair.

Numerical calculations show that in both theorems the smallness of the parameter (b resp. h) is essential for symmetry breaking: For thick annuli and for dumbbells with thick handle the (numerically calculated) solutions are symmetric (see Figure

2 and 3 in [CGIKO]). (All numerical calculations were done by Imai and Ohnishi. The theoretical results were obtained by Chanillo, Grieser and Kurata.)

We now turn to questions of *regularity of the 'free boundary'*. Here the free boundary is defined as

$$\mathcal{F} = \{u = t\},$$

where t is defined by (15). Note that $\partial D \subset \mathcal{F}$, but the reverse inclusion is not clear a priori.

Note first that one may expect that for certain domains Ω and certain values of the parameters there are solutions with non-smooth free boundary: For example, if one varies the thickness b of an annular domain continuously and considers solutions $D = D(b)$ with fixed α and $\delta = A/|\Omega|$ then the topological type of $D(b)$ will change with increasing b, according to the (numerically calculated) Figures 2(a) and 2(b) in [CGIKO]. This suggests that for a certain transition value $b = b_1$ the boundary is not a smooth curve; instead, it might be a curve that intersects itself. Such a picture was found indeed in a numerical calculation. (Assuming sufficiently regular dependance of suitable solutions $D(b)$ on b the existence of such a b_1 could also be proved rigorously.)

On the positive side we have high regularity in certain situations:

Theorem 4.5 (Theorem 8 in [CGK]) *Let (u, D) be an optimal pair. Let $x \in \mathcal{F}$, and assume $\nabla u(x) \neq 0$. Then, near x, $\mathcal{F}$ is a real analytic hypersurface and agrees with ∂D.*

The problem in the proof is that the potential χ_D is discontinuous at ∂D, so that u is not even C^2 there. In order to show that the level set $\{u = t\}$ is C^ω nevertheless, we introduce suitable coordinates (with u as one coordinate function) and analyze the resulting non-linear elliptic equation.

Using a perturbation argument and well-known properties of the first Dirichlet eigenfunction of the Laplace operator we conclude from this:

Theorem 4.6 (Theorem 9 in [CGIKO]) *Assume Ω is convex and has C^2 boundary. Then there is $\alpha_0 = \alpha_0(A, \Omega)$ such that for any solution D with $\alpha < \alpha_0$ the free boundary $\mathcal{F}$ is a convex and real analytic hypersurface which agrees with ∂D.*

It is very difficult to describe the singular set of $\mathcal{F}$ (i.e. the set of points $x \in \mathcal{F}$ with $\nabla u(x) = 0$) in general. Our results in this direction are based essentially on Hopf's Lemma: A function which is positive and superharmonic in a domain G and vanishes at a boundary point $x_0 \in \partial G$ must have positive normal derivative at x_0, if there is a ball $B \subset G$ with $x_0 \in \partial B$.

In the sequel we fix an optimal pair (u, D) for given data Ω, α, A, and we let t be the number in (15). Set

$$D^+ = \{u > t\}, \quad D^- = \{u < t\}.$$

Let $\Lambda = \Lambda(\alpha, A)$ be the optimal eigenvalue.

Applying the Hopf Lemma to $u - t$ and $t - u$ in D^+ and D^-, respectively, one obtains:

Lemma 4.7 (Lemma 2 in [CGK]) *(a) Let $x_0 \in \mathcal{F}$. If there is a ball $B \subset D^+$ with $x_0 \in \partial B$ then $\nabla u(x_0) \neq 0$; therefore, $\mathcal{F}$ is real analytic near x_0.*

(b) The same is true for D^- instead of D^+, assuming $\alpha \geq \Lambda(\alpha, A)$.

The condition in (b) is satisfied for large α: There is $\overline{\alpha}_\Omega(A)$ such that $\alpha \geq \Lambda(\alpha, A)$ if and only if $\alpha \geq \overline{\alpha}_\Omega(A)$ (see Proposition 10 in [CGIKO]).

From this one sees, for example, that $\mathcal{F}$ cannot have a 'cusp' which points into D^+, and if α is large then it cannot have a cusp pointing into D^-. In particular, $\mathcal{F}$ must contain regular points (choose an arbitrary point x in D^+, then any point $\mathcal{F}$ which is closest to x must be regular).

Lemma 4.7 is one of the central tools in the proof of our main regularity result on $\mathcal{F}$. Here we only state the most important claims from Theorem 9 and Proposition 2 in [CGK], assuming $\alpha > \Lambda(\alpha, A)$ for simplicity:

Theorem 4.8 (Theorem 9 and Proposition 2 in [CGK]) *For $\varepsilon > 0$ define*

$$K_\varepsilon = \{z \in \Omega : \mathrm{dist}\,(z, \mathcal{F}) = \varepsilon\},$$
$$F_\varepsilon = \{x \in \mathcal{F} : \mathrm{dist}\,(x, K_\varepsilon) = \varepsilon\}$$

and

$$\mathcal{E} = \mathcal{F} \setminus \bigcup_{\varepsilon > 0} F_\varepsilon$$

(the exceptional set). Assume $\alpha > \Lambda(\alpha, A)$. Then we have:

(i) $\mathcal{E}$ is a G_δ set.

(ii) $\mathcal{F} \setminus \mathcal{E}$ is a real analytic submanifold of $\mathbb{R}^n$ and is dense in $\mathcal{F}$.

We conjecture that in two dimensions the singular set of $\mathcal{F}$ is finite.

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