Some Extensions of Singular Mixture Copulas

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Abstract In [13] and [14] the family of Singular Mixture Copulas was introduced. We present and discuss two extensions of these copulas. Both extensions are based on an approach introduced by [12]. We study the dependence properties of the constructed copulas and show that the resulting copulas possess differing upper and lower tail dependence coefficients.

1 Introduction

Copulas are an effective and versatile tool for studying and modeling multivariate dependence. The term copula was first used in a mathematical sense by [17], although the history of copulas can be traced back to [7] and [10]. In the 1970s several authors rediscovered copulas under different names, among them [2] who refered to them as dependence functions. Since then copulas have gained popularity in theory as well as in applications, see, e.g., [1, 6, 9, 11, 15, 16, 18].

In [5] it was suggested that the "search for families of copulas having properties desirable for specific applications" should be one of the directions of future investigation in copula theory. It was also mentioned that these families of copulas should exhibit "different asymmetries, (non-exchangeable copulas, copulas with different tail behaviour, etc.". As a contribution to this field of research [13] and [14] introduced a family of copulas – Singular Mixture Copulas. These copulas were constructed via a convex sum[1] of certain singular copulas. It was also shown in [13]

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[1] See [16], Section 3.2.

that these copulas can be used to model the dependence between the flood levels of gauging stations along the German North Sea coast. In this paper we want to present an extension of Singular Mixture Copulas and thus overcome some drawbacks of the aforementioned construction, such as the restricted support of Singular Mixture Copulas. To this end we make use of an approach that was first studied by [12] (see also [8, 15]): Let C be an arbitrary copula, then C can be extented to a parametric family of copulas $C_{\alpha,\beta}$ by setting

$$C_{\alpha,\beta}(u,v) = u^{1-\alpha}v^{1-\beta}C(u^{\alpha},v^{\beta}),$$

where $0 \leq \alpha,\beta \leq 1$. We study the resulting copulas and take a look at their mathematical properties, especially with respect to dependence.

This paper is organized as follows. In Sections 2 and 3 we summarize the construction and some important properties of Singular Mixture Copulas. In Sections 4 and 5 we present two extensions of Singular Mixture Copulas that are based on Khoudraji's device mentioned above.

2 Singular Copulas

In [13, 14] we introduced a method of constructing singular copulas. This construction uses two distribution functions F and G on $[0,1]$ which fulfill the equation

$$\alpha F(x) + (1-\alpha)G(x) = x \tag{1}$$

for all $x \in [0,1]$, where α is a constant in $(0,1)$. The function G is given by

$$G(x) = \frac{x - \alpha F(x)}{1-\alpha}. \tag{2}$$

Let X be a random variable with a continuous uniform distribution over $[0,1]$, and let I be a random variable, independent of X, with a binomial $B(1,\alpha)$-distribution. Define the random variable Y via

$$Y := I \cdot F^{-1}(X) + (1-I) \cdot G^{-1}(X). \tag{3}$$

Then the random variable Y also follows a continuous uniform distribution over $[0,1]$. The distribution function of (X,Y) is the singular copula given by

$$C_{XY}(x,y) = \alpha \min(x,F(y)) + (1-\alpha)\min(x,G(y)).$$

The following lemma gives necessary and sufficient conditions for F to guarantee that G is also a distribution function.

Lemma 2.1. *Let F be an absolutely continuous distribution function on $[0,1]$. Then G given by (2) is an absolutely continuous distribution function on $[0,1]$ if and only if $F'(x) \leq \frac{1}{\alpha}$ for all $x \in [0,1]$.*

Proof. From $F(0) = 0$ and $F(1) = 1$ it follows immediately that $G(0) = 0$ and $G(1) = 1$. From equation (2) we have

$$G'(x) = \frac{1 - \alpha F'(x)}{1 - \alpha}, \tag{4}$$

so that $G'(x) \geq 0 \Leftrightarrow F'(x) \leq \frac{1}{\alpha}$, which completes the proof. □

The assumption of absolute continuity of F is essential, as the following example shows.

Example 2.1 *Let F be the distribution function of the Cantor distribution. This function is also known as the Cantor function.*[2] *Then F is an almost everywhere differentiable distribution function on $[0,1]$ with $F'(x) = 0 \leq \frac{1}{\alpha}$ for all $x \in [0,1]$ and any $\alpha \in (0,1)$. However, F is not absolutely continuous. It holds that $F(x) = \frac{1}{2}$ for all $x \in [\frac{1}{3}, \frac{2}{3}]$. For $\alpha = \frac{3}{4}$, we can conclude that,*

$$G\left(\frac{1}{3}\right) = \frac{\frac{1}{3} - \frac{3}{4} \cdot \frac{1}{2}}{\frac{1}{4}} = \frac{4}{3} - \frac{3}{2} = -\frac{1}{6} < 0.$$

Consequently, the function G is not a distribution function on $[0,1]$.

We denote the class of functions that fulfill the properties in Lemma 2.1 by $\mathscr{F}_\alpha$, i.e.,

$$\mathscr{F}_\alpha := \{F : [0,1] \to [0,1] \mid F \text{ is abs. cont.}, F(0) = 0, F(1) = 1, 0 \leq F'(x) \leq \tfrac{1}{\alpha}\}.$$

Remark 2.1 *The copula C_{XY} is a special case of the construction presented in [4] for the choice of $f_1 = f_2 = id_{[0,1]}$, $g_1 = F$, $g_2 = G$, $A(u,v) = B(u,v) = \min(u,v)$ and $H(x,y) = \alpha x + (1-\alpha)y$. In this setting Equation (1) corresponds to the assumptions in Theorems 1 and 2 of [4].*

The following statements show some properties of the copula C_{XY} which we will use later.

Proposition 2.1. *If α goes to zero then C_{XY} converges to the Fréchet-Hoeffding upper bound M^2.*

Proof. For $\alpha = 0$ the function G is given by $G(x) = x$ and therefore C_{XY} is given by $C_{XY}(x,y) = \min(x, G(y)) = \min(x,y) = M^2(x,y)$. □

Theorem 2.1 *For any $\alpha \in (0,1)$ and any $F \in \mathscr{F}_\alpha$ the copula C_{XY} is positively quadrant dependent.*

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[2] See [3] for more information about the Cantor function.

Proof. We have to show that $C_{XY}(x,y) \geq xy$ holds for all $(x,y) \in [0,1]^2$. Due to the representation of C_{XY} we consider four cases.
Case 1:

$$C_{XY}(x,y) = \alpha x + (1-\alpha)x = x \geq xy.$$

Case 2:

$$C_{XY}(x,y) = \alpha F(y) + (1-\alpha)G(y) = \alpha F(y) + y - \alpha F(y) = y \geq xy.$$

Case 3:

$$C_{XY}(x,y) = \alpha x + (1-\alpha)G(y) = y + \alpha(x - F(y)).$$

It is easily seen that $y + \alpha(x - F(y)) \geq xy$ is equivalent to

$$\frac{\alpha x - xy}{\alpha} \geq F(y) - \frac{y}{\alpha}. \tag{5}$$

For $y \leq \alpha$ the left-hand side of (5) is positive and the right-hand side is negative, since $F'(y) \leq \frac{1}{\alpha}$ for all $y \in [0,1]$. For $y > \alpha$ the following holds

$$\frac{\alpha x - xy + y}{\alpha} = \frac{\alpha x + y(1-x)}{\alpha} > \frac{\alpha x + \alpha(1-x)}{\alpha} = 1 \geq F(y).$$

Case 4:

$$C_{XY}(x,y) = \alpha F(y) + (1-\alpha)x = x + \alpha(F(y) - x).$$

It is easily seen that $x + \alpha(F(y) - x) \geq xy$ is equivalent to

$$F(y) \geq x \cdot \frac{y - (1-\alpha)}{\alpha}. \tag{6}$$

For $y \leq 1-\alpha$ the right-hand side of (6) is negative, therefore the desired inequality holds. For $y > 1-\alpha$ we can conclude from $F'(y) \leq \frac{1}{\alpha}$ for all $y \in [0,1]$ that

$$F(y) \geq \frac{y-(1-\alpha)}{\alpha} \geq x \cdot \frac{y-(1-\alpha)}{\alpha}. \square$$

3 Singular Mixture Copulas

Consider a family $\{F_\omega\} \subset \mathscr{F}_\alpha$ of distribution functions, then – using the construction above – for a fixed ω we can construct the singular copula $\check{C}_\omega$ given by

$$\check{C}_\omega(x,y) = \alpha \min(x, F_\omega(y)) + (1-\alpha)\min(x, G_\omega(y)).$$

If Ω is a real-valued random variable and $F_\omega \in \mathscr{F}_\alpha$ for all observations ω of Ω, then the convex sum of $\{\check{C}_\omega\}$ is given by

$$\begin{aligned}\dot{C}(x,y) &= \int \check{C}_\omega(x,y)\mathbb{P}^\Omega(d\omega)\\ &= \alpha\int \min(x,F_\omega(y))\mathbb{P}^\Omega(d\omega) + (1-\alpha)\int \min(x,G_\omega(y))\mathbb{P}^\Omega(d\omega).\end{aligned}$$

These copulas were introduced in [13, 14] as Singular Mixture Copulas. A special case considered the family of distribution functions F_ω given by

$$F_\omega(y) = \omega y^2 + (1-\omega)y \tag{7}$$

with $\omega \in [-1,1]$. Let $0 < \alpha \le \frac{1}{2}$ then F_ω is an element of $\mathscr{F}_\alpha$ for all $\omega \in [-1,1]$. Let Ω be a random variable with values in $[-1,1]$ then the Singular Mixture Copula resulting from the family $\{F_\omega\}_{\omega\in[-1,1]}$ is given by

$$\begin{aligned} C_\alpha(x,y) &= \mathbb{P}(X \le x, Y \le y)\\ &= \begin{cases} x & ,(x,y)\in A_1,\\ x+\alpha\left((x-y)\left(F_\Omega(\beta)-1\right)+(y^2-y)\int_\beta^1 \omega\mathbb{P}^\Omega(d\omega)\right) & ,(x,y)\in A_2,\\ \alpha\left((x-y)F_\Omega(\beta)+y+(y^2-y)\int_\beta^1 \omega\mathbb{P}^\Omega(d\omega)\right) & \\ +(1-\alpha)\left(x+(y-x)F_\Omega(b)\right)+\alpha(y-y^2)\int_{-1}^b \omega\mathbb{P}^\Omega(d\omega) & ,(x,y)\in A_3,\\ \alpha(x-y)F_\Omega(\beta)+y+\alpha(y-y^2)\int_{-1}^\beta \omega\mathbb{P}^\Omega(d\omega) & ,(x,y)\in A_4,\\ y & ,(x,y)\in A_5, \end{cases}\end{aligned} \tag{8}$$

where $\beta = \frac{x-y}{y^2-y}$, $b = \beta\,\frac{\alpha-1}{\alpha}$ and

$$\begin{aligned} A_1 &= \left\{(x,y)\in[0,1]^2 \,|x<y^2\right\},\\ A_2 &= \left\{(x,y)\in[0,1]^2 \,|y^2\le x<\frac{-\alpha}{1-\alpha}(y-y^2)+y\right\},\\ A_3 &= \left\{(x,y)\in[0,1]^2 \,|\frac{-\alpha}{1-\alpha}(y-y^2)+y\le x<\frac{\alpha}{1-\alpha}(y-y^2)+y\right\},\\ A_4 &= \left\{(x,y)\in[0,1]^2 \,|\frac{\alpha}{1-\alpha}(y-y^2)+y\le x<2y-y^2\right\},\\ A_5 &= \left\{(x,y)\in[0,1]^2 \,|2y-y^2\le x\right\}. \end{aligned}$$

The density of the copula is given by

$$c_\alpha(x,y) = \begin{cases} 0 & ,(x,y)\in A_1,\\ \alpha f_\Omega(\beta)\frac{y^2-2xy+x}{(y^2-y)^2} & ,(x,y)\in A_2,\\ \frac{y^2-2xy+x}{(y^2-y)^2}\left(\alpha f_\Omega(\beta)+\frac{(1-\alpha)^2}{\alpha}f_\Omega(b)\right) & ,(x,y)\in A_3,\\ \alpha f_\Omega(\beta)\frac{y^2-2xy+x}{(y^2-y)^2} & ,(x,y)\in A_4,\\ 0 & ,(x,y)\in A_5. \end{cases}$$

Depending on the choice of the family of distribution functions the resulting Singular Mixture Copula can be absolutely continuous, singular or can possess an absolutely continuous part and a singular part. An example of an absolutely continuous Singular Mixture Copula was given above. If $F_\omega = F$ for all ω, then the resulting Singular Mixture Copula is singular and it is equal to the singular copula presented in Section 2. As another example consider a family of distribution functions given by

$$\tilde{F}_\omega(x) = \begin{cases} \frac{1}{2}F_\omega(2x), & 0 \le x \le \frac{1}{2}, \\ x, & \frac{1}{2} \le x \le 1, \end{cases}$$

where F_ω is given by (7). Then obviously $\tilde{F}_\omega \in \mathscr{F}_\alpha$ for all $\omega \in [-1,1]$. Figure 1 shows a scatter plot of simulated points from this copula, which we denote with $\tilde{C}$, with a uniform mixing distribution and $\alpha = \frac{1}{2}$.

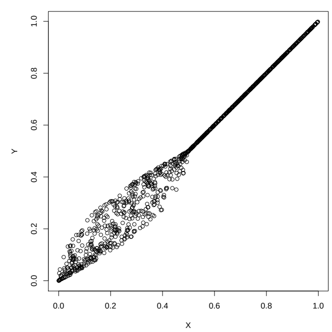

Fig. 1 Scatter plot of simulated points from the copula $\tilde{C}$.

The copula $\tilde{C}$ clearly has a singular part and an absolutely continuous part. Moreover, it is the ordinal sum of the (absolutely continuous) Singular Mixture Copula presented above and the Fréchet-Hoeffding upper bound with respect to $\{[0,\frac{1}{2}],[\frac{1}{2},1]\}$.

The following propositions show some properties of Singular Mixture Copulas.

Proposition 3.1. *If α goes to zero then $\dot{C}$ converges to M^2.*

Proof. The statement follows immediately from Proposition 2.1 and the construction of the copula $\dot{C}$. □

Proposition 3.2. *The Singular Mixture Copula $\dot{C}$ is positively quadrant dependent.*

Proof. In order to proof the statement we have to show that

$$\dot{C}(x,y) \ge xy \text{ for all } x,y \in [0,1].$$

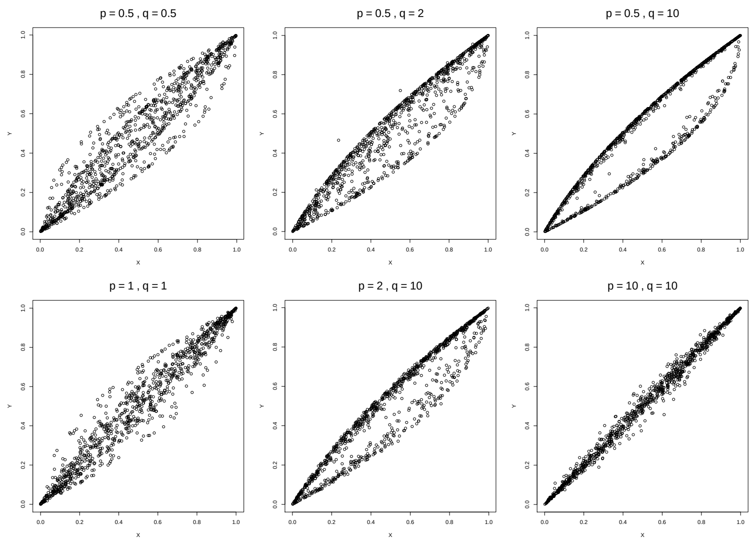

Fig. 2 Scatter plots of simulated points from a Singular Mixture Copula as in (8) for $\alpha = 0.3$ and with generalized beta mixing distribution (with shape parameters p and q) .

By construction of $\dot{C}$ we have

$$\dot{C}(x,y) = \int \check{C}_\omega(x,y)\mathbb{P}^\Omega(d\omega) \geq \int xy\mathbb{P}^\Omega(d\omega) = xy \text{ for all } x,y \in [0,1],$$

because all copulas $\check{C}_\omega$ are positively quadrant dependent (see Theorem 2.1). □

Proposition 3.3. *The copula C_α has upper and lower tail dependence given by*

$$\lambda_U = 1 - \alpha\mathbb{E}(|\Omega|) = \lambda_L.$$

Proof. The proof is straightforward. □

4 First Extension

Figure 2 shows that the support of the copula C_α is very restricted. To overcome this problem of Singular Mixture Copulas we now want to investigate an extension of the copula C_α that is based on the construction presented in [12]. Let a_1 and a_2 be two constants in $(0,1]$ and let C_α be the Singular Mixture Copula defined in Section 3, then C^*_α given by

$$C^*_\alpha(u,v) = u^{1-a_1}v^{1-a_2}C_\alpha(u^{a_1}, v^{a_2})$$

is a copula. Of course, for $a_1 = a_2 = 1$ it holds that $C^*_\alpha = C_\alpha$, so we omit this case.

Remark 4.1 *The above construction also works for $a_1 = 0$ and $a_2 = 0$, respectively. However in both cases the resulting copula is the independence copula. Exemplary for $a_1 = 0$ we receive*

$$C^*_\alpha(u,v) = uv^{1-a_2}C_\alpha(u^0, v^{a_2}) = uv^{1-a_2}v^{a_2} = uv.$$

The tail behavior of the C^*_α copulas differs from that of Singular Mixture Copulas, as the following theorems show.

Theorem 4.1 *For any $(a_1,a_2) \in (0,1]^2\backslash\{1,1\}$, the tail dependence coefficient of the copula C^*_α (as defined above) equals 0.*

Proof. By definiton,

$$\lambda_L(C^*_\alpha) = \lim_{u\searrow 0} \frac{C^*_\alpha(u,u)}{u} = \lim_{u\searrow 0} u^{1-a_1-a_2}C_\alpha(u^{a_1}, u^{a_2}).$$

Due to the piecewise representation of C_α there are several cases to consider depending on the choice of a_1 and a_2. Instead of determining the choices of a_1 and a_2 that lead to a specific case, we will simply calculate the above limit for all cases. This approach is more convenient, because - as we will see - most of the limits are the same - so there is no need for a distinction. We will denote the different cases by $A_1,\ldots,A_5$, as in the representation of C_α in Section 3.
A_1:

$$\frac{C^*_\alpha(u,u)}{u} = u^{1-a_1-a_2}\cdot u^{a_1} = u^{1-a_2} \longrightarrow 0 \text{ for } a_2 < 1.$$

For $a_2 = 1$ it would hold that $u^{a_1} \geq u^2$ for all $u \in [0,1]$. Consequently, case A_1 cannot occur when $a_2 = 1$.
A_2:

$$\begin{aligned}\frac{C^*_\alpha(u,u)}{u} &= u^{1-a_1-a_2}\left(u^{a_1} + \alpha\left((u^{a_1}-u^{a_2})(F_\Omega(\beta)-1) + (u^{2a_2}-u^{a_2})\int_\beta^1 \omega\mathbb{P}^\Omega(d\omega)\right)\right)\\ &= u^{1-a_2} + \alpha\left((u^{1-a_2}-u^{1-a_1})(F_\Omega(\beta)-1) + (u^{1-a_1+a_2}-u^{1-a_1})\int_\beta^1 \omega\mathbb{P}^\Omega(d\omega)\right)\\ &\longrightarrow 0 \text{ for } a_2 < 1.\end{aligned}$$

When $a_1 = 1$, notice that $\beta = (u-u^{a_2})/(u^{2a_2}-u^{a_2}) = (u^{1-a_2}-1)/(u^{a_2}-1) \longrightarrow 1$. For $a_2 = 1$ case A_2 cannot occur: The right-hand derivative of u^{a_1} at $u = 0$ equals infinity, therefore $u^{a_1} > \frac{-\alpha}{1-\alpha}(u-u^2)+u$ for sufficient small (positive) u.
A_3:

$$\begin{aligned}\frac{C^*_\alpha(u,u)}{u} = u^{1-a_1-a_2}\Bigg(&\alpha\left((u^{a_1}-u^{a_2})F_\Omega(\beta) + u^{a_2} + (u^{2a_2}-u^{a_2})\int_\beta^1 \omega\mathbb{P}^\Omega(d\omega)\right)\\ &+ (1-\alpha)(u^{a_1} + (u^{a_2}-u^{a_1})F_\Omega(b)) + \alpha(u^{a_2}-u^{2a_2})\int_{-1}^b \omega\mathbb{P}^\Omega(d\omega)\Bigg)\end{aligned}$$

$$= \alpha\left((u^{1-a_2}-u^{1-a_1})F_\Omega(\beta)+u^{1-a_1}+(u^{1-a_1+a_2}-u^{1-a_1})\int_\beta^1 \omega\mathbb{P}^\Omega(d\omega)\right)$$
$$+(1-\alpha)\left(u^{1-a_2}+(u^{1-a_1}-u^{1-a_2})F_\Omega(b)\right)+\alpha(u^{1-a_1}-u^{1-a_1+a_2})\int_{-1}^b \omega\mathbb{P}^\Omega(d\omega)$$
$$\longrightarrow 0 \text{ for } a_1,a_2<1.$$

For $a_1=1$ or $a_2=1$ case A_3 cannot occur: For $a_1=1$ the right-hand derivative of $\frac{-\alpha}{1-\alpha}(u^{a_2}-u^{2a_2})+u^{a_2}$ at $u=0$ equals infinity, therefore $\frac{-\alpha}{1-\alpha}(u^{a_2}-u^{2a_2})+u^{a_2}>u$ for sufficient small (positive) u. For $a_2=1$ the right-hand derivative of u^{a_1} at $u=0$ equals infinity, therefore $u^{a_1}>u+\frac{\alpha}{1-\alpha}(u-u^2)$ for sufficient small (positive) u.
A_4:

$$\frac{C^*_\alpha(u,u)}{u}=u^{1-a_1-a_2}\left(\alpha(u^{a_1}-u^{a_2})F_\Omega(\beta)+u^{a_2}+\alpha(u^{a_2}-u^{2a_2})\int_{-1}^\beta \omega\mathbb{P}^\Omega(d\omega)\right)$$
$$=\alpha(u^{1-a_2}-u^{1-a_1})F_\Omega(\beta)+u^{1-a_1}+\alpha(u^{1-a_1}-u^{1-a_1+a_2})\int_{-1}^\beta \omega\mathbb{P}^\Omega(d\omega)$$
$$\longrightarrow 0 \text{ for } a_1<1,$$

for $a_2=1$ notice that $\beta=(u^{a_1}-u)/(u^2-u)=(u^{a_1-1}-1)/(u-1)\longrightarrow-\infty$. For $a_1=1$ case A_4 cannot occur: The right-hand derivative of $\frac{\alpha}{1-\alpha}(u^{a_2}-u^{2a_2})+u^{a_2}$ at $u=0$ equals infinity, therefore $\frac{\alpha}{1-\alpha}(u^{a_2}-u^{2a_2})+u^{a_2}>u$ for sufficient small (positive) u.
A_5:

$$\frac{C^*_\alpha(u,u)}{u}=u^{1-a_1-a_2}\cdot u^{a_2}=u^{1-a_1}\longrightarrow 0 \text{ for } a_1<1.$$

For $a_1=1$ case A_5 cannot occur: The right-hand derivative of $2u^{a_2}-u^{2a_2}$ at $u=0$ equals infinity, therefore $2u^{a_2}-u^{2a_2}>u$ for sufficient small (positive) u.

Since all limits exist and are equal to zero, the proof is complete. □

Theorem 4.2 *The upper tail dependence coefficient of the copula C^*_α (as defined above) is given by*

$$\lambda_U(C^*_\alpha)=\begin{cases} a_2, & (a_1,a_2)\in B_1,\\ a_2+\alpha(a_2-a_1)(F_\Omega(\gamma)-1)-\alpha a_2\int_\gamma^1 \omega\mathbb{P}^\Omega(d\omega), & (a_1,a_2)\in B_2,\\ a_2+(a_1-a_2)\left(\alpha(1-F_\Omega(\gamma))+(1-\alpha)F_\Omega(\delta)\right) & \\ +\alpha a_2\left(\int_{-1}^\delta \omega\mathbb{P}^\Omega(d\omega)-\int_\gamma^1 \omega\mathbb{P}^\Omega(d\omega)\right), & (a_1,a_2)\in B_3,\\ a_1+\alpha(a_2-a_1)F_\Omega(\gamma)+\alpha a_2\int_{-1}^\gamma \omega\mathbb{P}^\Omega(d\omega), & (a_1,a_2)\in B_4,\end{cases}$$

where $\gamma:=\frac{a_1-a_2}{a_2}$, $\delta:=\gamma\cdot\frac{\alpha-1}{\alpha}$ *and*

$$B_1=\{(a_1,a_2)\in(0,1]^2 \mid a_1>2a_2\},$$

$$B_2 = \left\{ (a_1, a_2) \in (0,1]^2 \mid \frac{a_2}{1-\alpha} < a_1 \le 2a_2 \right\},$$
$$B_3 = \left\{ (a_1, a_2) \in (0,1]^2 \mid a_2 \frac{1-2\alpha}{1-\alpha} < a_1 \le \frac{a_2}{1-\alpha} \right\},$$
$$B_4 = \left\{ (a_1, a_2) \in (0,1]^2 \mid a_2 \frac{1-2\alpha}{1-\alpha} \ge a_1 \right\}.$$

Proof. The upper tail dependence coefficient of C^*_α is given by

$$\lambda_U(C^*_\alpha) = 2 - \lim_{u \nearrow 1} \frac{1 - C^*_\alpha(u,u)}{1-u} = 2 - \lim_{u \nearrow 1} \frac{1 - u^{2-a_1-a_2} C_\alpha(u^{a_1}, u^{a_2})}{1-u}.$$

Due to the piecewise representation of C_α (see Section 3) we have to distinguish several cases. It is easily seen that $u^{a_1} < u^{2a_2}$ for $u \in [0,1)$ if and only if $a_1 > 2a_2$. Therefore, if $(a_1,a_2) \in B_1$ then $C_\alpha(u^{a_1}, u^{a_2}) = u^{a_1}$ and consequently

$$\lambda_U(C^*_\alpha) = 2 - \lim_{u \nearrow 1} \frac{1 - u^{2-a_1-a_2} u^{a1}}{1-u} = a_2.$$

As a next step we have to determine (a_1,a_2) such that $u^{a_1} < -\frac{\alpha}{1-\alpha}(u^{a_2} - u^{2a_2}) + u^{a_2}$ holds for $u \in (1-\varepsilon, 1)$ for some $\varepsilon > 0$. Since both u^{a_1} and $-\frac{\alpha}{1-\alpha}(u^{a_2} - u^{2a_2}) + u^{a_2}$ are equal to 1 for $u = 1$ this can be done by comparing their derivatives at $u = 1$. It is $(u^{a_1})'(1) = a_1$ and $(-\frac{\alpha}{1-\alpha}(u^{a_2} - u^{2a_2}) + u^{a_2})'(1) = \frac{a_2}{1-\alpha}$ and consequently $u^{a_1} < -\frac{\alpha}{1-\alpha}(u^{a_2} - u^{2a_2}) + u^{a_2}$ holds for $u \in (1-\varepsilon, 1)$ for some $\varepsilon > 0$ if and only if $\frac{a_2}{1-\alpha} < a_1$. Hence, if $(a_1,a_2) \in B_2$ then $C_\alpha(u^{a_1}, u^{a_2}) = u^{a_1} + \alpha\left((u^{a_1} - u^{a_2})(F_\Omega(\beta) - 1) + (u^{2a_2} - u^{a_2}) \int_\beta^1 \omega \mathbb{P}^\Omega(d\omega)\right)$ where β is given by

$$\beta = \frac{u^{a_1} - u^{a_2}}{u^{2a_2} - u^{a_2}} \text{ with } \lim_{u \nearrow 1} \frac{u^{a_1} - u^{a_2}}{u^{2a_2} - u^{a_2}} = \frac{a_1 - a_2}{a_2} = \gamma.$$

Consequently,

$$\begin{aligned}
\lambda_U(C^*_\alpha) &= 2 - \lim_{u \nearrow 1} \frac{1 - u^{2-a_1-a_2}\left(u^{a_1} + \alpha\left((u^{a_1} - u^{a_2})(F_\Omega(\beta) - 1) + (u^{2a_2} - u^{a_2}) \int_\beta^1 \omega \mathbb{P}^\Omega(d\omega)\right)\right)}{1-u} \\
&= 2 - (2 - a_2) + \alpha(a_2 - a_1) \lim_{u \nearrow 1} (F_\Omega(\beta) - 1) - \alpha a_2 \lim_{u \nearrow 1} \int_\beta^1 \omega \mathbb{P}^\Omega(d\omega) \\
&= a_2 + \alpha\left((a_2 - a_1)(F_\Omega(\gamma) - 1) - a_2 \int_\gamma^1 \omega \mathbb{P}^\Omega(d\omega)\right).
\end{aligned}$$

With analogous arguments we can conclude that $u^{a_1} < \frac{\alpha}{1-\alpha}(u^{a_2} - u^{2a_2}) + u^{a_2}$ holds for $u \in (1-\varepsilon, 1)$ for some $\varepsilon > 0$ if and only if $a_2 \frac{1-2\alpha}{1-\alpha} < a_1$. Therefore, if $(a_1,a_2) \in B_3$ then

$$\begin{aligned}\lambda_U(C^*_\alpha) &= 2-(2-a_2)+\alpha(a_1-a_2)+\alpha(a_2-a_1)\lim_{u\nearrow 1}F_\Omega(\beta)-\alpha a_2\lim_{u\nearrow 1}\int_\beta^1 \omega\mathbb{P}^\Omega(d\omega)\\ &\quad+(1-\alpha)(a_1-a_2)\lim_{u\nearrow 1}F_\Omega(b)+\alpha a_2\lim_{u\nearrow 1}\int_{-1}^b \omega\mathbb{P}^\Omega(d\omega)\\ &= a_2+(a_1-a_2)\left(\alpha(1-F_\Omega(\gamma))+(1-\alpha)F_\Omega(\delta)\right)\\ &\quad+\alpha a_2\left(\int_{-1}^\delta \omega\mathbb{P}^\Omega(d\omega)-\int_\gamma^1 \omega\mathbb{P}^\Omega(d\omega)\right),\end{aligned}$$

where $b=\beta\cdot\frac{\alpha-1}{\alpha}$ with β as above and $\delta:=\lim_{u\nearrow 1}b=\gamma\cdot\frac{\alpha-1}{\alpha}$.

By comparing derivatives, we can conclude that $2u^{a_2}-u^{2a_2}\le u^{a_1}$ holds for $u\in(1-\varepsilon,1)$ for some $\varepsilon>0$ if and only if $a_1\le 0$ which would violate the aforementioned assumptions. Hence, if $(a_1,a_2)\in B_4$ then

$$\begin{aligned}\lambda_U(C^*_\alpha) &= 2-(2-a_1)+\alpha(a_2-a_1)\lim_{u\nearrow 1}F_\Omega(\beta)+\alpha a_2\lim_{u\nearrow 1}\int_{-1}^\beta \omega\mathbb{P}^\Omega(d\omega)\\ &= a_1+\alpha(a_2-a_1)F_\Omega(\gamma)+\alpha a_2\int_{-1}^\gamma \omega\mathbb{P}^\Omega(d\omega).\square\end{aligned}$$

Corollary 4.1 *If $a_1=a_2=a$, then the copula C^*_α has upper tail dependence given by*

$$\lambda_U(C^*_\alpha)=a\left(1-\alpha\mathbb{E}(|\Omega|)\right)=a\lambda_U(C_\alpha).$$

Proof. From Theorem 4.2 we can conclude

$$\lambda_U(C^*_\alpha)=a+\alpha a\left(\int_{-1}^0 \omega\mathbb{P}^\Omega(d\omega)-\int_0^1 \omega\mathbb{P}^\Omega(d\omega)\right)=a\left(1-\alpha\mathbb{E}(|\Omega|)\right).\square$$

Proposition 4.1. *The copula C^*_α is positively quadrant dependent.*

Proof. By the fact that C_α is positively quadrant dependent (see Proposition 3.2),

$$C^*_\alpha(u,v)=u^{1-a_1}v^{1-a_2}C_\alpha(u^{a_1},v^{a_2})\ge u^{1-a_1}v^{1-a_2}u^{a_1}v^{a_2}=uv \text{ for all } u,v\in[0,1].\square$$

Figure 3 shows that the C^*_α copulas exhibit even more asymmetry than Singular Mixture Copulas. This is not surprising since the construction used was introduced by [12] to construct asymmetric copulas from exchangeable copulas.

Moreover, this construction overcomes the drawback of a very restricted support (compare Figure 2 with Figure 3) which was a major disadvantage of Singular Mixture Copulas. Consequently, the copulas described in this section should find broader application.

The increased flexibility of the C^*_α copulas is also emphasized by the following proposition which shows that C^*_α copulas include both the Fréchet-Hoeffding upper bound and the independence copula as a limiting case.

Proposition 4.2. *The Fréchet-Hoeffding upper bound M^2 and the independence copula Π^2 are limiting cases of a series of C^*_α copulas.*

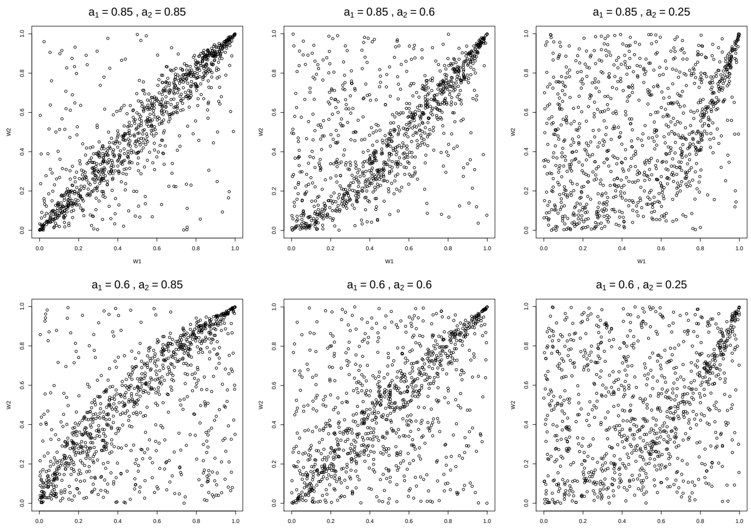

Fig. 3 Scatter plots of simulated points from the copula C^*_α for $\alpha = 0.3$ and different values for a_k. The underlying mixture distribution is a $\mathscr{U}(-1,1)$-distribution.

Proof. Let C^*_{α,a_1,a_2} denote the copula given by $C^*_{\alpha,a_1,a_2}(u,v) = u^{1-a_1}v^{1-a_2}C_\alpha(u^{a_1},v^{a_2})$, then clearly

$$\lim_{a_1\to 0}\lim_{a_2\to 0} C^*_{\alpha,a_1,a_2}(u,v) = uvC_\alpha(1,1) = uv = \Pi^2(u,v).$$

On the other hand,

$$\lim_{a_1\to 1}\lim_{a_2\to 1} C^*_{\alpha,a_1,a_2}(u,v) = C_\alpha(u,v),$$

and Proposition 3.1 showed that $\lim_{\alpha\to 0} C_\alpha(u,v) = M^2(u,v)$. $\square$

5 Second Extension

Following the approach of [12] it is also possible to construct a new copula using two Singular Mixture Copulas C_α and C_β via

$$C^\star(u,v) = C_\alpha(u^{1-a_1},v^{1-a_2})C_\beta(u^{a_1},v^{a_2})$$

with $a_1,a_2 \in [0,1]$.

Proposition 5.1. *The copula $C^\star$ is positively quadrant dependent.*

Proof. By the fact that both C_α and C_β are positively quadrant dependent (see Proposition 3.2),

$C^\star(u,v) = C_\alpha(u^{1-a_1}, v^{1-a_2})C_\beta(u^{a_1}, v^{a_2}) \geq u^{1-a_1}v^{1-a_2}u^{a_1}v^{a_2} = uv$ for all $u,v \in [0,1]$.□

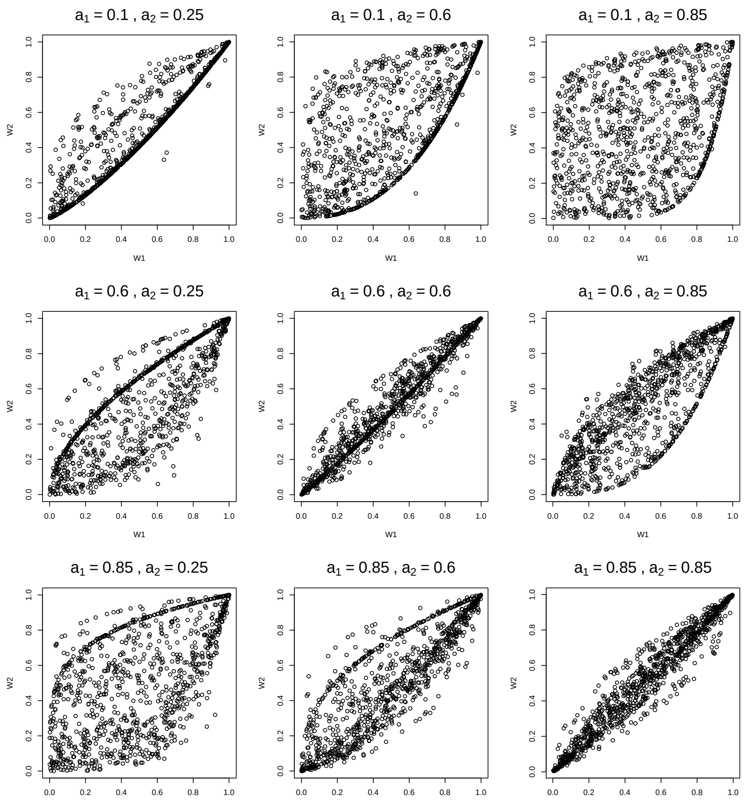

Fig. 4 Scatter plots of simulated points from the copula $C^\star$ for $\alpha = 0.3$, $\beta = 0.1$ and different values of a_k. The underlying mixture distributions are two $\mathscr{U}(-1,1)$-distributions.

Like the C^*_α copulas, the $C^\star$ copulas include both the Fréchet-Hoeffding upper bound and the independence copula as a limiting case as the following proposition shows.

Proposition 5.2. *The Fréchet-Hoeffding upper bound M^2 and the independence copula Π^2 are limiting cases of a series of $C^\star$ copulas.*

Proof. Let $C^\star_{\alpha,\beta,a_1,a_2}(u,v) = C_\alpha(u^{1-a_1}, v^{1-a_2})C_\beta(u^{a_1}, v^{a_2})$, then clearly

$$\lim_{a_1\to 0}\lim_{a_2\to 1} C^\star_{\alpha,\beta,a_1,a_2}(u,v) = C_\alpha(u,1)C_\beta(1,v) = uv = \Pi^2(u,v).$$

On the other hand,

$$\lim_{a_1\to 0}\lim_{a_2\to 0} C^{\star}_{\alpha,\beta,a_1,a_2}(u,v) = C_\alpha(u,v),$$

and Proposition 3.1 showed that $\lim_{\alpha\to 0} C_\alpha(u,v) = M^2(u,v)$. □

As Figure 4 shows, $C^{\star}$ copulas possess quite asymmetric shapes. This copula construction also overcomes – to some extent – the drawback of the restricted support. In contrast to the C^*_α construction it is possible to create copulas which distribute probability mass only on a restricted area, but this area is much less restricted than the corresponding area in the Singular Mixture Copula approach.

At first glance Figure 4 might seem to show that $C^{\star}$ can possess a singular component. Nevertheless, this is not true. Since C_α and C_β are absolutely continuous copulas it is apparent from its construction that $C^{\star}$ is absolutely continuous, too. What seems to be a singular component is in fact a very narrow band in which probability mass is distributed.

6 Concluding Remarks

In this paper we presented and discussed two extensions of Singular Mixture Copulas. These extensions are based on the approach introduced in [12]. We showed that the constructed copulas can overcome some drawbacks of Singular Mixture Copulas and thus offer a more flexible tool for modelling stochastic dependence. We also showed that the copula C^*_α possesses a form of asymmetry in the way that it exhibits no lower tail dependence yet upper tail dependence.

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