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Semigroups and Probability:

From representation theorems to Poisson approximation

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Conference in honor of Paul Deheuvels
Paris, June 20-21, 2013

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Agenda

- Introduction
- Probabilistic representations of operator semigroups
- The semigroup approach to Poisson approximation
- Bibliography

Introduction

Important relationships between

semigroups	$\leftrightarrow$	**probability**
Feller semigroups	$\leftrightarrow$	Markov processes
convolution semigroups	$\leftrightarrow$	Poisson approximation
representation theorems	$\leftrightarrow$	Bochner and Pettis integral
approximation theorems	$\leftrightarrow$	law of large numbers

Introduction

Semigroups $\{T(t)|t \geq 0\} \subseteq \mathcal{L}[\mathcal{X},\mathcal{X}]$ of class (C_0) on a Banach space $\mathcal{X}$:

- $T(0) = \mathrm{I}$ (identity)
- $T(s+t) = T(s) \circ T(t)$ for $s,t \geq 0$
- $\lim_{t \downarrow 0} \|T(t)f - f\| = 0$ for all $f \in \mathcal{X}$

There exist constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq M e^{\omega t}$ for $t \geq 0$

Infinitesimal generator:

$Af(x) = \lim_{t \downarrow 0} \frac{1}{t}[T(t) - \mathrm{I}]f(x)$ for $f \in \mathcal{D}(A)$ and $AT(t)f = T(t)Af = \frac{d}{dt}T(t)f$

Introduction

Feller semigroups $\{T(t)|t \geq 0\}$ for standard Lévy processes $\{X_t|t \geq 0\}$:

$$T(t)f(x) = E\left[f\left(x + X_t\right)\right] \text{ for } f \in \mathcal{X} = UCB(\mathbb{R})$$

Infinitesimal generator:

$$Af(x) = \lim_{t \downarrow 0} \frac{1}{t}\left[T(t) - \mathrm{I}\right] f(x) = \lim_{t \downarrow 0} \frac{E\left[f\left(x + X_t\right) - f(x)\right]}{t} \text{ for } f \in \mathcal{D}(A)$$

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$$T(t)f(x) = E\left[f\left(x + X_t\right)\right] \text{ for } f \in \mathcal{X} = UCB(\mathbb{R})$$

Example: Brownian motion: $P^{X_t} = \mathcal{N}(\mu t, \sigma^2 t)$ with $\mu \in \mathbb{R}$, $\sigma > 0$:

$$Af(x) = \lim_{t \downarrow 0} \frac{E\left[f\left(x + X_t\right) - f(x)\right]}{t} = \mu f'(x) + \frac{\sigma^2}{2} f''(x)$$

for $f \in \mathcal{D}(A) = \{f \in \mathcal{X} | f', f'' \in \mathcal{X}\}$

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Introduction

Feller semigroups $\{T(t)|t\geq 0\}$ for standard Lévy processes $\{X_t|t\geq 0\}$:

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Example: Brownian motion: $P^{X_t}=\mathcal{N}(\mu t,\sigma^2 t)$ with $\mu\in\mathbb{R},\ \sigma>0$:

$$\begin{aligned}\frac{E\left[f\left(x+X_t\right)-f(x)\right]}{t}&=\frac{1}{t}E\left[X_t\right]f'(x)+\frac{1}{2t}E\left[X_t^2\right]f''(x)+\mathcal{O}\left(t^2\right)\\&=\mu f'(x)+\frac{\sigma^2}{2}f''(x)+\mathcal{O}\left(t^2\right)\end{aligned}$$

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Feller semigroups $\{T(t)|t \geq 0\}$ for standard Lévy processes $\{X_t|t \geq 0\}$:

$$T(t)f(x) = E\left[f\left(x + X_t\right)\right] \text{ for } f \in \mathcal{X} = UCB(\mathbb{R})$$

Example: Gamma process: $P^{X_t} = \Gamma(t,\lambda)$ with $\lambda > 0$:

$$Af(x) = \lim_{t \downarrow 0} \frac{E\left[f\left(x + X_t\right) - f(x)\right]}{t} = \int_0^\infty e^{-\lambda u} \frac{f(x+u) - f(x)}{u} du$$

for $f \in D(A) = \{f \in \mathcal{X} | f' \in \mathcal{X}\}$ [pure jump process]

Introduction

Feller semigroups $\{T(t)|t \geq 0\}$ for Lévy processes $\{X_t|t \geq 0\}$:

$$T(t)f(x) = E\left[f\left(x + X_t\right)\right] \text{ for } f \in \mathcal{X} = UCB(\mathbb{R})$$

Example: Poisson process: $P^{X_t} = \mathcal{P}(\lambda t)$ with $\lambda > 0$:

$$Af(x) = \lim_{t \downarrow 0} \frac{E\left[f\left(x + X_t\right) - f(x)\right]}{t} = \lambda\left[f(x+1) - f(x)\right]$$

for $f \in \mathcal{D}(A) = \mathcal{X}$

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Introduction

Convolution semigroups $\{T(t)|t \geq 0\}$ on $\mathcal{X} = \ell^p$ with $p \in \{1, \infty\}$:

$$T(t)f = \mathfrak{p}(t) * f \ \text{ for } f \in \mathcal{X}$$

where $\mathfrak{p}(t) = \left(\mathfrak{p}(t)(0), \mathfrak{p}(t)(1), \mathfrak{p}(t)(2), \cdots\right) \in \ell^1$ is an infinitely divisible discrete distribution, i.e.

$$\mathfrak{p}(s+t) = \mathfrak{p}(s) * \mathfrak{p}(t) \ \text{ for } s,t \geq 0.$$

Introduction

Convolution semigroups $\{T(t)|t \geq 0\}$ on $\mathcal{X} = \ell^p$ with $p \in \{1, \infty\}$:

$$T(t)f = \mathfrak{p}(t) * f \text{ for } f \in \mathcal{X}$$

Example: Negative binomial convolution semigroup: $\mathfrak{p}(t) = NB(t,p)$ with $0 < p < 1$:

$$\mathfrak{p}(t)(n) = \binom{t+n-1}{n} p^t (1-p)^n = \frac{\Gamma(t+n)}{\Gamma(t) \cdot n!} p^t (1-p)^n$$

$$Af(n) = \lim_{t \downarrow 0} \frac{1}{t}[T(t) - \mathrm{I}]f(n) = \begin{cases} -\ln(p) \cdot f(0) & \text{if } n = 0 \\ \displaystyle\sum_{k=1}^{n} \frac{(1-p)^k}{k}[f(n-k) - f(n)] & \text{if } n > 0 \end{cases} \text{ for } f \in \mathcal{X}$$

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Introduction

Convolution semigroups $\{T(t)|t \geq 0\}$ on $\mathcal{X} = \ell^p$ with $p \in \{1, \infty\}$:

$$T(t)f = \mathfrak{p}(t) * f \text{ for } f \in \mathcal{X}$$

Example: Poisson convolution semigroup: $\mathfrak{p}(t) = \mathcal{P}(\lambda t)$ with $\lambda > 0$:

$$Af(n) = \lim_{t \downarrow 0} \frac{1}{t}[T(t) - \mathrm{I}]f(n) = \begin{cases} -\lambda f(0) & \text{if } n = 0 \\ \lambda[f(n-1) - f(n)] & \text{if } n > 0 \end{cases} \text{ for } f \in \mathcal{X}$$

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Introduction

Representation theorems for $\{T(t)|t \geq 0\}$:

History (excerpt): Hille (1942), Widder (1946), Yosida (1948), Kendall (1954), Trotter (1958), Kato (1959), Chung (1962), Ditzian (1969), Butzer & Hahn (1980), Shaw (1980), Pfeifer (1984-1986)

If A is bounded:

$$T(t)f = e^{tA}f = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k f \text{ for } f \in \mathcal{D}(A) = \mathcal{X}$$

$$T(t)f = e^{tA}f = \lim_{n\to\infty}\left(\mathrm{I} + \frac{t}{n}A\right)^n f \text{ for } f \in \mathcal{D}(A) = \mathcal{X}$$

$$T(t)f = e^{tA}f = \lim_{n\to\infty}\left(\mathrm{I} - \frac{t}{n}A\right)^{-n} f \text{ for } f \in \mathcal{D}(A) = \mathcal{X}$$

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General case (A unbounded):

Hille' s first exponential formula: with $A_h = \frac{1}{h}\left[T(h) - \mathrm{I}\right]$:

$$T(t)f = \lim_{h \downarrow 0} \exp\left(tA_h\right)f \;\text{ for }\; f \in \mathcal{X}$$

Hille-Yosida: with resolvent $R(\lambda; A)f = (\lambda\mathrm{I} - A)^{-1}f = \int_0^\infty e^{-\lambda t}T(t)f\,dt,\ \lambda > \omega$:

$$T(t)f = \lim_{\lambda \to \infty} \exp\left(tB_\lambda\right)f \;\text{ for }\; f \in \mathcal{X}$$

with

$$B_\lambda = \lambda\left[\lambda R(\lambda; A) - \mathrm{I}\right] \;\text{ for }\; \lambda > \omega$$

Introduction

General case (A unbounded): with $A_h = \frac{1}{h}\left[T(h) - \mathbf{I}\right]$:

Kendall:

$$T(t)f = \lim_{h \downarrow 0}\left(\mathbf{I} + \frac{t}{n}A_{1/n}\right)^{n} f \text{ for } f \in \mathcal{X}$$

Shaw:

$$T(t)f = \lim_{h \downarrow 0}\left(\mathbf{I} - \frac{t}{n}A_{1/n}\right)^{-n} f \text{ for } f \in \mathcal{X}$$

[particular cases of Chernov's product formula]

Probabilistic representations of operator semigroups

Some notation:

$$\varphi_N(t) := E\left[t^N\right] = \sum_{n=0}^{\infty} t^n P(N=n)$$

[probability generating function for a non-negative integer valued random variable N]

$$\psi_X(t) := E\left[e^{tX}\right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} E\left[X^n\right]$$

[moment generating function for a non-negative real valued random variable X]

Probabilistic representations of operator semigroups

Main Representation Theorem (Pfeifer 1984): Let N be a non-negative integer-valued random variable with $E(N)=\zeta$ and $Y>0$ be a real-valued random variable with $E(Y)=\gamma$ such that $\varphi_N(\delta_1)<\infty$ and $\psi_Y(\delta_2)<\infty$ for some $\delta_1>1$ and $\delta_2>0$. Then for sufficiently large n, $\varphi_N\left(E\left[T\left(\frac{Y}{n}\right)\right]\right)\in\mathcal{L}[\mathcal{X},\mathcal{X}]$ with

$$\left\|\varphi_N\left(E\left[T\left(\frac{Y}{n}\right)\right]\right)\right\|\leq M\varphi_N\left(\psi_Y\left(\frac{\omega}{n}\right)\right),$$

and

$$T(\xi)f=\lim_{n\to\infty}\left\{\varphi_N\left(E\left[T\left(\frac{Y}{n}\right)\right]\right)\right\}^n f \text{ for } f\in\mathcal{X} \text{ with } \xi=\zeta\gamma.$$

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Probabilistic representations of operator semigroups

Corollary: comprises (all) known representation theorems: for example:

Choose $P^N = B(1,\xi)$ [binomial distribution] and $Y \equiv 1$, then

$$\left\{\varphi_N\left(E\left[T\left(\frac{Y}{n}\right)\right]\right)\right\}^n = \left(\mathrm{I} + \frac{\xi}{n} A_{1/n}\right)^n \quad \text{(Kendall)}$$

Choose $P^N = NB(1,\xi)$ [negative binomial distribution] and $Y \equiv 1$, then

$$\left\{\varphi_N\left(E\left[T\left(\frac{Y}{n}\right)\right]\right)\right\}^n = \left(\mathrm{I} - \frac{\xi}{n} A_{1/n}\right)^{-n} \quad \text{(Shaw)}$$

Probabilistic representations of operator semigroups

Corollary: comprises (all) known representation theorems: for example:

Hille's first exponential formula: Choose $P^N = \mathcal{P}(\xi)$ and $Y \equiv 1$, then

$$\left\{\varphi_N\left(E\left[T\left(\frac{Y}{n}\right)\right]\right)\right\}^n = \exp\left(\xi n\left[T\left(\frac{1}{n}\right)-\mathrm{I}\right]\right) = \exp\left(\xi A_{1/n}\right)$$

Probabilistic representations of operator semigroups

Corollary: comprises (all) known representation theorems: for example:

Hille-Yosida: Choose $P^N = \mathcal{P}(\xi)$ and $P^Y = \mathcal{E}(1)$, then

$$E\left[T\left(\frac{Y}{n}\right)\right]f = \int_0^\infty e^{-u}T\left(\frac{u}{n}\right)f\,du = n\int_0^\infty e^{-nv}T(v)f\,dv = nR(n;A)f \quad \text{for } f \in \mathcal{X}$$

and hence

$$\left\{\varphi_N\left(E\left[T\left(\frac{Y}{n}\right)\right]\right)\right\}^n = \exp\left(\xi n\left[nR(n;A) - \mathrm{I}\right]\right) = \exp\left(\xi B_n\right)$$

Probabilistic representations of operator semigroups

Idea of proof: Let $\{Y_n\}_{n\in\mathbb{N}}$ be i.i.d. as Y, independent of N. Consider the random sum $X=\sum_{k=1}^{N} Y_k$ with $\psi_X(t)=\varphi_N(\psi_Y(t))$. Then in some sense

$$
\begin{aligned}
E[T(X)] &= \sum_{n=0}^{\infty} E\left[T\left(\sum_{k=1}^{n} Y_k\right)\right]\cdot P(N=n) \\
&= \sum_{n=0}^{\infty} E\left[T(Y_1)\circ T(Y_2)\circ\cdots\circ T(Y_n)\right]\cdot P(N=n) \\
&= \sum_{n=0}^{\infty} E\left[T(Y)\right]^n\cdot P(N=n) = \varphi_N\left(E\left[T(Y)\right]\right)
\end{aligned}
$$

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Probabilistic representations of operator semigroups

Now take i.i.d. copies $\{X_n\}_{n\in\mathbb{N}}$ of X and consider $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ which converges a.s. (and thus also in probability) to $\xi = \zeta\gamma$. By the law of large numbers, it follows that

$$T(\xi)f = \lim_{n\to\infty} E\left[T\left(\bar{X}_n\right)\right]f = \lim_{n\to\infty} E\left[T\left(\frac{X}{n}\right)\right]^n f = \lim_{n\to\infty}\left\{\varphi_N\left(E\left[T\left(\frac{Y}{n}\right)\right]\right)\right\}^n f \quad \text{for } f \in \mathcal{X}.$$

Idea of proof: modulus of continuity

$$\omega(f,\xi,\delta) := \sup\left\{\left\|\left(T(t)-T(\xi)\right)f\right\| \,\middle|\, |t-\xi| < \delta, t > 0\right\}$$

[Note that for $f \in \mathcal{X}$ and $\varepsilon > 0$, there exists a $\delta > 0$ with $\omega(f,\xi,\delta) < \varepsilon$]

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Probabilistic representations of operator semigroups

For all $\delta > 0$, we have

$$\left\|E\left[T\left(\bar{X}_n\right)\right]f - T(\xi)f\right\| = \left\|E\left[T\left(\bar{X}_n\right) - T(\xi)\right]f\right\| \le E\left\|\left[T\left(\bar{X}_n\right) - T(\xi)\right]f\right\|$$

$$= \int \left\|\left[T\left(\bar{X}_n\right) - T(\xi)\right]f\right\| dP = \int_{\left\{\left|\bar{X}_n - \xi\right| < \delta\right\} \cup \left\{\left|\bar{X}_n - \xi\right| \ge \delta\right\}} \left\|\left[T\left(\bar{X}_n\right) - T(\xi)\right]f\right\| dP$$

$$\le \omega(f,\xi,\delta) + \|f\| \int_{\left\{\left|\bar{X}_n - \xi\right| \ge \delta\right\}} \left\{\left\|T\left(\bar{X}_n\right)\right\| + \|T(\xi)\|\right\} dP$$

$$\le \omega(f,\xi,\delta) + \|f\| P\left(\left|\bar{X}_n - \xi\right| \ge \delta\right) M \left\{E\left[e^{\omega \bar{X}_n}\right] + e^{\omega\xi}\right\} \le 2\omega(f,\xi,\delta)$$

for sufficiently large n.

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Probabilistic representations of operator semigroups

Crucial point: In which sense does $E\left[T(X)\right]$ exist, and does there hold

$$E\left[T(X+Y)\right]=E\left[T(X)\circ T(Y)\right]=E\left[T(X)\right]\circ E\left[T(Y)\right] \quad (*)$$

for independent random variables X,Y?

Bad News Theorem (Pfeifer 1984): If $\liminf_{t\downarrow z}\left\|T(t)-T(z)\right\|>0$ for some $z>0$ and the semigroup is injective in a neighbourhood of zero, then $t\mapsto T(t)$ is neither Borel-measurable nor separably valued, hence $E\left[T(X)\right]$ does in general not exist as a Bochner expectation.

Example: semigroup of translations: $T(t)f=f(\bullet+t)$ for $f\in\mathcal{X}=UCB(\mathbb{R})$ with $\left\|T(t)-T(z)\right\|=2$ whenever $t\neq z$.

Solution: (modified) Pettis integral (uses a suitable subset of the dual space of $\mathcal{L}[\mathcal{X},\mathcal{X}]$ and the Hahn-Banach-Theorem) $\rightarrow$ (*) can be justified!

Probabilistic representations of operator semigroups

Consequence: Main Representation Theorem (and extensions) can be used to find estimates for the rate of convergence, central idea:

$$E[T(X)]f - T(\xi)f = E(X-\xi)T(\xi)Af + \frac{1}{2}E(X-\xi)^2 T(\xi)A^2 f + R \approx \frac{\sigma^2}{2}T(\xi)A^2 f$$

with $\sigma^2 = Var(X)$; more precisely (among other results):

$$\left\| E[T(X)]f - T(\xi)f - \frac{\sigma^2}{2}T(\xi)A^2 f \right\| \le \frac{M}{6}\left\|A^3 f\right\| \left\{ e^{\omega\xi} E|X-\xi|^3 + \omega \left\{ E(X-\xi)^6 \right\}^{2/3} \{\psi_X(3\omega)\}^{1/3} \right\}$$

for $f \in \mathcal{D}(A^3)$ → starting point for joint work on Poisson approximation

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The semigroup approach to Poisson approximation

History (excerpt): Le Cam (1960), Franken (1964), Chen (1974), Serfling (1975, 1978), Arenbaev (1976), Shorgin (1977), Presman (1984), Barbour & Hall (1984), Serfozo (1985), Deheuvels and Pfeifer (1986-1989)

Startup framework: Let $X_1,\cdots,X_n$ be independent binomially distributed over $\{0,1\}$ with $P(X_i=1)=p_i\in(0,1)$ and T be Poisson distributed with parameter $\lambda>0$. Define $S:=\sum_{i=1}^{n}X_i$. Then for a large class of probability metrics ρ, we have

$$\rho\left(P^S,P^T\right)=\left\|\left\{\prod_{i=1}^{n}(\mathrm{I}+p_iA)-e^{\lambda A}\right\}f\right\|_\rho$$

for a suitable Banach space $\mathcal{X}$ with norm $\|\bullet\|_\rho$ and a suitable $f\in\mathcal{X}$. Here A is the generator of the Poisson convolution semigroup.

The semigroup approach to Poisson approximation

Examples:

total variation:

$$\rho\left(P^S,P^T\right)=\sup_{A\subseteq\mathbb{Z}^+}\left|P(S\in A)-P(T\in A)\right|:\quad \mathcal{X}=\ell^1,\left\|\bullet\right\|_\rho=\frac{1}{2}\left\|\bullet\right\|_{\ell^1},f=(1,0,0,\cdots)=g$$

Kolmogorov metric:

$$\rho\left(P^S,P^T\right)=\sup_{m\in\mathbb{Z}^+}\left|P(S\le m)-P(T\le m)\right|:\quad \mathcal{X}=\ell^\infty,\left\|\bullet\right\|_\rho=\left\|\bullet\right\|_{\ell^\infty},f=(1,1,1,\cdots)=h$$

Fortet-Mourier (Wasserstein) metric:

$$\rho\left(P^S,P^T\right)=\sum_{k=0}^{\infty}\left|P(S\le m)-P(T\le m)\right|:\quad \mathcal{X}=\ell^1,\left\|\bullet\right\|_\rho=\left\|\bullet\right\|_{\ell^1},f=(1,1,1,\cdots)=h$$

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The semigroup approach to Poisson approximation

Theorem (Deheuvels & Pfeifer 1986): Under the assumptions of the startup framework, it holds

$$\rho\left(P^S,P^T\right)=\left\|\left\{\prod_{i=1}^{n}(\mathrm{I}+p_iA)-e^{\lambda A}\right\}f\right\|_\rho=\frac{s}{2}\left\|e^{tA}\left(2\delta A+A^2\right)f\right\|_\rho+\mathcal{O}\left(\max\left\{v,s^2,(\lambda-t)^2\right\}\right)$$

if $s=\mathcal{O}(1)$ with $t=\sum_{k=1}^{n}p_k,\ s=\sum_{k=1}^{n}p_k^2,\ v=\sum_{k=1}^{n}p_k^3$ and $\delta=\dfrac{\lambda-t}{s}$.

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The semigroup approach to Poisson approximation

Particular cases: $\lambda = \sum_{k=1}^{n} p_k = t$ [i.e. $E(S) = E(T)$]:

$$\rho\left(P^S, P^T\right) = \left\| \left\{ \prod_{i=1}^{n} (\mathrm{I} + p_i A) - e^{\lambda A} \right\} f \right\|_\rho \sim \frac{s}{2} \left\| e^{tA} A^2 f \right\|_\rho$$

total variation:

$$\left\| e^{tA} A^2 f \right\|_\rho = \frac{1}{2} \sum_{k=0}^{\infty} e^{-t} \frac{t^{k-2}}{k!} \left| t^2 - 2kt + k(k-1) \right| = e^{-t} \left\{ \frac{t^{a-1}(a-t)}{a!} + \frac{t^{b-1}(b-t)}{b!} \right\} \sim \frac{2}{t\sqrt{2\pi e}}$$

with $a = \left\lfloor t + \frac{1}{2} + \sqrt{t + \frac{1}{4}} \right\rfloor$, $b = \left\lfloor t + \frac{1}{2} - \sqrt{t + \frac{1}{4}} \right\rfloor$, so $\rho\left(P^S, P^T\right) \sim \frac{1}{\sqrt{2\pi e}} \frac{\sum_{k=1}^{n} p_k^2}{\sum_{k=1}^{n} p_k}$

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The semigroup approach to Poisson approximation

Particular cases: $\lambda = \sum_{k=1}^{n} p_k = t$ [i.e. $E(S) = E(T)$]:

$$\rho\left(P^S, P^T\right) = \left\| \left\{ \prod_{i=1}^{n} (\mathrm{I} + p_i A) - e^{\lambda A} \right\} f \right\|_\rho \sim \frac{s}{2} \left\| e^{tA} A^2 f \right\|_\rho$$

Kolmogorov metric:

$$\left\| e^{tA} A^2 f \right\|_\rho = e^{-t} \sup_{k \in \mathbb{Z}^+} \frac{t^{k-1}}{k!} |t-k| = e^{-t} \max\left\{ \frac{t^{a-1}(a-t)}{a!}, \frac{t^{b-1}(b-t)}{b!} \right\} \sim \frac{1}{t\sqrt{2\pi e}}$$

with $a = \left[t + \frac{1}{2} + \sqrt{t + \frac{1}{4}} \right]$, $b = \left[t + \frac{1}{2} - \sqrt{t + \frac{1}{4}} \right]$, so $\rho\left(P^S, P^T\right) \sim \frac{1}{2\sqrt{2\pi e}} \frac{\sum_{k=1}^{n} p_k^2}{\sum_{k=1}^{n} p_k}$

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The semigroup approach to Poisson approximation

Particular cases: $\lambda = \sum_{k=1}^{n} p_k = t$ [i.e. $E(S) = E(T)$]:

$$\rho\left(P^S, P^T\right) = \left\| \left\{ \prod_{i=1}^{n} (\mathrm{I} + p_i A) - e^{\lambda A} \right\} f \right\|_\rho \sim \frac{s}{2} \left\| e^{tA} A^2 f \right\|_\rho$$

Fortet-Mourier (Wasserstein) metric:

$$\left\| e^{tA} A^2 f \right\|_\rho = \sum_{k=0}^{\infty} e^{-t} \frac{t^{k-1}}{k!} |t-k| = 2e^{-t} \frac{t^{\lfloor t \rfloor}}{\lfloor t \rfloor !} \sim \frac{2}{\sqrt{2\pi t}}$$

so $$\rho\left(P^S, P^T\right) \sim \frac{1}{\sqrt{2\pi}} \frac{\sum_{k=1}^{n} p_k^2}{\sqrt{\sum_{k=1}^{n} p_k}}$$

The semigroup approach to Poisson approximation

Question: What is an (asymptotically) "optimal" choice of λ?

Answer: Minimize $\Delta(t,\delta,f) := \frac{s}{2}\left\| e^{tA}\left(2\delta A + A^2\right) f \right\|_\rho$ w.r.t. δ!

The solution is of the form $\delta = \delta(t) \in \left[0,\frac{1}{2}\right]$ with $\lambda = t + \delta(t) s$

total variation:

$$\delta(t) = \begin{cases} \frac{1}{2} & \text{if } 0 < t \le \sqrt{2} \\ \frac{1}{2} - \frac{3}{2t}\frac{2-t}{3-t} & \text{if } \sqrt{2} < t \le \sqrt[3]{6} \\ 0 & \text{if } \sqrt[3]{6} < t \le 2 \end{cases}$$

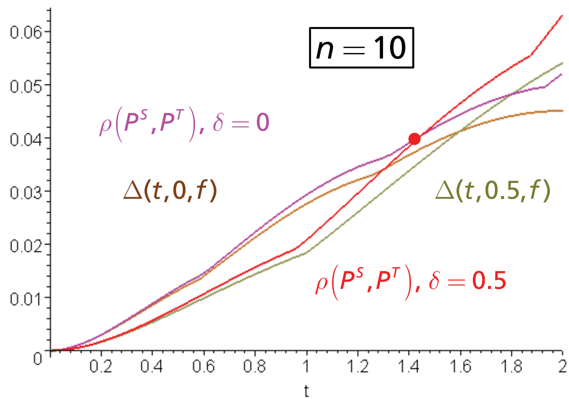
The semigroup approach to Poisson approximation

Question: What is an (asymptotically) "optimal" choice of λ?

Answer: Minimize $\Delta(t,\delta,f) := \frac{s}{2}\left\| e^{tA}\left(2\delta A + A^2\right)f\right\|_\rho$ w.r.t. δ!

Examples:

$$p_k = \frac{t}{n}, k = 1,\cdots,n,\ s = \frac{t^2}{n}$$



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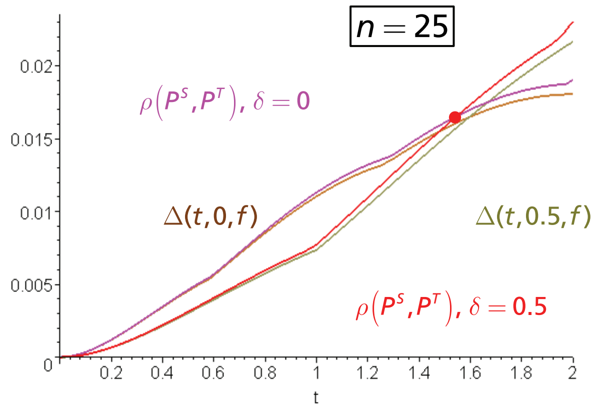
The semigroup approach to Poisson approximation

Question: What is an (asymptotically) "optimal" choice of λ?

Answer: Minimize $\Delta(t,\delta,f) := \frac{s}{2}\left\| e^{tA}\left(2\delta A + A^2\right) f \right\|_\rho$ w.r.t. δ!

Examples:

$$p_k = \frac{t}{n}, k = 1,\cdots,n,\ s = \frac{t^2}{n}$$



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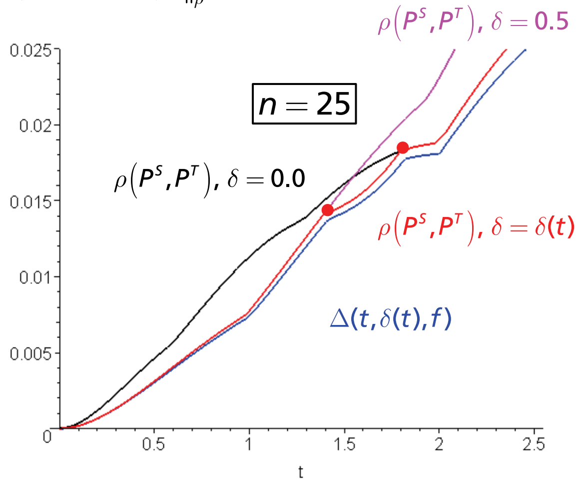
$$\delta(t) = \begin{cases} \frac{1}{2} & \text{if } 0 < t \le \sqrt{2} \\ \frac{1}{2} - \frac{3}{2t}\frac{2-t}{3-t} & \text{if } \sqrt{2} < t \le \sqrt[3]{6} \\ 0 & \text{if } \sqrt[3]{6} < t \le 2 \end{cases}$$



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The solution is of the form $\delta = \delta(t) \in \left[0, \frac{1}{2}\right]$ with $\lambda = t + \delta(t)s$

Kolmogorov metric:

$$\delta(t) = \begin{cases} \frac{1}{2} - \frac{1}{2(1+t)} & \text{if } 0 < t \leq \sqrt{3} - 1 \\ \frac{1}{2} - \frac{t}{2+t^2} & \text{if } \sqrt{3} - 1 < t \leq 1 \end{cases}$$

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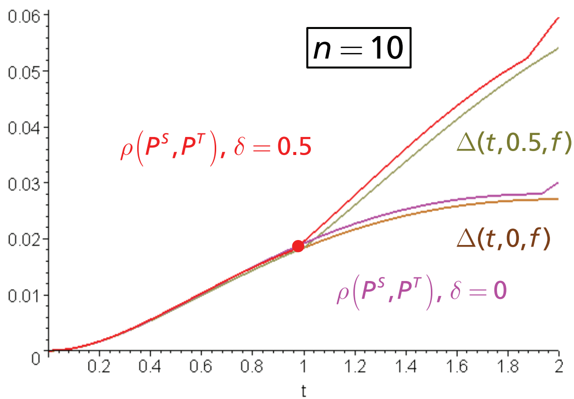

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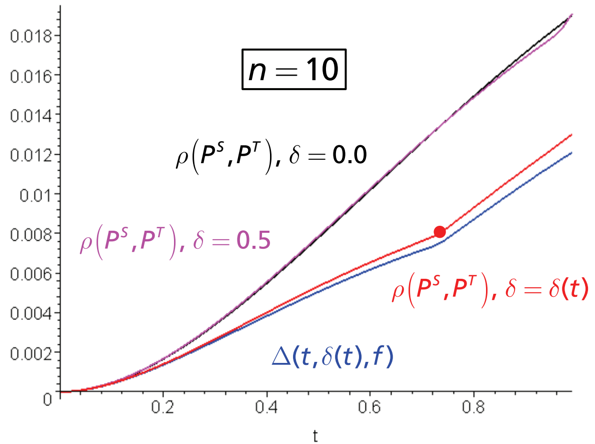

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Fortet-Mourier metric:

$$\delta(t) = \begin{cases} \frac{1}{2} & \text{if } 0 < t \le \ln 2 \\ 0 & \text{if } \ln 2 < t \le 1 \\ \frac{1}{2} - \frac{1}{2t} & \text{if } 1 < t \le \alpha \\ 0 & \text{if } \alpha < t \le 2 \end{cases}$$

where $\alpha = 1.6784...$ is the positive root of $2(1+\alpha) = e^{\alpha}$.

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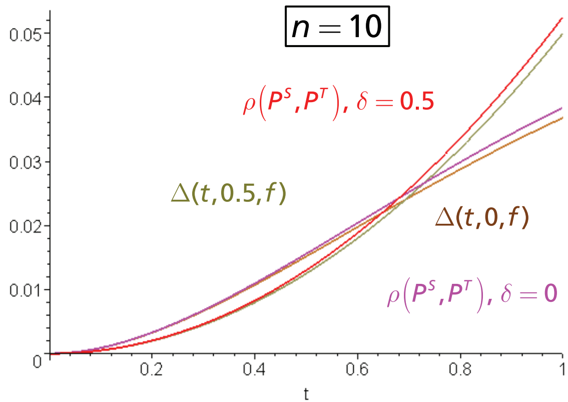

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Examples:

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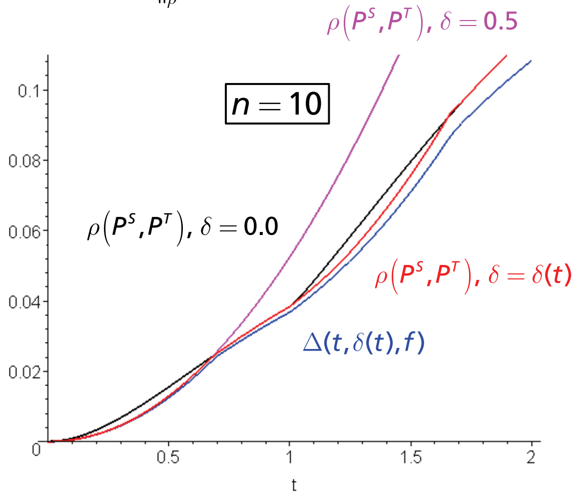

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Answer: Minimize $\Delta(t,\delta,f) := \frac{s}{2}\left\| e^{tA}\left(2\delta A + A^2\right)f \right\|_\rho$ w.r.t. δ!

The solution is of the form $\delta = \delta(t)$ with $\lambda = t + \delta(t)s$ (dependent on the underlying metric!)

For the total variation and the Fortet-Mourier metric, $\delta(t) = \frac{1}{2}$ is optimal for small values of t with $\lambda = t + \frac{s}{2} \sim \sum_{k=1}^{n} -\ln\left(1 - p_k\right)$ [Coupling approach, Serfling 1978] but e.g. not for the Kolmogorov metric!

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$$\|e^{tA}(2\delta A + A^2)g\|_{\ell^1} =$$

$$e^{-t}\sum_{k=0}^{\infty}\frac{t^{k-2}}{k!}\left|k^2 - 2k(t+\tfrac{1}{2}-\delta t) + t(t-2\delta t)\right| =$$

$$2\,e^{-t}\left\{\frac{t^{c-1}(c-(1-2\delta)t)}{c!} + \frac{t^{d-1}((1-2\delta)t-d)}{d!}\right\}$$

$$\sim \frac{2}{t\sqrt{2\pi}}\left\{\zeta\exp\left(-\frac{1}{2\zeta^2}\right) + \frac{1}{\zeta}\exp\left(-\tfrac{1}{2}\zeta^2\right)\right\}$$

$$\geq \frac{4}{t\sqrt{2\pi e}} \qquad (t\to\infty)$$

where $c = \left[\!\left[t - r + \sqrt{t+r^2} \right]\!\right]$, $d = \left[\!\left[t - r - \sqrt{t+r^2} \right]\!\right]$

and $r = \delta t - \frac{1}{2}$, $\zeta = \delta\sqrt{t} + \sqrt{1+\delta^2 t}$

$$\|e^{tA}(2\delta A + A^2)h\|_{\ell^\infty} = e^{-t}\sup_{n\geq 0}\frac{t^{n-1}}{n!}\left|(1-2\delta)t - n\right| =$$

$$e^{-t}\max\left\{\frac{t^{c-1}(c-(1-2\delta)t)}{c!}, \frac{t^{d-1}((1-2\delta)t-d)}{d!}\right\}$$

$$\sim \frac{1}{t\sqrt{2\pi}}\max\left\{\zeta\exp\left(-\frac{1}{2\zeta^2}\right), \frac{1}{\zeta}\exp\left(-\tfrac{1}{2}\zeta^2\right)\right\}$$

$$\sim \frac{\zeta}{t\sqrt{2\pi}}\exp\left(-\frac{1}{2\zeta^2}\right) \qquad (t\to\infty)$$

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$$\left\| e^{tA}(2\delta A + A^2)h \right\|_{\ell^1} = e^{-t}\sum_{k=0}^{\infty}\frac{t^{k-1}}{k!}\left|k-(1-2\delta)t\right| =$$

$$2\delta - 4\delta\sum_{k=0}^{N} e^{-t}\frac{t^k}{k!} + 2e^{-t}\frac{t^N}{N!}$$

$$\sim \frac{2}{\sqrt{2\pi(1-2\delta)t}}\exp\left(-\frac{2\delta^2 t}{1-2\delta}\right) + 2\delta - 4\delta\,\phi(-2\delta\sqrt{t}) \quad (t\to\infty)$$

where $N = [\![(1-2\delta)t]\!]$ and $\phi(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)du$,

and for $\delta > 0$,

$$4\delta\,\phi(-2\delta\sqrt{t}) \sim \frac{2}{\sqrt{2\pi t}}\exp(-2\delta^2 t) \qquad (t\to\infty).$$

The semigroup approach to Poisson approximation

Extensions:

- multinomial distributions
- point processes
- Markov chains
- mixed distributions
- Poisson-stopped sums
- weighted metrics
- signed measures
- …

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