Lecture 2: Some basic principles of the b-calculus

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General principles for studying singular problems

Preliminary step: Put problem in the form $(X, \mathcal{V})$.
(This may involve blow-ups, e.g. cone $\rightsquigarrow X$)

General principles for studying $(X, \mathcal{V})$

- Split into geometric and analytic aspects:
  - Geometry encodes singular structure
  - Analysis: conormal distributions ('hide' Fourier transform)

  This helps when studying complicated singularities.
- Separate different types of singular behavior by blow-ups
- Describe operators via their Schwartz kernels

Motivation: Significance of Schwartz kernel

Schwartz kernel $K_A \in \mathcal{D}'(M^2)$ of a linear operator A on $C^\infty(M)$:

$$(Af)(p) = \int_M K_A(p, p')\, f(p')\, dp'$$

'Knowing' $K_{P^{-1}}$ yields info. on solutions of $Pu = f$, e.g. ($\operatorname{supp} f$ compact):

$P = \Delta$ on $\mathbb{R}^n$, $n \geq 3$: $K_{P^{-1}}(p, p') = c_n |p - p'|^{2-n}$.

- 'Order -2' singularity at diagonal $p = p' \Rightarrow$ regularity of u
- Polynomial decay as $|p - p'| \to \infty \Rightarrow$ polynomial decay of u at ∞

$P = \Delta + 1$: $K_{P^{-1}}(p, p') = \hat{h}(p - p'), h(\xi) = \frac{1}{|\xi|^2 + 1}$.

- 'Order -2' singularity at diagonal $p = p' \Rightarrow$ regularity of u
- Exponential decay as $|p - p'| \to \infty \Rightarrow$ rapid decay of u at ∞

Basics: Geometry

- Spaces: **Manifolds with corners, p-submanifolds**
- Operation on spaces: **Blow-up**

Definition/Lemma: Blow-up

Let Z be a manifold, $p \in Z$. Then there is a unique manifold with boundary, denoted $[Z, p]$, and smooth map $\beta : [Z, p] \to Z$ with

- $[Z, p] \setminus \beta^{-1}(p) \to Z \setminus \{p\}$ is a diffeomorphism
- locally near p, $\beta =$ polar coordinates $\mathbb{R}_+ \times S^{n-1} \to \mathbb{R}^n$

Generalizations:

- Replace p by a closed submanifold Y, get $[Z, Y]$
- $Z =$ manifold with corners, Y a p-submanifold
- Iterate

Useful coordinates on $[Z, Y]$: projective coordinates

Basics: Analysis

Definition: Conormal distribution

Z manifold, $D \subset Z$ closed submanifold. A distribution $u \in \mathcal{D}'(Z)$ is **conormal with respect to** D if

- u is smooth on $Z \setminus D$, and
- in any local coordinates with $D = \{z = 0\}$

$$u(y,z) = \int_{\mathbb{R}^{N-k}} e^{iz\zeta} a(y,\zeta)\, d\zeta$$

where a is a symbol. $(N = \dim Z, k = \dim D)$

(principal) symbol of u: a (mod lower order terms) , well-defined as function on N^*D

Warning: no relation to 'conormal symbol' of cone calculus

Examples: δ, δ', $p.v.\frac{1}{x}$, Schwartz kernels of (pseudo)-differential operators

Generalization: Z manifold with corners, D p-submanifold

Basics: Classical ΨDO calculus

X = compact smooth manifold

Operators	Symbols (on T^*X)	Schwartz kernels (in $\mathcal{D}'(X^2)$)
$\mathrm{Diff}^*(X)$	homog. polynomials in ξ	δ-type at D_X
$\Psi^*(X)$	homog. functions in ξ	Conormal w.r.t. D_X

- **Composition Theorem:** $\Psi^*(X)$ is closed under products and the Symbol map $\sigma_* : \Psi^*(X) \to S^*(T^*X)$ preserves products
- There is a **short exact symbol sequence** $0 \to \Psi^{m-1}(X) \to \Psi^m(X) \to S^{[m]}(T^*X) \to 0$
- Asymptotic completeness

Theorem

These properties give parametrix construction: $P \in \Psi^m(X)$ elliptic $\Rightarrow$ $\exists Q \in \Psi^{-m}(X)$ with $PQ - I, QP - I \in \Psi^{-\infty}(X)$.

Basics: Classical ΨDO calculus

Functional analysis:

- $P \in \Psi^m(X)$ bounded $H^s(X) \to H^{s-m}(X)$
- $R \in \Psi^{-\infty}(X) \Rightarrow K_R$ smooth $\Rightarrow R$ compact operator

Corollary

$P \in \Psi^m(X)$ elliptic, then

- elliptic regularity: $Pu = f$, $f \in H^{s-m}(X) \Rightarrow u \in H^s(X)$
- P Fredholm

What we'll get: $\mathcal{V}$ – ΨDO calculus

X = compact manifold with corners, $\mathcal{V}$ Lie algebra of vector fields

Operators	Symbols (on ${}^{\mathcal{V}}T^*X$)	Schwartz kernels (in $\mathcal{D}'(X^2_{\mathcal{V}})$)
$\mathrm{Diff}^*_{\mathcal{V}}(X)$	homog. polynomials in ξ	δ-type at $D_{X,\mathcal{V}}$
$\Psi^*_{\mathcal{V}}(X)$	homog. functions in ξ	Conormal w.r.t. $D_{X,\mathcal{V}}$

- **Composition Theorem:** $\Psi^*_{\mathcal{V}}(X)$ is closed under products and the Symbol map ${}^{\mathcal{V}}\sigma_* : \Psi^*_{\mathcal{V}}(X) \to S^*({}^{\mathcal{V}}T^*X)$ preserves products
- There is a **short exact symbol sequence** $0 \to \Psi^{m-1}_{\mathcal{V}}(X) \to \Psi^m_{\mathcal{V}}(X) \to S^{[m]}({}^{\mathcal{V}}T^*X) \to 0$
- Asymptotic completeness

Theorem

These properties give parametrix construction: $P \in \Psi^m_{\mathcal{V}}(X)$ elliptic $\Rightarrow$ $\exists Q \in \Psi^{-m}_{\mathcal{V}}(X)$ with $PQ - I, QP - I \in \Psi^{-\infty}_{\mathcal{V}}(X)$.

What we'll get: $\mathcal{V}$ – ΨDO calculus

Functional analysis:

- $P \in \Psi^m_{\mathcal{V}}(X)$ bounded $H^s_{\mathcal{V}}(X) \to H^{s-m}_{\mathcal{V}}(X)$
- $R \in \Psi^{-\infty}_{\mathcal{V}}(X) \Rightarrow K_R$ smooth (but $\not\Rightarrow R$ compact operator)

Corollary

$P \in \Psi^m_{\mathcal{V}}(X)$ $\mathcal{V}$-elliptic, then

- 'small' elliptic regularity: $Pu = f$, $f \in H^{s-m}_{\mathcal{V}}(X) \Rightarrow u \in H^s_{\mathcal{V}}(X)$

To get compact errors (hence Fredholm P), need **larger calculus** or **stronger ellipticity condition**.

Main steps in building a $\mathcal{V}$ – ΨDO calculus

1. Construct **double space** $X^2_{\mathcal{V}}$. Requirements:
   - Diagonal D_X lifts to p-submanifold $D_{X,\mathcal{V}}$
   - For any $V \in \mathcal{V}$, the vector field $V \times 0$ on X^2 lifts smoothly to $X^2_{\mathcal{V}}$
   - These lifts span the normal space to $D_{X,\mathcal{V}}$
2. Define **small $\mathcal{V}$-calculus** $\Psi^*_{\mathcal{V}}(X)$ via Schwartz kernels on $X^2_{\mathcal{V}}$:
   - conormal w.r.t. $D_{X,\mathcal{V}}$ (uniformly to the boundary)
   - vanish to all orders at all faces except those intersecting $D_{X,\mathcal{V}}$
   - symbols are functions on ${}^{\mathcal{V}}T^*X \cong N^*D_{X,\mathcal{V}}$

   $\rightsquigarrow$ can invert $\mathcal{V}$-elliptic operators up to **smoothing** errors.
3. Identify obstruction to compactness of smoothing operators.
   $\rightsquigarrow$ normal, indicial operator
4. If needed, enlarge calculus by including inverses of normal operator
   $\rightsquigarrow$ get compact errors

Example: b-calculus

The problem:
$X =$ cpct manifold with boundary, $\mathcal{V}_b = \{\text{vector fields tangent to } \partial X\}$
(spanned by $x\partial_x, \partial_{y_i}$ near boundary)

The solution:

1. **Double space:** $X_b^2 := [X^2, (\partial X)^2]$ (check lifts of $x\partial_x, \partial_{y_i}$)
2. **Small b-calculus:** $\Psi_b^*(X)$

Motivating example

$P = x\partial_x + c$ on $X = \mathbb{R}_+ = [0, \infty)$.
(only analyze behavior near $x = 0$)
Kernels of inverses: $K_Q(x, x') = \left(\frac{x'}{x}\right)^c (H(x - x') + \text{const})$

The example also shows:

- The small calculus is not enough.
- Separation: different kinds of singular behavior of K_Q are separated