A SEMIGROUP-THEORETIC PROOF OF POISSON'S LIMIT LAW

D. Pfeifer

Communicated by Jerome A. Goldstein

1. INTRODUCTION

Operator-theoretic proofs of limit laws in probability theory date back at least to Trotter's [4] famous proof of the Central Limit Theorem, while the general semigroup-theoretic approach seems to have been introduced by Goldstein ([2], [3]), who gave proofs for the Central Limit Theorem and the Law of Large Numbers. The purpose of this paper is to present a semigroup-theoretic proof of Poisson's Limit Law (including uniform rates of convergence), which turns out to be a special case of an elementary representation theorem for certain contraction semigroups.

2. MAIN RESULT

THEOREM. Let B be a linear contraction operator on a Banach space X. Then the (bounded) linear operator A = B - I (where I denotes the identity operator) generates a contraction semigroup $T(\xi) = e^{\xi A}$, $\xi \geq 0$, and for any sequence $a_n(\xi)$ with $\lim_{n \to \infty} n \cdot a_n(\xi) = \xi$, $\xi > 0$, we have

\begin{equation}
T(\xi) = s - \lim_{n \to \infty} \left( I + a_n(\xi)A \right)^n, \text{ uniformly in } \xi > 0
\end{equation}

in every bounded interval provided $n \cdot a_n(\xi)$ converges uniformly in every bounded interval.

Moreover, for sufficiently large $n > \xi$ (such that $0 < a_n(\xi) < 1$) and every $f \in X$, we have
(2) \[ \| T(\xi)f - (I + a_n(\xi)A)^n f \| \leq |n \cdot a_n(\xi) - \xi| \| A f \| + \frac{\xi^2}{2n} \| A^2 f \| . \]

**PROOF.** The contraction property of \( T(\xi) \) follows from
\[ \| T(\xi) \| = \| e^{-\xi e^{\xi B}} \| \leq e^{\xi e^{\xi B}} \| f \| \leq 1. \]
Further, since \( n \cdot a_n(\xi) \to \xi \), for sufficiently large \( n \) as indicated, both
\[ I + a_n(\xi)A = (1 - a_n(\xi))I + a_n(\xi)B \]
and
\[ I + \frac{\xi}{n} A = (1 - \frac{\xi}{n})I + \frac{\xi}{n} B \]
also are contraction operators. Since for commuting contraction operators \( U, V \),
\[ \| U^n f - V^n f \| = \| \sum_{k=0}^{n-1} U^{n-k-1}V^{k+1}f \| \leq \sum_{k=0}^{n-1} \| U^{n-k-1}(U - V)V^k f \| = n \| U f - V f \| , \]
we get
\[ \| T(\xi)f - (I + a_n(\xi)A)^n f \| \leq n \| T(\xi)f - (I + \frac{\xi}{n} A f) + n \| a_n(\xi) - \frac{\xi}{n} \| A f \| . \]
But
\[ T(\xi) = I + \frac{\xi}{n} A + \int_0^{\xi/n} (\frac{\xi}{n} - s)T(s)A^2 ds \]
(c.f. Butzer-Berens [1], Prop. 1.1.6), hence
\[ \| T(\xi)f - (I + \frac{\xi}{n} A)f \| \leq \| A^2 f \| \int_0^{\xi/n} (\frac{\xi}{n} - s)ds = \frac{\xi^2}{2n^2} \| A^2 f \| . \]
This proves (2), from which (1) is immediately obvious.

**COROLLARY** (Poisson's Limit Law).
Let \( B(n, \xi) \), \( n \in \mathbb{N} \), \( 0 < \xi < 1 \) denote the binomial distribution over \( 0, 1, \ldots, n \) with mean \( \xi \) and let \( P(\xi) \), \( \xi > 0 \) denote the Poisson distribution over \( \mathbb{N}_0 = \{0,1,2,\ldots\} \) with mean \( \xi \). Then for every sequence \( a_n(\xi) \in (0,1) \) with \( \lim_{n \to \infty} n \cdot a_n(\xi) = \xi \), \( \xi > 0 \), we have
\[ B(n,a_n(\xi)) \xrightarrow{\text{d}} P(\xi) , \]
where \( \xrightarrow{\text{d}} \) denotes convergence in distribution. Moreover,
(3) \[ \sup_{S \subseteq \mathbb{N}_0} |P(\xi;S) - B(n,a_n(\xi);S)| \leq |n \cdot a_n(\xi) - \xi| + \frac{\xi^2}{n}. \]

**PROOF.** Let \( X = 1 \), and for \( f \in l^\infty \) write \( f = (f(0), f(1), \ldots) \). Let further the contraction \( B \) be defined by \( Bf = \varepsilon_1 * f \)

where \( \varepsilon_k, k \in \mathbb{N}_0 \), is the unit mass at \( k \) and \( * \) denotes convolution. (In fact, \( B \) is a kind of shift operator with \( (Bf)(n) = f(n-1), n \geq 1 \).) Then with \( A = B - I \),

\[ P(\xi) * f = e^{-\xi} \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \varepsilon_k * f = e^{\xi A} f, \]

i.e. \( A \) is the generator of the Poisson convolution semigroup. On the other hand,

\[ B(n,a_n(\xi)) * f = ((1-a_n(\xi))\varepsilon_0 + a_n(\xi)\varepsilon_1)^n * f = (I + a_n(\xi)A)^nf, \]

hence by the Theorem,

\[ \|P(\xi) * f - B(n,a_n(\xi)) * f\|_\infty \rightarrow 0 \quad \text{for all } f \in l^\infty \]

which implies convergence in distribution. Further, for a finite set \( S \subseteq \mathbb{N}_0 \) with \( a = \max(S) \) let \( f_S \in l^\infty \) be such that \( f_S(a-k) = 1 \) if \( k \in S \) and 0 otherwise. Then

\[ |P(\xi;S) - B(n,a_n(\xi);S)| \leq \|P(\xi) * f_S - B(n,a_n(\xi)) * f_S\|_\infty \]

\[ = \|T(\xi)f_S - (I + a_n(\xi)A)^nf_S\| \leq |n \cdot a_n(\xi) - \xi| + \frac{\xi^2}{n} \]

since for every \( f \in \{0,1\}^\mathbb{N}_0 \), \( \|Af\| \leq 1 \) and \( \|A^2f\| \leq 2 \). A simple approximation argument for general \( A \subseteq \mathbb{N}_0 \) now completes the proof of (3).
REFERENCES


Institut für Statistik und Wirtschaftsmathematik
Technical University Aachen
Wüllnerstr. 3
5100 Aachen
West-Germany