Angle Trisection, the Heptagon, and the Triskaidecagon

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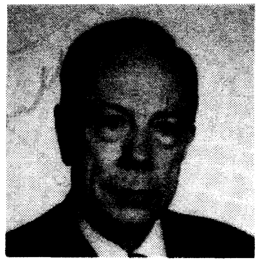

Dr. Gleason graduated from Yale in 1942 and then served four years in the Navy. After the war he went to Harvard as a Junior Fellow in the Society of Fellows. Except for an interlude in the Navy from 1950–52, he has been at Harvard ever since. He now holds the Hollis Professorship of Mathematicks and Natural Philosophy, a chair that was endowed in 1727. Although he has no doctor's degree, he says that George Mackey was the equivalent of his dissertation supervisor. He has worked in several areas including topological groups, Banach algebras, finite geometries, and coding theory. He received the Newcomb Cleveland prize of the AAAS in 1952. He is a member of the National Academy of Sciences and is a former president of the American Mathematical Society.

To David Vernon Widder on his 90th birthday

In 1796, Gauss discovered how to construct a regular 17-gon using only ruler and compass. Gauss also showed that regular polygons with 257 or 65537 sides can be constructed. He published these results in his famous *Disquisitiones Arithmeticae* [**3**, section VIII] in 1801. There he gave an analysis of the fields $\mathbf{Q}(\xi)$, where ξ is a complex pth root of unity, p an odd prime, and from this analysis he deduced that a regular n-gon can be constructed if n has the form $2^m p_1 p_2 \cdots p_k$, where $p_1, p_2, \ldots, p_k$ are distinct Fermat primes, that is, odd primes that are one greater than a power of two. He did not give explicit geometric constructions. Gauss also stated [**3**, p. 459] very emphatically that no other regular polygons are constructible, but he never published a proof of this fact. A proof was eventually published by Wantzel [**10**] in 1837. Since no Fermat primes larger than 65537 have been discovered, the list of constructible regular polygons remains as Gauss left it.[2]

If we enlarge our kit of construction tools, other regular polygons may become constructible.[3] For example, given an Archimedean spiral, we can divide any angle into any number of equal parts, and hence draw any regular polygon. Suppose we allow ourselves to trisect angles in addition to the standard ruler and compass constructions, what do we gain? Obviously, we can construct regular polygons with $9, 27, 81, \ldots$ sides, but it is certainly not obvious that we can also make a regular polygon with seven sides. In what follows we shall explore the relation between angle trisections and solving cubic equations and then determine which regular polygons can be constructed with the aid of an angle-trisector. We begin by showing how to draw a regular heptagon.

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1. The author thanks Ethan Bolker and Persi Diaconis who made valuable suggestions and Lloyd Schoenbach who made the figures.

2. Since $2^{rs} + 1$, with r odd and greater than 1, has the nontrivial factor $2^s + 1$, it is clear that $2^q + 1$ can be prime only if q itself is a power of 2. Thus all Fermat primes have the form $2^{2^k} + 1$. The known Fermat primes $3, 5, 17, 257, 65537$ correspond to $k = 0, 1, 2, 3, 4$. Euler observed that $2^{2^5} + 1 = 641 \times 6700417$, thus disproving Fermat's conjecture that all values of k would yield primes. More recently, it has been shown that $2^{2^k} + 1$ is composite for $6 \leqslant k \leqslant 19$ and many larger values. In many cases the complete factorization has not been found. See [**2**].

3. Both [**1**] and [**6**] contain much information about constructions using extraordinary tools. For an explanation of the standard "ruler and compass" constructions see [**5**].

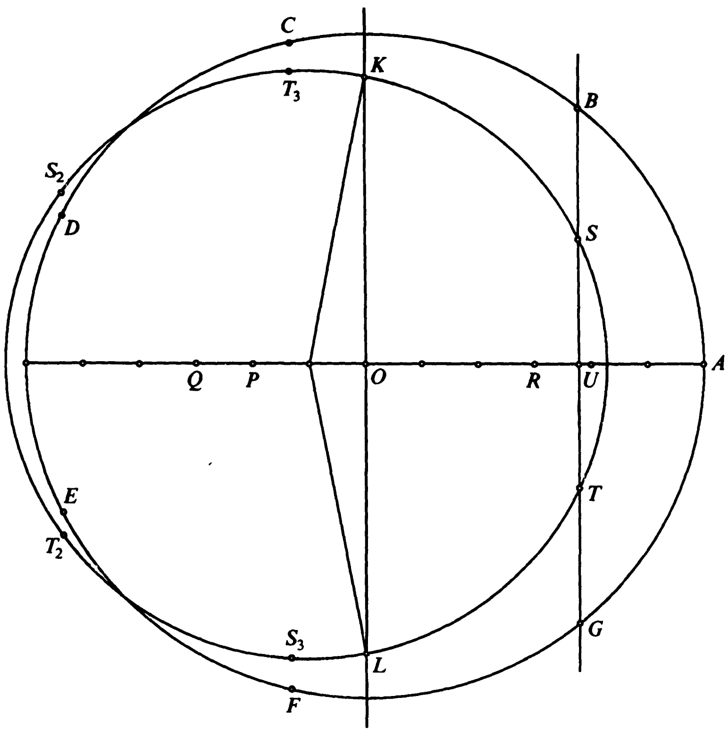

FIG. 1. Construction of a regular heptagon.

Start with a circle $\mathscr{C}$ of radius 6 centered at the origin of a Cartesian coordinate system. Mark $A(6,0)$, $Q(-3,0)$, and $R(3,0)$. Locate $K(0,\sqrt{27})$ and $L(0,-\sqrt{27})$, the vertices of the equilateral triangles with base QR. With center $P(-1,0)$ draw the arc from K to L and trisect it at S and T. The points B and G at which the line ST meets circle $\mathscr{C}$ are vertices of a regular heptagon $ABCDEFG$. The remaining vertices can be found by laying off the arc AB successively around the circle.

Alternatively, we may draw the major arc from K to L and trisect it at S_2 and T_2; then S_2T_2 meets $\mathscr{C}$ at D and E. Finally, if we think of arc KL as going all the way around P, back to K, and then on to L, and trisect this arc at S_3 and T_3, we find C and F.

Proof of the construction. Let U be the point where ST meets the x-axis. It is evident from the construction that $PK=\sqrt{28}$, $\cos\angle APK=1/\sqrt{28}$, $PU=\sqrt{28}\cos\angle APS=\sqrt{28}\cos(1/3)\angle APK$. The construction is correct if $OU=6\cos 2\pi/7$; or equivalently, if $PU=1+6\cos 2\pi/7$. Thus we need only establish the identity $\sqrt{28}\cos((1/3)\arccos(1/\sqrt{28}))=1+6\cos(2\pi/7)$. To do so, let $\xi=\cos 2\pi/7+i\sin 2\pi/7$ be the principal seventh root of 1. Put $\eta=\xi+\xi^{-1}=2\cos 2\pi/7$. Then $\eta^3+\eta^2-2\eta-1=\xi^3+\xi^2+\xi+1+\xi^{-1}+\xi^{-2}+\xi^{-3}=0$.

Thus η is a root of the equation

$$X^3 + X^2 - 2X - 1 = 0. \tag{1}$$

Setting $X = (Y - 1)/3$, we find that $1 + 6\cos 2\pi/7$ $(= 1 + 3\eta)$ is a root of

$$Y^3 - 21Y - 7 = 0. \tag{2}$$

The substitution $Y = \sqrt{28}\cos\theta$ reduces this to

$$7\sqrt{28}\,(4\cos^3\theta - 3\cos\theta) = 7;$$

whence

$$\cos 3\theta = \frac{1}{\sqrt{28}}.$$

This leads to six determinations of θ modulo 2π, which pair off to give three roots of (2). We leave it to the reader to check that $1 + 3\eta$ corresponds to the choice $\theta = 1/3 \arccos 1/\sqrt{28}$. The other roots of (2) are, of course, $1 + 6\cos 4\pi/7$ and $1 + 6\cos 6\pi/7$; these correspond to different determinations of θ and lead to the alternative constructions of the points C, D, E, and F mentioned above.

Plemelj [6] gave a different construction of the regular heptagon $ABCDEFG$, given A and the center O:

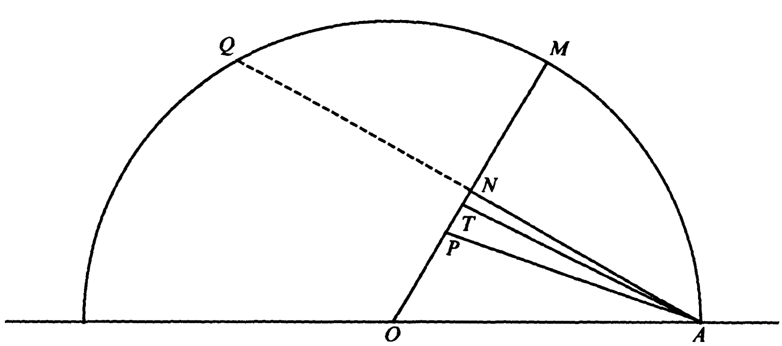

FIG. 2. Plemelj's construction.

Draw the circle with center O passing through A and on it find M so that $AM = OA$. Bisect and trisect OM at N and P, respectively, and find T on NP so that $\angle NAT = \frac{1}{3}\angle NAP$. Then AT equals the side of the required heptagon, which can easily be completed by laying off this segment around the circle.

Plemelj goes on to note that because the angle to be trisected is so small, the simplest approximate trisection will introduce an error too small to be noticeable in any practical case. If we take T to be one-third of the way from N to P, then AT will be too long by less than one part in 20000. Even if we take T at N, AT will be too short by only about one part in 400. Since AN is half of AQ, the side of an inscribed equilateral triangle, we have the rule: The side of the inscribed regular heptagon is (approximately) half of the side of the inscribed equilateral triangle. According to Tropfke **[8]**, this approximation was used by the tenth-century Arabian mathematician Abul Wafa Mohamed, and was also known to Heron of Alexandria, and perhaps even to earlier mathematicians.

To validate Plemelj's construction, we must prove that $AT = OA(2\sin\pi/7)$. Since $2\cos 2\pi/7 = 2 - (2\sin\pi/7)^2$, it follows from (1) that $2\sin\pi/7$ is a root of the equation

$$(2 - X^2)^3 + (2 - X^2)^2 - 2(2 - X^2) - 1 = 0.$$

The other roots are $-2\sin\pi/7$, $\pm 2\sin 2\pi/7$, and $\pm 2\sin 3\pi/7$.
This equation factors:

$$\left(X^3 + \sqrt{7}(X^2 - 1)\right)\left(X^3 - \sqrt{7}(X^2 - 1)\right) = 0.$$

The zeros of the first factor are $2\sin\pi/7$, $-2\sin 2\pi/7$, and $-2\sin 3\pi/7$. If we write the corresponding equation in the form

$$\left(\frac{1}{X}\right)^3 - \frac{1}{X} = \frac{1}{\sqrt{7}}, \tag{3}$$

and make the standard substitution $1/X = 2\sqrt{1/3}\cos\psi$, then (3) becomes $\cos 3\psi = \sqrt{27/28}$. The desired root corresponds to the choice

$$\psi = \alpha = \tfrac{1}{3}\operatorname{arc\,cos}\sqrt{27/28} = \tfrac{1}{3}\operatorname{arc\,tan} 1/3\sqrt{3},$$

and we have finally

$$2\sin\frac{\pi}{7}\cos\alpha = \frac{1}{2}\sqrt{3}.$$

From FIGURE 2 we see that $\angle NAP = \operatorname{arc\,tan} 1/3\sqrt{3}$, so $\angle NAT = \alpha$. Hence we have $AT\cos\alpha = AN = (1/2)\sqrt{3}\,OA$. Comparing these equations, we see that $AT = OA(2\sin\pi/7)$, as required.

Note that $\angle OKP$ in FIGURE 1 is the same as $\angle NAP$ in FIGURE 2. Thus $\angle NAP$ (FIGURE 2) is the complement of $\angle OPK$ (FIGURE 1), so $\angle OPS = \pi/6 - \alpha$. Thus the trisections required in the two constructions are equivalent in the sense that either trisector can readily be constructed from the other by ruler and compass.

It is interesting to pursue the other two roots of (3). They are given, of course, by changing the determination of ψ from α to $\alpha + 2\pi/3$ and to $\alpha - 2\pi/3$. We have

$$\left(-2\sin\frac{2\pi}{7}\right)\cos\left(\alpha + \frac{2\pi}{3}\right) = \left(-2\sin\frac{3\pi}{7}\right)\cos\left(\alpha - \frac{2\pi}{3}\right) = \frac{1}{2}\sqrt{3}.$$

Absorbing the unwanted minus signs into the cosines, we find

$$2\sin\frac{2\pi}{7}\sin\left(\frac{\pi}{6} + \alpha\right) = 2\sin\frac{3\pi}{7}\sin\left(\frac{\pi}{6} - \alpha\right) = \frac{1}{2}\sqrt{3}.$$

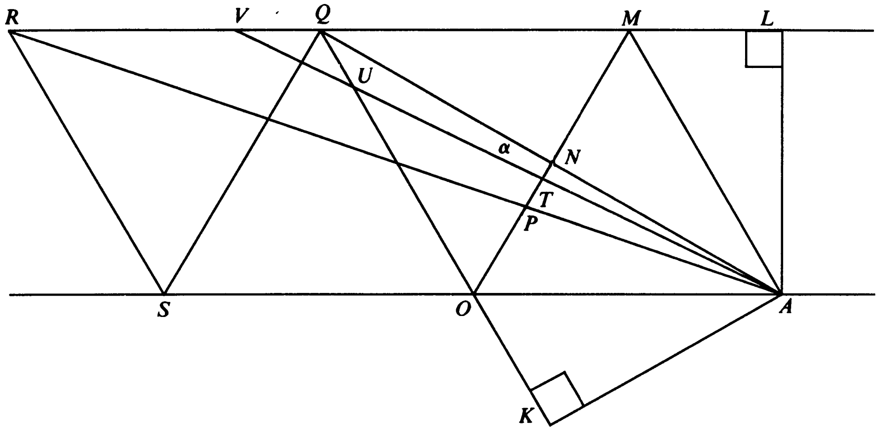

FIG. 3. Plemelj's construction extended. O, A, M, Q, R, S are points of an equilateral triangular lattice. Line AV is chosen so that $\angle QAV = 1/3\angle QAR$. Then AT, AU, and AV equal, respectively, the side, short diagonal, and long diagonal of the regular heptagon inscribed in a circle of radius OA.

In FIGURE 3 we have $AU\sin(\pi/6 + \alpha) = AK = (1/2)\sqrt{3}\,OA$ and $AV\sin(\pi/6 - \alpha) = AL = (1/2)\sqrt{3}\,OA$. Hence $AU = OA(2\sin 2\pi/7)$ and $AV = OA(2\sin 3\pi/7)$. This shows that AU and AV are equal, respectively, to the short and long diagonals of the heptagon.

Now let us consider the theory underlying these constructions. We begin by reviewing the solution of the cubic equation with real coefficients. The term in X^2 can always be removed by translating the roots and, to avoid fractions later on, we write the equation in the form

$$X^3 - 3pX + 2q = 0. \tag{4}$$

Since this equation has odd degree, it must have at least one real root. The nature of the other roots can be found by considering the quantity[4]

$$D = q^2 - p^3.$$

If $D > 0$, there are two conjugate complex roots and one real root; if $D < 0$, there are three distinct real roots; and if $D = 0$, there is one double root, q/p, and a simple root, $-2q/p$ (unless $p = q = 0$, in which case 0 is obviously a triple root). These facts are easily deduced by examining the critical points of the polynomial function in (4). Note that D cannot be negative unless p is positive.

For positive D, Cardano's formula gives the unique real root

$$\sqrt[3]{-q+\sqrt{D}} + \sqrt[3]{-q-\sqrt{D}}\,.$$

This root could be constructed from p and q by ruler and compass and the extraction of one cube root. (The second cube root can be constructed from the first by ruler and compass, since the product of the two cube roots is p.)

When $D < 0$, we have what is known as the *casus irreducibilis* and a seeming paradox: although all the roots are real, they cannot be found by radicals without leaving the real domain.[5] Cardano's formula remains valid, but it involves the cube roots of complex numbers.

To find the cube root of a complex number c, we must in general find the cube root of the real number $|c|$ and trisect the polar angle of c. In this special situation, however, only the trisection requires special tools since the absolute value of either cubic radicand is $p^{3/2}$, for which the cube root can easily be constructed.

When we know that all the roots are real, we can jump directly to the trisection problem by setting $X = 2\sqrt{p}\cos\theta$ in (4), which converts this equation into $\cos 3\theta = -qp^{-3/2}$. Note that the hypothesis $D < 0$ guarantees that $|qp^{-3/2}| < 1$, so there is a constructible angle to trisect. As before, the six determinations of θ modulo 2π lead to the three desired roots. After one value of θ has been obtained, the others can easily be found by adding and subtracting $2\pi/3$; hence one angle-trisection will suffice to find all three roots.

Conversely, any cubic equation that can be solved by an angle trisection must have all its roots real, since the method will produce three roots if it produces any. Thus we have the fundamental result:

THEOREM 1. *A real cubic equation can be solved geometrically using ruler, compass, and angle-trisector if and only if its roots are all real.*

In particular, an angle trisector will not help us to duplicate the cube, because the equation to be solved, $X^3 = 2$, has only one real root.

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4. This discussion is usually given in terms of the discriminant of the equation (i.e., the square of the product of all the differences of the roots). The discriminant of (4) is not D, but $-108D$; consequently positive D corresponds to negative discriminant and vice versa.

5. See [9, p. 180] for a proof of this fact. See [4] for an extended discussion of the solution of equations by real radicals.

We are now in position to describe precisely what can be constructed with ruler, compass, and angle-trisector.

Associated with any geometric figure (i.e., a finite collection of points, lines, and circles) in a Cartesian coordinate plane[6] is a certain subfield of the real numbers, namely, the field generated by the coordinates of all the points and the coefficients of the equations of all the lines and circles when written in the standard forms $y = mx + b$ (or $x = a$) and $x^2 + y^2 = ax + by + c$. Suppose the data of a construction problem[7] are associated with the field F_0 and the figure to be constructed is associated with the field G. The construction can be carried out with ruler and compass alone if and only if there is a tower of fields $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k$ such that $G \subseteq F_k$ and each F_i ($i = 1, \dots, k$) is obtained from F_{i-1} by adjoining the square root of some positive element of F_{i-1}. (See [**9**, p. 183ff.] for the proof of this standard result.) The intermediate fields F_i correspond to the original figure augmented by the successive points, lines, and circles used in the construction. When we allow the use of the trisector, the result is the same except that we now also allow ourselves to build F_i from F_{i-1} by adjoining a root of a cubic polynomial having coefficients in F_{i-1} and all real roots. The proof of this new theorem is virtually identical to the proof of the standard theorem, once Theorem 1 has been established.

It is convenient to describe the above situation by saying that the field F_k can be constructed from F_0.

For regular polygons we have the following theorem:

THEOREM 2. *A regular polygon of n sides can be constructed by ruler, compass, and angle-trisector if and only if the prime factorization of n is* $2^r 3^s p_1 p_2 \cdots p_k$, *where* $p_1, p_2, \dots, p_k$ *are distinct primes* (> 3) *each of the form*[8] $2^t 3^u + 1$. (*We include the possibility* $k = 0$; *i.e.*, $n = 2^r 3^s$.)

The proof uses the following lemma.

LEMMA. *Suppose K is a real field and L is a normal extension of K of degree* 3. *Then L can be constructed from K by ruler, compass, and angle-trisector.*

Proof. We know that $L = K[\beta]$, where β is a zero of some irreducible cubic polynomial $p(X)$ with coefficients in K. Since L is a normal extension of K, all of

6. We assume that the points $(0,0)$ and $(1,0)$ are points of the figure.

7. If a coordinate system is not given in the plane of the data, first construct one, choosing two given points as $(0,0)$ and $(1,0)$. The data must include two points, or at least two points of intersection must be immediately at hand, else no constructions can be carried out. (We do not accept the instruction "choose a point at random" because that complicates the analysis of constructions considerably. See [5].)

8. Altogether there are 41 primes of this form less than one million; namely, 2, 3, 5, 7, 13, 17, 19, 37, 73, 97, 109, 163, 193, 257, 433, 487, 577, 769, 1153, 1297, 1459, 2593, 2917, 3457, 3889, 10369, 12289, 17497, 18433, 39367, 52489, 65537, 139969, 147457, 209953, 331777, 472393, 629857, 746497, 839809, and 995329. It is reasonable to conjecture that there are infinitely many; probably about $9t$ of them less than 10 .

the zeros of $p(X)$ lie in L, and any one of them generates L. But one of the zeros, say γ, is real and $L = K[\gamma]$. Thus, L is a real field, and the zeros of $p(X)$ are all real. Hence the lemma follows from Theorem 1.

Proof of Theorem 2. Since the proof is very similar to the well known proof for ruler and compass alone, we give only the most important steps.

Suppose n is an integer, at least 3. Let $\xi = e^{2\pi i/n} = \cos 2\pi/n + i \sin 2\pi/n$, and $\eta = \xi + \xi^{-1} = 2\cos 2\pi/n$. The Galois group over $\mathbb{Q}$ of the cyclotomic field $\mathbb{Q}(\xi)$ is abelian with $\varphi(n)$ elements, where φ is Euler's phi function. Consequently, every field between $\mathbb{Q}$ and $\mathbb{Q}(\xi)$ is normal over $\mathbb{Q}$ with abelian Galois group. In particular, the real field $\mathbb{Q}(\eta)$ is normal over $\mathbb{Q}$. Since ξ has degree 2 over $\mathbb{Q}(\eta)$, the degree of $\mathbb{Q}(\eta)$ over $\mathbb{Q}$ is $(1/2)\varphi(n)$.

Now suppose n has the form stated in Theorem 2. Then $\varphi(n) = 2^v 3^w$ for some integers v and w, so the Galois group of $\mathbb{Q}(\eta)$ has $2^{v-1}3^w$ elements. This group will, therefore, have a composition series of length $v + w - 1$ with all quotients isomorphic either to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Correspondingly, there is a tower

$$F_0 = \mathbb{Q} \subseteq F_1 \subseteq \cdots \subseteq F_{v+w-1} = \mathbb{Q}(\eta)$$

of real fields, each normal over its predecessor of degree 2 or 3. Applying the lemma, we see that the field $\mathbb{Q}(\eta)$ can be constructed using ruler, compass, and angle-trisector. This means we can construct a segment of length $\cos 2\pi/n$, and from this we can easily construct a regular n-gon. Looking back, we see that we will have to use the angle-trisector exactly w times.

Conversely, suppose a regular n-gon can be constructed with ruler, compass and trisector. Then η can be constructed, so it must lie in some field of degree $2^a 3^b$ over $\mathbb{Q}$. Hence η itself has degree $2^c 3^d$, and $\varphi(n) = 2^{c+1}3^d$. But this implies that n has the form given in the theorem.

As an application of our theory, consider the next new prime, $13 = 2^2 \cdot 3 + 1$. Theorem 2 tells us that the regular triskaidecagon can be constructed using one angle trisection. There are many ways to proceed; none seem geometrically perspicuous.

The numbers $2\cos 2\pi k/13$, $k = 1, \ldots, 6$ are the zeros of the polynomial

$$X^6 + X^5 - 5X^4 - 4X^3 + 6X^2 + 3X - 1,$$

which factors over the field $\mathbb{Q}(\sqrt{13})$ to

$$\left(X^3 - X - 1 + \lambda(X^2 - 1)\right)\left(X^3 - X - 1 + \bar{\lambda}(X^2 - 1)\right),$$

where $\lambda = (1 - \sqrt{13})/2$ and $\bar{\lambda} = (1 + \sqrt{13})/2$. The first factor has the zero $2\cos 2\pi/13$. Then $2\lambda + 12\cos 2\pi/13$ is a zero of a cubic polynomial having no quadratic term which we can find as above. After considerable computation we obtain

$$12\cos\frac{2\pi}{13} = \sqrt{13} - 1 + \sqrt{104 - 8\sqrt{13}}\,\cos\frac{1}{3}\arctan\frac{\sqrt{3}\,(\sqrt{13}+1)}{7 - \sqrt{13}},$$

which leads to the following construction, quite similar to the first one for the heptagon:

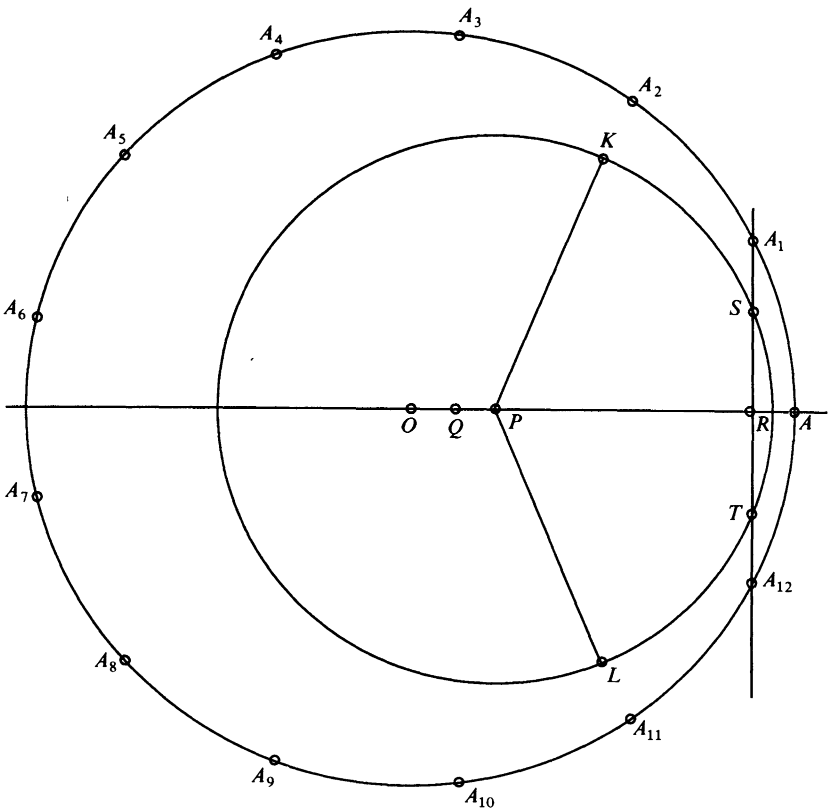

FIG. 4. Construction of a regular triskaidecagon.

Let $\mathscr{D}$ be the circle of radius 12 with center at the origin. Mark $A(12,0)$, $P(\sqrt{13}-1,0)$, $Q(5-\sqrt{13},0)$, and $R(7+\sqrt{13},0)$. Locate $K(6,\sqrt{3}(\sqrt{13}+1))$ and $L(6,-\sqrt{3}(\sqrt{13}+1))$, the vertices of the equilateral triangles having base QR. With center P draw the arc KL and trisect it at S and T. The line ST meets $\mathscr{D}$ at A_1 and A_{12}, vertices of the regular triskaidecagon $AA_1A_2 \cdots A_{12}$.

Note that a segment of length $\sqrt{13}$ is easily found as the hypotenuse of a right triangle with sides 2 and 3.

The figure suggests two approximate constructions. The point R appears to be very near the line ST. Indeed, the line through R perpendicular to the x-axis meets $\mathscr{D}$ at a point about twelve minutes of arc from A_1. Even closer, the line PK meets $\mathscr{D}$ at a point within three minutes of A_2.

The next new prime, 19, requires two trisections. The details are left to the reader!

Possible generalizations of Theorem 2 immediately suggest themselves. To construct a regular 11-gon, we must solve the fifth-degree equation having the root $2\cos 2\pi/11$. Can this be done by quinsecting some angle? Gauss answered all such questions. Here is his final statement on the subject [**3**, p. 450][9]:

> As a result the division of the whole circle into n [a prime] parts requires, *first*, the division of the whole circle into $n-1$ parts; *second*, the division into $n-1$ parts of another arc which can be constructed as soon as the first division is accomplished; *third*, the extraction of one square root, and it can be shown that this is always $\sqrt{n}$.

It follows easily from this that a regular n-gon (n need no longer be prime) can be constructed if, in addition to ruler and compass, equipment is available to p-sect any angle for every prime p that divides $\varphi(n)$. Thus, ruler, compass, and angle quinsector will suffice to construct a regular 11-gon, 41-gon, or 101-gon.

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