A SEMIGROUP SETTING FOR DISTANCE MEASURES IN CONNEXION WITH POISSON APPROXIMATION

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1. INTRODUCTION

A classical theorem of probability theory due to Poisson [6] says that if $B(n,p)$ are binomial distributions over $\{0,1,\ldots,n\}$ with success parameter $p \in (0,1)$ and $Po(\lambda)$ are Poisson distributions over $\mathbb{Z}^+$ with mean $\lambda > 0$, then

$$(1) \qquad B(n,\tfrac{\lambda}{n}) \xrightarrow{\mathcal{D}} Po(\lambda) \qquad\qquad (n \to \infty)$$

where $\xrightarrow{\mathcal{D}}$ means convergence in distribution. However, relation (1) does not give any impression on the speed of convergence with respect to some appropriate distance measure, say a metric on the set $\mathcal{P}$ of all probability measures over $\mathbb{Z}^+$. The two most important and well-investigated metrics in connexion with Poisson approximation are the total variation distance d defined by

$$(2) \qquad d(P,Q) = \sup_{A\subset\mathbb{Z}^+} |P(A) - Q(A)| = \frac{1}{2}\sum_{k=0}^{\infty} |P(\{k\}) - Q(\{k\})|, \qquad P,Q \in \mathcal{P}$$

and the cumulative distribution distance d_0 given by

$$(3) \qquad d_0(P,Q) = \sup_{n \in \mathbb{Z}^+} \left| \sum_{k=0}^{n} P(\{k\}) - \sum_{k=0}^{n} Q(\{k\}) \right|, \quad P,Q \in \mathscr{P}$$

(see Serfling [8],[9]). By means of d and d_0, several estimations for the rate of convergence in (1) have been worked out in the literature, using different approaches such as operator methods (Le Cam [3], Chen [1], Presman [7]), special probabilistic methods such as coupling techniques (Serfling [8],[9]), semigroup methods (Pfeifer [4], Deheuvels and Pfeifer [2]), and others, however with some emphasis on the total variation distance d. For instance, in the light of (1), we have

$$(4) \qquad d(B(n, \tfrac{\lambda}{n}) ; Po(\lambda)) \leq \frac{\lambda^2}{n}$$

$$(5) \qquad d_0(B(n, \tfrac{\lambda}{n}); Po(\lambda)) \leq \frac{1}{2} \frac{\lambda^2}{n}$$

for $n \geq \lambda$ (see Serfling [9]). It is the aim of the present paper to show that in connexion with Poisson approximation, both metrics can in a natural way be seen within the same semigroup framework introduced in Pfeifer [4] by specializing on the underlying Banach space as ℓ^1 and ℓ^∞, resp. This allows for an immediate translation of results obtained for the total variation distance d to d_0 and vice versa.

2. THE SEMIGROUP SETTING

We begin with a restatement of a well-known theorem in the theory of operator semigroups(cf. also Pfeifer [4]).

<u>THEOREM 1.</u> Let B be a linear contraction on some Banach space X. Then $A = B - I$ (I stands for the identity operator) is the (bounded) generator of a contraction semigroup $T(\xi) = e^{\xi A}$, and

(6) $\quad \|T(\xi)f - (I + \frac{\xi}{n}A)^n f\| \leq \frac{\xi^2}{2n}\|A^2 f\|, \qquad f \in X.$

For our considerations, $X = \ell^1$ or $X = \ell^\infty$ will be a convenient choice. In either case, the convolution $f * g$ for $f \in \ell^1$, $g \in X$ is defined by

(7) $$f * g(n) = \sum_{k=0}^{n} f(k)g(n-k), \qquad n \geq 0,$$

where $f = (f(0), f(1), \ldots)$. Then again $f * g \in X$, and

(8) $$\|f * g\|_X \leq \|f\|_{\ell^1} \|g\|_X .$$

Further, probability measures $P \in \mathscr{P}$ will be indentified with the probability vector $(P(\{0\}), P(\{1\}), \ldots) \in \ell^1$. Let ε_k, $k \in \mathbb{Z}^+$ denote the Dirac measure (or unit mass) at k. Then

(9) $$B f = \varepsilon_1 * f, \qquad f \in X$$

defines a contraction on X, and with $A = B - I$,

(10) $$T(\xi)f = e^{\xi A} f = \sum_{k=0}^{\infty} e^{-\xi} \frac{\xi^k}{k!} \varepsilon_k * f = Po(\xi) * f, \qquad \xi > 0,\ f \in X.$$

Also, for $n \geq \xi$,

(11) $$(I + \frac{\xi}{n}A)^n f = B(n, \frac{\xi}{n}) * f, \qquad \xi > 0,\ f \in X.$$

This gives rise to the following semigroup representation for the metrics d and d_0 (for simplicity being restricted to the situation under (1)).

<u>LEMMA.</u> Let $f = (1,0,0,\ldots) \in \ell^1$ and $g = (1,1,1,\ldots) \in \ell^\infty$ be fixed. Then for $0 < \xi \leq n$,

$$(12)\quad d(B(n,\tfrac{\xi}{n});Po(\xi)) = \frac{1}{2}\,\|T(\xi)f - (I+\tfrac{\xi}{n}A)^n f\|_{\ell^1}$$

$$(13)\quad d_0(B(n,\tfrac{\xi}{n});Po(\xi)) = \|T(\xi)g - (I+\tfrac{\xi}{n}A)^n g\|_{\ell^\infty}.$$

PROOF. Straightforward using (2),(3),(10) and (11). ■
Theorem 1 now allows for a direct estimation of d and d_0 by semigroup methods.

THEOREM 2. For $0 < \xi \leq n$, we have

$$(14)\quad d(B(n,\tfrac{\xi}{n});\ Po(\xi)) \leq \frac{\xi^2}{n}$$

$$(15)\quad d_0(B(n,\tfrac{\xi}{n});\ Po(\xi)) \leq \frac{1}{2}\,\frac{\xi^2}{n}.$$

PROOF. Obvious from the Lemma and Theorem 1 since

$$(16)\quad A^2 f = (1,-2,1,0,0,\ldots),\ \text{hence } \|A^2 f\|_{\ell^1} = 4,\ \text{and}$$

$$(17)\quad A^2 g = (1,-1,0,0,\ldots)\quad \text{hence } \|A^2 g\|_{\ell^\infty} = 1.\ ■$$

It should be pointed out that although a ℓ^∞-approach is also possible for (12) (see Pfeifer [4],[5]), the present approach is more natural in that it avoids otherwise necessary approximation arguments.

CONCLUDING REMARKS.

The preceding arguments are not only restricted to the simple situation under (1) (see for example Deheuvels and Pfeifer [2], and Pfeifer [4],[5]). In the light of the above Lemma, the results there obtained for d by semigroup methods easily also carry over to corresponding results involving d_0 by simple change of the underlying Banach space. Also, using a further Taylor expansion in Theorem 1 more sophisticated estimations for d and d_0 are possible (see [2] and [5]).

REFERENCES

[1] Chen, L.H.Y.: *An approximation theorem for convolutions of probability measures.* Ann. Prob. 3(1975), 992-999.

[2] Deheuvels, P. and Pfeifer, D.: *A semi-group approach to Poisson approximation.* (1984). To appear.

[3] Le Cam, L.: *An approximation theorem for the Poisson binomial distribution.* Pacific J. Math. 10(1960), 1181-1197.

[4] Pfeifer, D.: *A semi-group theoretic proof of Poisson's limit law.* Semigroup Forum 26(1983), 379-383.

[5] Pfeifer, D.: *Approximation-theoretic aspects of probabilistic representations for operator semigroups.* (1984). To appear in: J. Approx. Theory.

[6] Poisson, S.D.: *Recherches sur la Probabilité des Jugements en Matière Criminelle et en Matière Civile Précédées des Règles Générales du Calcul des Probabilités.* Bachelier, Paris (1837).

[7] Presman, E.L.: *Approximation of binomial distributions by infinitely divisible ones.* Theory Prob. Appl. 28(1984), 393-402.

[8] Serfling, R.J.: *A general Poisson approximation theorem.* Ann. Prob. 3(1975), 726-731.

[9] Serfling, R.J.: *Some elementary results on Poisson approximation in a sequence of Bernoulli trials.* SIAM Review 20(1978), 567-579.

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