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Proposal for correction of the SCR calculation bias in Solvency II

Martin Hampel · Dietmar Pfeifer

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Abstract In the context of Solvency II the Solvency Capital Requirement (SCR) is a well known financial demand which will have to be fulfilled by all European insurance companies to assure a theoretical ruin probability of 0.005 or less.

A standard formula for the calculation of the SCR will be provided. Its current state is given by the Technical Specifications of the 5th Quantitative Impact Study. Every European insurance company will be obligated to use the provided standard formula if they do not legitimate an internal risk model.

The standard formula uses a lognormal distribution which is parameterized with a mean of 1 and a standard deviation parameter. The latter can be set corresponding to the market-wide estimations or corresponding to the data of the company.

We favor the possibility for insurance companies to take into account their individual risk situation and believe that the restriction of a mean of 1 is not appropriate. We therefore introduce a correction formula and propose its implementation into the formula for the undertaking-specific parameter. Using the correction formula leads to the same SCR as taking into account both the individual mean and the individual standard deviation.

1 Introduction

The project Solvency II pursues the harmonization and advancement of the directives for European insurers. Beside the reform of the directives a standard formula for the Solvency Capital Requirement (SCR) calculation will be provided. Every European insurance company will be obligated to use this formula if they do not legitimate an internal risk model.

M. Hampel (✉) · D. Pfeifer
Department of Mathematics, Carl von Ossietzky University Oldenburg,
Carl von Ossietzky Straße 9-11, 26111 Oldenburg, Germany
e-mail: Martin.Hampel@uni-oldenburg.de

The current state of the standard formula is given by the Technical Specifications of the 5th Quantitative Impact Study (QIS). For premium and reserve risk the SCR is given by

$$(F_X^{-1}(0.995) - 1)V = F_X^{-1}(0.995)V - V,$$

where $F_X^{-1}(0.995)$ denotes the 0.995-quantile of a lognormally distributed random variable X and V stands for the volume measure. To interpret this formula we assume the premium and reserve risk to be approximately lognormally distributed.

The SCR is therefore given by the difference of the 0.995-quantile of the (absolute) risk $X \cdot V$ and the volume measure V, which is approximately equal to the amount of premiums. Therefore X is the loss ratio. In QIS 5 the expected loss ratio is assumed to be 1. Thus, the use of the standard formula generally leads to a systematic bias in the SCR calculation. This has already been shown and discussed for the standard formula in QIS 4 in Hampel (2011). A correction is therefore needed if the SCR estimation shall adequately reflect the underlying risk in any given case.

We will show that in all economically relevant cases a standard deviation $\tilde{s}$ exists such that the usage of $\tilde{s}$ as standard deviation parameter eliminates the bias which results from the assumption of an expectation of 1. Thereby, $\tilde{s}$ is obtained by a manageable transformation of the true 0.995-quantile of the lognormally distributed random variable X. For the calculation only a simple spread sheet technique is needed, e.g. MS Excel.

This paper is organized in two main parts. First, we analyze the standard formula of QIS 5 and put it into the context of statistical distribution theory. We then derive the (transformation) formula for $\tilde{s}$ and propose to use $\tilde{s}$ as undertaking-specific parameter of the loss ratio X. We think that due to its simplicity and serviceability this approach is very valuable since there seems to be no ambition for dropping the assumption of an expectation of 1.

2 Systematic bias in QIS 5

The following assumptions are made for the sake of the examples' transparency:

Assumption 1

1. *We only consider premium and reserve risk in non-life insurance.*
2. *We do not concern ourselves with estimation problems and concentrate on the systematic bias. Thus, the parameters of the distribution are used instead of their estimations.*

In the following X denotes a lognormally distributed random variable.

Proposition 1 *For $X \sim \mathcal{LN}(\mu, \sigma^2)$ we have*

$$E(X) = e^{(\mu + \frac{\sigma^2}{2})}$$

$$\operatorname{Var}(X) = \left(e^{\sigma^2} - 1\right)e^{2\mu+\sigma^2}$$

and therefore

$$\mu = \ln(E(X)) - \frac{\sigma^2}{2}, \tag{1}$$

$$\sigma^2 = \ln\left(\frac{\operatorname{Var}(X)}{E(X)^2} + 1\right). \tag{2}$$

Proof For the first two statements see Bronstein et al. (2005) on p. 782. The last two statements follow directly. □

As can be seen by (1) and (2) the distribution of X in Proposition 1 is uniquely defined by the first two moments of X. Corresponding to CEIOPS (2010) we define $F^{-1}_{m,s^2}(u)$, $u \in (0, 1)$, as the u-quantile of a lognormally distributed random variable with expectation m and standard deviation s if we don't state otherwise. Aside, $q_u := \Phi^{-1}(u)$, $u \in (0, 1)$, denotes the u-quantile of the standard normal distribution.

Proposition 2 *The u-quantile of a lognormally distributed random variable X with expectation $m \in \mathbb{R}^+$ and standard deviation $s \in \mathbb{R}^+$ is given by*

$$F^{-1}_{m,s^2}(u) = \exp\left(q_u\sqrt{\ln\left(\frac{s^2}{m^2} + 1\right)} + \ln(m) - \frac{1}{2}\ln\left(\frac{s^2}{m^2} + 1\right)\right).$$

Alternatively the quantile is given by

$$\begin{aligned} F^{-1}_{m,s^2}(u) &= \frac{\exp\left(q_u\sqrt{\ln(\frac{s^2}{m^2} + 1)}\right)}{\sqrt{\frac{s^2}{m^2} + 1}} \cdot m \\ &= F^{-1}_{1,(\frac{s}{m})^2}(u) \cdot m. \end{aligned}$$

Proof This conclusion follows directly using (1) and (2): For $u \in (0, 1)$ we have

$$\begin{aligned} u = P(X \le t) = P\left(Z \le \frac{\ln(t) - \mu}{\sigma}\right) &= \Phi\left(\frac{\ln(t) - \mu}{\sigma}\right) \\ \Leftrightarrow \quad t &= \exp\left(\sigma\Phi^{-1}(u) + \mu\right) \\ &= \exp\left(q_u\sqrt{\ln\left(\frac{s^2}{m^2} + 1\right)} + \ln(m) - \frac{1}{2}\ln\left(\frac{s^2}{m^2} + 1\right)\right). \end{aligned}$$

Here, Z denotes a standard normally distributed random variable with its cumulative distribution function Φ. □

Lemma 1 *Let $X_i \sim \mathcal{LN}(\mu_i, \sigma_i^2)$, $i = 1, 2$. Moreover, F_X denotes the distribution function of the random variable X. For $\mu_1 < \mu_2$, $0 < \sigma_1 = \sigma_2$ the following relation holds*:

$$F_{X_1}(t) > F_{X_2}(t), \quad t \in \mathbb{R}^+.$$

Proof Using Propositions 1 and 2 we have for $u \in (0, 1)$

$$F_{X_1}^{-1}(u) = \exp(\Phi^{-1}(u)\sigma_1 + \mu_1) < \exp(\Phi^{-1}(u)\sigma_1 + \mu_2) = F_{X_2}^{-1}(u).$$

Due to the strict monotonicity and surjectivity of Φ^{-1} a $\delta > 0$ exists such that

$$t := \exp(\Phi^{-1}(u + \delta)\sigma_1 + \mu_1) = \exp(\Phi^{-1}(u)\sigma_1 + \mu_2)$$

and therefore

$$F_{X_2}(t) = u \text{ and } F_{X_1}(t) = F_{X_2}(t) + \delta.$$

Finally we get $F_{X_1}(t) = F_{X_2}(t) + \delta(t)$, $\delta(t) > 0$, for all $t \in \mathbb{R}^+$, which is equivalent to the statement. □

Theorem 1 *Let X_1 and X_2 be two lognormally distributed random variables with expectations m_1 and m_2 and standard deviations s_1 and s_2. Then a $u \in (0, 1)$ exists such that*

$$F^{-1}_{m_1, s_1^2}(u) = F^{-1}_{m_2, s_2^2}(u)$$

iff

$$F_{X_1}(t) = F_{X_2}(t) \quad \textit{for all } t \textit{ or } \sigma_1 \neq \sigma_2.$$

In case of $\sigma_1 \neq \sigma_2$ there is exactly one such $u \in (0, 1)$ and it is given by

$$u = \Phi\left(\frac{\mu_2 - \mu_1}{\sigma_1 - \sigma_2}\right).$$

In particular, if $\frac{s_1}{m_1} = \frac{s_2}{m_2}$ with $m_1 \neq m_2$ the distribution functions are shifted in the sense of Lemma 1.

Proof Corresponding to Proposition 1 and Proposition 2 the difference of the u-quantiles is zero iff

$$\begin{aligned} f(u) &:= \ln\big(F^{-1}_{m_1, s_1^2}(u)\big) - \ln\big(F^{-1}_{m_2, s_2^2}(u)\big) \\ &= q_u(\sigma_1 - \sigma_2) + \mu_1 - \mu_2 \\ &= q_u(\sigma_1 - \sigma_2) + \ln(m_1) - \frac{1}{2}\sigma_1^2 - \ln(m_2) + \frac{1}{2}\sigma_2^2 \end{aligned} \tag{3}$$

is zero. And $f(u) = 0$ iff

$$\text{(i)} \quad \mu_2 = \mu_1 \quad \text{and} \quad \sigma_1 = \sigma_2$$

$$\text{or} \quad \text{(ii)} \quad q_u = \frac{\mu_2 - \mu_1}{\sigma_1 - \sigma_2} \quad \text{and} \quad \sigma_1 \neq \sigma_2.$$

(i) Is equivalent to the equality of the distribution functions of X_1 and X_2. Because of continuity and strict monotonicity we have that ii) is equivalent to

$$u = \Phi\left(\frac{\mu_2 - \mu_1}{\sigma_1 - \sigma_2}\right) \quad \text{and} \quad \sigma_1 \neq \sigma_2.$$

Moreover, by term (3) and Proposition 1 the distribution functions are shifted in the sense of Lemma 1 if $\frac{s_1}{m_1} = \frac{s_2}{m_2}$ and $m_1 \neq m_2$. □

Theorem 1 proves that in general the assumption of an expectation of 1 results in a systematic bias in quantile calculation if $m \neq 1$. Moreover, for different coefficients of variation the distribution functions cross each other exactly once. Depending on the distortion caused by the assumption this can lead to unpleasant phenomena as shown in Example 1.

In the following the bias in quantile calculation is given by

$$B(m, s^2, u) := F^{-1}_{m,s^2}(u) - F^{-1}_{1,s^2}(u), \quad u \in (0, 1).$$

Example 1 We have a look at three hypothetical insurance companies with lognormally distributed risks $S_1, S_2, S_3 \sim S$ with expectation $E(S) = 0.33$ and variance $\text{Var}(S) = 0.48^2$. The premiums are given by 0.7, 1, 1.3 and we assume the premiums to be constant from year to year. Table 1 shows the results in SCR calculation using the standard formula of QIS 5 in comparison with the true values.

Concerning the risk of shortage of premiums, C1 is the most endangered company and for C1 the standard formula systematically underestimates the SCR. In contrast, C3 has the lowest risk of shortage and the calculation leads to a systematic overestimation.

Please note that all three companies have an expected loss ratio of less than 100%. Especially for companies with an expected loss ratio of more than 100% the underes-

Table 1 Bias in SCR calculation

	C1	C2	C3
m	0.47	0.33	0.25
s	0.69	0.48	0.37
True 0.995-quantile	4.161	2.913	2.240
0.995-quantile (QIS 5)	4.081	2.913	2.356
Bias	*0.080*	*0.000*	*−0.115*
	1.9%	*0.0%*	*−5.2%*
True SCR	2.213	1.913	1.613
SCR (QIS 5)	2.157	1.913	1.763
Bias	*0.056*	*0.000*	*−0.150*
	2.5%	*0.0%*	*−9.3%*

timation can be noticeably more substantial.[1] One might say that the overestimation is used for conservative estimation, but one should explore in how far the bias holds as a risk buffer. For example, there exist parameters which might be considered as economically relevant, such that the overestimation is substantially too large for such an argumentation. Here we do not want to discuss which choices of parameters are economically relevant since such a discussion is more exhausting and more complex than the implementation of the correction formula proposed in Theorem 2. Note that the "range of economically relevant parameters" should cover worst cases as well.

The fact that some insurance companies get an advantage of a reduction of the SCR as in the stated example cannot be justified from a mathematical point of view. The standard formula should ensure that the insurer's probability of ruin has to be lower than 0.005 as postulated. From a political point of view it might be tolerable to deflect from this (initial) directive in order to influence the behavior of the market participants. But notice that the endangered company is favored by the standard formula compared to the insurance company with the lowest risk of shortage. This circumstance is also disputable under the political point of view as the consumer protection is one central issue of the reform. Overall, the fact that the standard formula can systematically underestimate the SCR and hence can lead to irregularity of the directives in Solvency II is undesirable in terms of consistency.

We conclude that the risk buffer should be adjusted in a mathematically justified relation to the underlying risk and that it seems necessary to question the plausibility of the assumption of $m = 1$.

The question regarding the maximal bias for economically relevant cases is still unanswered. But as said before it is easier to integrate a correction formula as proposed in the next section than to dispute about the "set of economically relevant parameters".

For additional examples see (Hampel 2011).

In the next section we show that the systematic bias $B(m, s^2, u)$ can be corrected with little effort in economically relevant cases, for which we assume to fulfill the relations $s < 1.5$ and $m < 3$.

3 Correction formula

The insurance company is allowed to use an undertaking-specific parameter estimation to improve the risk assessment. QIS 5 still assumes the expected loss ratio to be 100% for premium and reserve risk.[2] Hence, it is not sufficient to predict the standard deviation of the loss ratio X. One should rather than to calculate a parameterization such that the resulting 0.995-quantile is equal to the one without an assumption of an expected loss ratio of 100%.

[1] Cf. Hampel (2011), on p. 12.

[2] CEIOPS (2010), p. 198, p. 244 and following.

Theorem 2 *Let X be a lognormally distributed random variable with expectation $m \in \mathbb{R}^+$ and standard deviation $s \in \mathbb{R}^+$. Then*

$$F^{-1}_{m,s^2}(u) = F^{-1}_{1,\tilde{s}^2}(u)$$

iff

$$\tilde{s}^2 = \exp\Big(\big(q_u \pm \sqrt{-2\ln(F^{-1}_{m,s^2}(u)) + q_u^2}\big)^2\Big) - 1 \quad \textit{and} \quad \tilde{s}^2 \in \mathbb{R}.$$

Proof Corresponding to Proposition 2 the problem of finding $\tilde{s}^2$, such that

$$F^{-1}_{m,s^2}(u) = F^{-1}_{1,\tilde{s}^2}(u) \tag{4}$$

holds, is equivalent to the problem of finding $\tilde{s}^2$ such that

$$\ln(F^{-1}_{m,s^2}(u)) = q_u\sqrt{\ln(\tilde{s}^2+1)} - \frac{1}{2}\ln(\tilde{s}^2+1).$$

By substitution $x := \sqrt{\ln(\tilde{s}^2+1)}$ we obtain

$$\ln(F^{-1}_{m,s^2}(u)) = q_u \cdot x - \frac{x^2}{2} = \frac{-1}{2}((x-q_u)^2 - q_u^2).$$

If we assume equation (4) then we get by reasons of consistency

$$x = q_u \pm \sqrt{-2\ln(F^{-1}_{m,s^2}(u)) + q_u^2} \quad \text{with} -2\ln(F^{-1}_{m,s^2}(u)) + q_u^2 \geq 0$$

and thus

$$\tilde{s}^2 = \exp\Big(\big(q_u \pm \sqrt{-2\ln(F^{-1}_{m,s^2}(u)) + q_u^2}\big)^2\Big) - 1$$

and $\tilde{s}^2 \in \mathbb{R}$.[3] The other implication holds because

$$\ln(\tilde{s}^2+1) = \Big(q_u \pm \sqrt{-2\ln(F^{-1}_{m,s^2}(u)) + q_u^2}\Big)^2$$

and therefore

$$\begin{aligned}\ln(F^{-1}_{1,\tilde{s}^2}(u)) &= q_u^2 \pm \sqrt{-2\ln(F^{-1}_{m,s^2}(u)) + q_u^2}\,q_u - \frac{1}{2}\Big(q_u^2 - 2\ln(F^{-1}_{m,s^2}(u)) + q_u^2 \\ &\qquad \pm 2q_u\sqrt{-2\ln(F^{-1}_{m,s^2}(u)) + q_u^2}\Big) \\ &= \ln(F^{-1}_{m,s^2}(u)).\end{aligned}$$

□

[3] In case of $-2\ln(F^{-1}_{m,s^2}(u)) + q_u^2 < 0$ the element x and also the element $\tilde{s}^2$ are not elements of $\mathbb{R}$ and consequently $F^{-1}_{1,\tilde{s}^2}$ is not defined. This is inconsistent with the assumption.

Table 2 Example of the correction formula

	Method 1	Method 2	True values
m	1.00	1.00	0.47
$s, \tilde{s}$	0.69	0.70	0.69
μ	−0.19	−0.20	−1.32
σ	0.62	0.63	1.07
0.995-quantile	4.0812	4.1609	4.1609
SCR	2.1568	2.2126	2.2126
Bias	0.0558	0.0000	0.0000
	2.5%	0.0%	0.0%

In general, $\tilde{s}^2$ is not unique, but this does not affect the fact that the calculated quantile is appropriate. In Appendix we determine in detail for which parameters m and s we have $\tilde{s}^2 \in \mathbb{R}$, but Lemma 2 sufficiently guarantees $\tilde{s}^2 \in \mathbb{R}$ for economically relevant cases.

Lemma 2 *Let* $(s, m) \in (0, 1.5) \times (0, 3)$, *then* $\tilde{s}^2 \in \mathbb{R}$.

Proof See Appendix. □

For an illustration we consider Example 2.

Example 2 We consider insurance company C1 from Example 1 and now compare two different methods for the calculation of the SCR. The first one is given by the standard formula of QIS 5. The second method extends the first method by implementing the correction formula and using $\tilde{s}$ as standard deviation parameter instead of the standard deviation of X. We get the results in Table 2.

As seen in the proof of Theorem 2 the solution $\tilde{s}^2$ and $\tilde{s}$ respectively are not unique. The first solution for $\tilde{s}$ is $\tilde{s}_1 = 0.70$ and the other solution is $\tilde{s}_2 = 27430.4$. Of course, both parameters lead to the same quantile. Figure 1 illustrates the results graphically.

Definition 1 Let X be a lognormally distributed loss ratio with expectation m and standard deviation s. If $\frac{s}{m} = c$ holds, where c stands for the market-wide ordinary coefficient of variation of the loss ratio, we call X a market-wide ordinary loss ratio. In particular, the underlying absolute risk is determined by $S := V \cdot X$ and also has the coefficient of variation c.

Corollary 1 *If we assume the loss ratio to be market-wide ordinary* (*equivalent* $\frac{s}{m} = c$, *where* c *is the market-wide ordinary coefficient of variation of the loss ratio*) *then we have*

$$\tilde{s}^2 = \exp\Big(\Big(q_u \pm \sqrt{-2\ln(F^{-1}_{1,c^2}(u) \cdot m) + q_u^2}\Big)^2\Big) - 1. \tag{5}$$

Proof This corollary follows directly from the Proposition 2 and Theorem 2. □

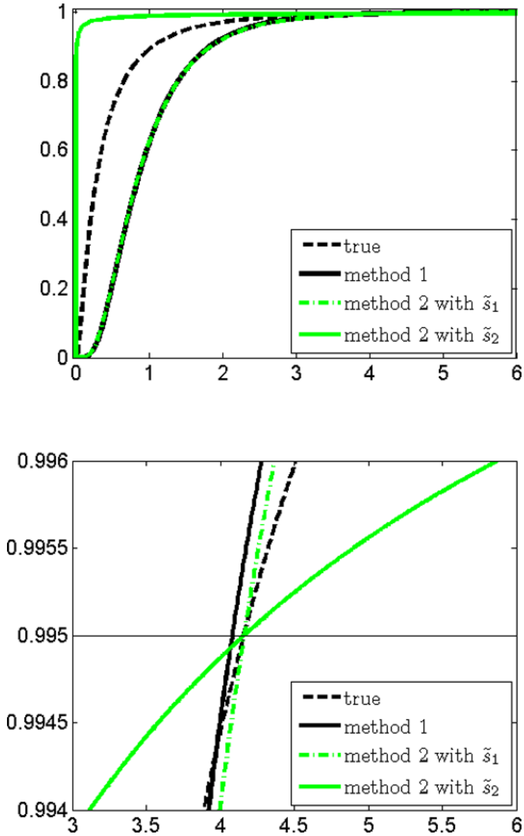

Fig. 1 Distribution functions of the different methods and the original distribution function

Hence, for a market-wide ordinary loss ratio only the estimate of the expectation is necessary. Please note that in accordance to QIS 5 the market-wide ordinary coefficient of variation of the loss ratio is equal to the market-wide ordinary standard deviation of the loss ratio. Therefore, the use of the variance formulas in QIS 5 and the stated market-wide parameters for the different lines of business lead to the market-wide ordinary coefficient of variation. This holds under the terms of QIS 5.

4 Conclusion

The standard formula for premium and reserve risk stated in QIS 5 assumes an expected loss ratio of 100%. This leads to a bias in quantile estimation if the true expectation differs from 1. Depending on the distortion caused by this assumption the bias can result in a systematic overestimation or in a systematic underestimation of the SCR. As discussed in this paper the legitimation of the assumption is questionable.

A correction is not only a benefit for the regularity of Solvency II such that the directives and the standard formula coincide. Furthermore, it eliminates the burden of an unjustifiable risk buffer and therefore might relieve many insurers.

For economically relevant cases we have shown that an $\tilde{s}^2 \in \mathbb{R}$ exists, which can easily be determined by the proposed formula, such that the quantile calculation formula in QIS 5 leads to the right quantile if $\tilde{s}$ is used as undertaking-specific parameter of the loss ratio instead of the true standard deviation. Therefore, implementing the formula for $\tilde{s}^2$ into the formula for the undertaking-specific parameter of the loss ratio eliminates the systematic bias without big efforts. In particular, for a market-wide ordinary loss ratio a correct calculation of the u-quantile can be done exclusively based on its expectation. Since the estimation of the mean is at hand, even small insurers with market-wide ordinary loss ratios can calculate their individual SCR.

Appendix

The existence of a real $\tilde{s}^2$ in Theorem 2 is important for the benefit from the correction formula. Therefore, we have to analyze for which $m \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$ the relation

$$0 \leq -2\ln(F^{-1}_{m,s^2}(u)) + q_u^2$$

holds.

Using Proposition 1 and Proposition 2 we have $\ln(F^{-1}_{m,s^2}(u)) = \sigma_{m,s} q_u + \mu_{m,s}$ with appropriate $\mu_{m,s}$ and $\sigma_{m,s}$, depending on m and s, and q_u defined as before. For simplicity in notation we write μ instead of $\mu_{m,s}$ and σ instead of $\sigma_{m,s}$. Then we have

$$\begin{aligned} -2\ln(F^{-1}_{m,s^2}(u)) + q_u^2 &= -2(\sigma q_u + \mu) + q_u^2 \\ &= \sigma^2 - 2\sigma q_u + q_u^2 - 2\ln(m) \\ &= (\sigma - q_u)^2 - \ln(m^2). \end{aligned} \tag{6}$$

For $m \leq 1$ and for all $s \in \mathbb{R}^+$ the term in (6) is obviously greater or equal to zero and therefore $\tilde{s}^2 \in \mathbb{R}$ for such m. Else ($m > 1$) there exists an interval I, given by $I = (\max\{0, q_u - \sqrt{\ln(m^2)}\}, q_u + \sqrt{\ln(m^2)})$, such that the term in (6) is smaller than zero iff $\sigma \in I$. Thus, $\tilde{s}^2$ is an element of $\mathbb{R}$ iff

$$\sigma \in I^c$$

respectively, iff

$$s^2 \in J^c, \quad J := \Big(m^2\big(e^{\max\{0, q_u - \sqrt{\ln(m^2)}\}^2} - 1\big), m^2\big(e^{\left(q_u + \sqrt{\ln(m^2)}\right)^2} - 1\big)\Big).$$

In particular, for $m \leq 3$ and $u = 0.995$ we get $q_u - \sqrt{\ln(3^2)} > 0$ and thus

$$m^2\big(e^{\max\{0, q_u - \sqrt{\ln(m^2)}\}^2} - 1\big) \geq e^{(q_u - \sqrt{\ln(3^2)})^2} - 1 =: a \approx 2.3062.$$

Therefore, we have $\tilde{s}^2 \in \mathbb{R}$ if $s \leq 1.5 < \sqrt{a}$.

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