Amplitude Equations for a Benthic Nutrient-Bacteria Model

Daniel Wetzel\(^1\), Ulrike Feudel\(2\) and Hannes Uecker\(1\)
\(^1\) Institut für Mathematik, \(^2\)ICBM

Abstract

We study a two-dimensional two-component reaction-diffusion system describing pattern formation in a benthic nutrient-bacteria system\(^3\). The model shows a variety of patterns including stripes, cold and hot spots. Here we describe analytically stability properties and transitions between different patterns by using amplitude equations.


Model Set Up

In dimensionless form the system is described by

\[
\begin{align*}
\partial_t u &= \left( \gamma + (1 - \gamma) \frac{u}{k + u} \right) - m u + \varepsilon + \Delta u \\
\partial_t v &= \left( \gamma + (1 - \gamma) \frac{v}{k + v} \right) - \sigma(y)(v_0 - v) + d \Delta v.
\end{align*}
\]

(1)

Domain \((x, y) \in [0, L_x] \times [0, L_y]\) with Neumann boundary conditions

- \(u\): population density of bacteria
- \(v\): bioirrigation (bifurcation parameter)
- \(\gamma = 0.25\): associated half saturation constant
- \(m = 0.3175\): mortality of bacteria
- \(c = 0.005\): bacteria inflow
- \(v_0 = 125\): nutrient concentration in the water
- \(d = 50\): diffusion constant

Let \(f\) be the reaction-term of (1) and \((u^*, v^*)\) a fixed point of \(f\). Taylor-expanding \(f\) around \((u^*, v^*)\) to third order and setting \(w = (u, v) - (u^*, v^*)\) we obtain

\[
\begin{align*}
\partial_t w &= Lw + B(w, u) + C(u, w, w),
\end{align*}
\]

(2)

where \(L\) is bilinear and symmetric, \(C\) is trilinear and symmetric, and \(L = J_f + \left( \frac{3}{2} \right) A \). It holds that \(L(w, x) \equiv L(\hat{\tilde{\Phi}}(x, y), \tilde{\phi}(x, y), J_f - \frac{2}{3} \Delta) \). This yields the eigenvalue problem \(\hat{\tilde{\Phi}}(\sigma) = \lambda(\sigma) \tilde{\phi}(\sigma)\). There is a Turing-bifurcation at \(\sigma_c \approx 0.1254\) with \(k_c \approx 0.19\).

Hexagonal Spots

Setting \(A_1 = A_2 = A_3 = A\) yields \(A = -\frac{1}{\sqrt{3}} \pm \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 - \frac{2.15}{\sqrt{3}}}\) such that

\[
\begin{align*}
\begin{pmatrix}
u \\
\end{pmatrix} &\approx
\begin{pmatrix}
u^* \\
\end{pmatrix} + 2A
\left(\cos(k_1 x) + \cos\left(\frac{\sqrt{3}}{2}(k_1 x + \sqrt{3} y)\right) + \cos\left(\frac{\sqrt{3}}{2}(-x - \sqrt{3} y)\right)\right) \Phi.
\end{align*}
\]

\((u^*, v^*)\) yields \(v_{\min} = 0.039\) for cold spots and \(v_{\max} = 0.06\) for stripes. The largest fixed point of \(x\)-dependent \((\sigma(1, x, y), e)\) yields \(\sigma(1, x, y) \approx 0.04\) and \(e \approx 0.04\) for the full \(x\)-dependent.

Main Bifurcation Branches with Comparison to Numerics for (1)

Pattern Layering in the Full Model

Describing the bioirrigation by the depth dependent function \(\sigma(y) = \frac{0.18}{\sqrt{y}}\langle\Phi\rangle\) and starting with a small random perturbation of \((u, v) \equiv (1, 1)\) we obtain quasi-stationary layering of patterns. Our analysis predicts transitions between patterns in the ranges \(y \in [265, 296]\) and \(y \in [380, 418]\).

Mixed Modes

In the \(\tau\)-range of stable stripes numerical solutions of the stationary problem for (4) also yields mixed modes with \(A_1 \neq A_2 \neq A_3 \neq 0\).

Conclusions. Amplitude formalism gives good predictions for pattern formation also far from onset and for non-stationary conditions.

Outlook.

Arclength continuation and further bifurcation analysis for stationary solutions of (1) with constant \(\sigma\). Derivation and justification of \((y\)-dependent\) modulation equations for the full \(y\)-dependent system.

Amplitude Equations for Constant \(\sigma\)

To approximate solutions of (2) analytically we make the ansatz

\[
w = \left( A_{j1} x + A_{j2} y + A_{j3} \right) \Phi + c.c.,
\]

(3)

where \(\Phi = \phi(k_c), e^j \equiv e^j(x, y) k_j, A_j = A_j(1) \in \mathbb{C}, j = 1, 2, 3, k_1 = k_c \left(\frac{i}{k_1}\right), k_2 = k_c \left(\frac{i}{k_2}\right)\), and \(k_3 = k_c \left(\frac{i}{k_3}\right)\).

Inserting (3) into (2) and sorting with respect to coefficients of \(c_1, c_2, c_3\) yields

\[
\begin{align*}
\begin{align*}
c_1 &= \partial A_1 = \lambda(k_c) A_1 + 2 \Delta A_1 3 + (3 \lambda A_2 - 6 |A_3|^2) A_2 c \\
c_2 &= \partial A_2 = \lambda(k_c) A_2 + 2 \Delta A_2 3 + (3 \lambda A_3 - 6 |A_3|^2) A_3 c \\
c_3 &= \partial A_3 = \lambda(k_c) A_3 + 2 \Delta A_3 3 + (3 \lambda A_1 - 6 |A_3|^2) A_1 c.
\end{align*}
\end{align*}
\]

(4)

\(b(\sigma) = (\Phi, \Phi^*)^T, c(\sigma) = (C(\Phi, \Phi), \Phi^*, \Phi^*)^T, \) and \(\Phi^*\) is the adjoint eigenvector of \(\hat{\tilde{\Phi}}(k_c)\).

Stripes

\(
A_1 = \pm \frac{1}{\sqrt{3}} + \frac{2.15}{\sqrt{3}}, A_2 = A_3 = 0 \Rightarrow w = A e \Phi + c.c.\) is an approx. solution of (2).

\[
A = \pm \frac{1}{\sqrt{3}} + \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 - \frac{2.15}{\sqrt{3}}}.
\]

For \(\sigma = 0.1\) we find \(u \approx 1 + \cos(0.19\pi)\) and \(v \approx 1 - 0.18 \cos(0.19\pi)\).