Relaxed plasma-vacuum states in cylinders

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Abstract

The variational principle for relaxed toroidal plasma-vacuum systems with pressure is applied to axially periodic circular cylinders. More precisely, equilibria with cylindrical symmetry are investigated for their potential to be a relaxed state. Such equilibria are characterized by their pinch ratio μ, the jump δ of the pitch angle of the magnetic field across the plasma–vacuum interface, a constant plasma pressure β, and the ratio l of wall radius over interface radius. In the limit of an infinitely long cylinder the necessary and sufficient condition for the equilibrium to be a relaxed state defines one or two intervals of allowed values of the pinch ratio μ, which depend still on the other parameters. These intervals are contained in the interval known from Taylor's theory, but are generally smaller. They are shrinking with increasing plasma pressure β or increasing radius ratio l. In particular, in the field-free limit case $\beta = 1$, for l exceeding the critical value $l_c \approx 4.983$, or for a vanishing δ, these intervals are zero.

1 Introduction

Taylor's theory of plasma relaxation [1, 2] describes the evolution of a magnetic field in a conducting fluid with small but finite resistivity and viscosity in a toroidal vessel with highly conducting wall in a particular simple way: the system relaxes to the state of lowest energy compatible with conservation of the total magnetic helicity and the total toroidal magnetic flux, and with the boundary condition that the wall be a magnetic surface. In these relaxed states the magnetic field $\mathbf{B}$ is a linear force-free field

$$\operatorname{curl} \mathbf{B} = \mu \, \mathbf{B}$$

with μ below some critical value μ_T, which depends only on the vessel. Contrary to the ideal situation in plasmas with finite resistivity individual flux tubes can no longer be identified and the infinite number of invariants of ideal magnetohydrodynamics lose their significance. Total helicity, however, is not affected by these topological considerations and should remain (at least approximately) a conserved quantity. In fact, analytical considerations [3] as well as numerical studies [4, chap. 7] show that total helicity dissipation is small compared to energy dissipation.

The determination of relaxed states for arbitrary toroidal vessels is a formidable task, which in general does not succeed in an explicit way (cf. [5]). However, in axially periodic circular cylinders, representing large aspect ratio tori, relaxed states as well as μ_T can be calculated explicitly [1, 6]. These calculations are the basis of the successful application of the theory to various toroidal pinch experiments [2], [4, chap. 9].

A plasma filling the vessel up to the wall is, of course, not realistic. Therefore, in [7] relaxed states in plasma-vacuum systems have been investigated. For simplicity it has been assumed that plasma and vacuum region are separated by a sharp, highly conducting interface. The constraint of helicity conservation in the plasma region then makes sense and a correspondingly extended variational principle has been formulated. According to this extended theory relaxed states are those states which minimize energy within the constraints of conserved total helicity and total toroidal flux in the plasma region as well as of total toroidal and poloidal flux in the vacuum region. (In a cylinder "toroidal" refers to the axial direction and "poloidal" to the azimuthal direction.) In [8] the theory has been further extended in order to include plasma pressure. The theory has been applied to a plane slab in [7] and those symmetric equilibria have been identified which are relaxed states. Here, symmetric means that all equilibrium quantities vary only in the direction perpendicular to the slab. It is the purpose of the present paper to apply the extended variational principle to axially periodic circular cylinders and to identify the cylinder-symmetric relaxed states in this system. These results allow a comparison of the extended theory with Taylor's original theory without vacuum region.

Section 2 summarizes the extended variational principle and presents a slightly modified version which is more appropriate for the subsequent application. Section 3 presents cylinder-symmetric equilibria with suitable parametrization. In section 4 the modified extended variational principle is applied to these equilibria and a dispersion relation is derived, which describes the relaxed states in the space of the equilibrium parameters. The evaluation of this dispersion relation is more complicated than in Taylor's theory and ultimately has to be done numerically. The corresponding results are presented in the sections 5 and 6 for the cases of vanishing and nonvanishing plasma pressure, respectively. An appendix contains the modifications of the plane slab–results in [7] due to a nonvanishing plasma pressure.

It was a remarkable prediction in [7] that pressure-less equilibria with a smooth interface and ergodic field lines on that surface are relaxed states only if the pitch angle of the magnetic field is discontinuous across the interface. This prediction is confirmed by the present analysis of pressure-less equilibria as well as of equilibria with pressure.

Magnetic relaxation has also been considered in the somewhat artificial situation of a viscous but perfectly conducting fluid [9, 10]. In that case magnetic helicity in any flux tube is a conserved quantity during the relaxation process. As a consequence the topology of knots and links of magnetic lines is also conserved, and it is argued in [10] that in relaxed states with nontrivial topology the presence of tangential discontinuities is rather the rule than the exception. In our situation the only conserved helicity is total helicity in the plasma region. Therefore, only at the plasma-vacuum interface a tangential discontinuity may arise. We show here that in the class of cylinder-symmetric plasma-vacuum equilibria all relaxed states have in fact tangential discontinuities at the interface. No other examples of discontinuous relaxed states are known to us.

2 Extended variational principle

We consider a highly conducting plasma in a toroidal region $\mathcal{P}$ surrounded by a vacuum region $\mathcal{V}$ that extends to a rigid toroidal wall $\mathcal{W}$. It is assumed that plasma and vacuum are separated by a sharp interface $\mathcal{I}$ and that the interface as well as the wall are perfectly conducting. This implies that the interface is a magnetic surface and that magnetic fluxes through any loops in $\mathcal{I}$ or $\mathcal{W}$ are conserved. According to [7, 8] relaxed states are states of lowest energy compatible with conservation of total magnetic helicity and mass in $\mathcal{P}$ as well as of magnetic fluxes through any loops in $\mathcal{I}$ or $\mathcal{W}$.

Denoting with B the magnitude of the magnetic field vector $\mathbf{B}$, with P the plasma pressure and with $\gamma = 5/3$ the ratio of specific heats, the potential energy of the plasma-vacuum system takes the form

$$U = \int_{\mathcal{P}} d^3\tau \, \frac{1}{\gamma - 1} \, P + \int_{\mathcal{P}\cup\mathcal{V}} d^3\tau \, \frac{1}{2} \, B^2 \, .$$

With ρ denoting the mass density and observing that the quantity $S = P/\rho^\gamma$ is constant in an isentropic ideal gas, the constraint of mass conservation takes the form

$$M = \int_{\mathcal{P}} d^3\tau \, P^{1/\gamma}. \tag{1}$$

A gauge–invariant form of the helicity of the magnetic field $\mathbf{B}$ which is appropriate for toroidal domains reads

$$H = \int_{\mathcal{P}} d^3\tau \, \mathbf{A} \cdot \mathrm{curl}\, \mathbf{A} - \oint_{\mathcal{C}_s} \mathbf{dl} \cdot \mathbf{A} \oint_{\mathcal{C}_l} \mathbf{dl} \cdot \mathbf{A} \, . \tag{2}$$

Here $\mathbf{A}$ is a vector potential for $\mathbf{B}$ and $\mathcal{C}_s$, $\mathcal{C}_l$ are fixed loops the short and the long way in $\mathcal{I}$.

The variational principle now takes the more precise form: the first variation of the functional U with respect to the variables $\mathbf{A}$ and P within the constraints (1), (2) as well as the flux conditions mentioned above must vanish in a relaxed state and the second variation has to be positive. Setting the first variation of the energy functional

$$W := U - \frac{1}{2}\,\mu\, H - \nu\, M, \tag{3}$$

to zero, where μ and ν are Lagrangian multipliers, yields the following system of equations (cf. [7, 8]):

$$\operatorname{curl}\mathbf{B} = \mu\,\mathbf{B}, \quad P = \text{const} \qquad \text{in } \mathcal{P}, \tag{4}$$

$$\operatorname{curl}\mathbf{B} = 0, \quad \operatorname{div}\mathbf{B} = 0 \qquad \text{in } \mathcal{V}, \tag{5}$$

$$\mathbf{n}\cdot\mathbf{B} = 0, \quad \langle P + \frac{1}{2}B^2\rangle = 0 \qquad \text{on } \mathcal{I}, \tag{6}$$

$$\mathbf{n}\cdot\mathbf{B} = 0 \qquad \text{on } \mathcal{W}. \tag{7}$$

Here μ is a constant, $\langle\ldots\rangle := \ldots|_{\mathcal{V}} - \ldots|_{\mathcal{P}}$ denotes the jump of a quantity $\ldots$ across $\mathcal{I}$ and $\mathbf{n}$ is the outward pointing unit normal vector. The boundary conditions on $\mathbf{n}\cdot\mathbf{B}$ at $\mathcal{I}$ and $\mathcal{W}$ (zero on $\mathcal{I}$ by assumption, and zero on $\mathcal{W}$ for simplicity) are due to the flux constraints with respect to shrinkable loops in $\mathcal{I}$ and $\mathcal{W}$, respectively. The flux constraints with respect to non-shrinkable loops fix the toroidal fluxes $\Psi_P^{(t)}$ and $\Psi_V^{(t)}$ in $\mathcal{P}$ and $\mathcal{V}$, respectively, and the poloidal flux $\Psi_V^{(p)}$ in $\mathcal{V}$:

$$\Psi_P^{(t)} = \text{const}, \quad \Psi_V^{(t)} = \text{const}, \quad \Psi_V^{(p)} = \text{const}. \tag{8}$$

Given the vessel with boundary $\mathcal{W}$ the system (4)–(7) constitutes a free-boundary problem for the determination of the relaxed state $\mathbf{B}$ together with the interface $\mathcal{I}$.

Denoting equilibrium quantities, i.e. a solution of the system (4)–(7), by capital letters and perturbations by lower case letters the second variation reads [7, 8]

$$\delta^2 W = \delta^2 W_P + \delta^2 W_I + \delta^2 W_V, \tag{9}$$

$$\delta^2 W_P = \int_{\mathcal{P}} d^3\tau\,(|\operatorname{curl}\mathbf{a}|^2 - \mu\,\mathbf{a}^*\cdot\operatorname{curl}\mathbf{a}), \tag{10}$$

$$\delta^2 W_I = \int_{\mathcal{I}} d^2\sigma\,|\xi|^2\,\langle B\,\mathbf{n}\cdot\nabla B\rangle\,, \tag{11}$$

$$\delta^2 W_V = \int_{\mathcal{V}} d^3\tau\,|\operatorname{curl}\mathbf{a}|^2. \tag{12}$$

The pressure variation has already been minimized to zero, $\xi = \mathbf{n}\cdot\boldsymbol{\xi}$ denotes the normal displacement of $\mathcal{I}$ and * means complex conjugation.

In [7] the positivity condition on $\delta^2 W$ has been reformulated in terms of an eigenvalue problem. This problem reads

$$\operatorname{curl}\operatorname{curl}\mathbf{a} = \alpha \operatorname{curl}\mathbf{a} \qquad \text{in } \mathcal{P}, \tag{13}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{a} = 0 \qquad \text{in } \mathcal{V}, \tag{14}$$

$$\mu \langle \mathbf{B}\cdot\operatorname{curl}\mathbf{a}\rangle + \alpha \langle B\,\mathbf{n}\cdot\nabla B\rangle\,\xi = 0 \qquad \text{on } \mathcal{I}, \tag{15}$$

$$\mathbf{n}\times\mathbf{a}_{P,V} + \xi\,\mathbf{B}_{P,V} = 0 \qquad \text{on } \mathcal{I}, \tag{16}$$

$$\mathbf{n}\times\mathbf{a} = 0 \qquad \text{on } \mathcal{W}. \tag{17}$$

Plasma quantities and vacuum quantities at $\mathcal{I}$ are distinguished by subscripts P and V whenever necessary. The equilibrium is a relaxed state if no eigenvalue α is between zero and μ. The eigenvalue problem can likewise be written in terms of the perturbing magnetic field $\mathbf{b} = \operatorname{curl}\mathbf{a}$ (cf. [7]). However, the eigenvalue parameter enters the problem in a highly implicit form which makes its determination difficult if nontrivial geometries (such as cylindrical geometry considered below) are involved. Therefore, we present yet a slightly different method for the evaluation of the second energy variation.

We minimize $\delta^2 W$ with respect to $\mathbf{a}$ while keeping fixed the displacement ξ of the interface and according to (16) the tangential components of $\mathbf{a}$. No normalization appears in this inhomogeneous variational problem and the Euler–Lagrange equations for $\mathbf{a}$ read:

$$\operatorname{curl}\operatorname{curl}\mathbf{a} = \mu \operatorname{curl}\mathbf{a} \qquad \text{in } \mathcal{P}, \tag{18}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{a} = 0 \qquad \text{in } \mathcal{V}, \tag{19}$$

$$\mathbf{n}\times\mathbf{a} = 0 \qquad \text{on } \mathcal{W}. \tag{20}$$

Inserting (18)–(20) into the energy functional (9)–(12), $\delta^2 W$ reduces to an interface integral of the form

$$\delta^2 W = \int_{\mathcal{I}} \mathbf{d}^2\boldsymbol{\sigma}\,\langle \xi^*\mathbf{B}\cdot\mathbf{b} + |\xi|^2 B(\mathbf{n}\cdot\nabla)B\rangle. \tag{21}$$

Here, ξ is a yet undetermined test function, whereas the perturbing magnetic field $\mathbf{b} = \operatorname{curl}\mathbf{a}$ is determined by the following system of equations corresponding to (18)–(20) and (16):

$$\operatorname{curl}\mathbf{b} = \mu\,\mathbf{b} \qquad \text{in } \mathcal{P}, \tag{22}$$

$$\operatorname{curl}\mathbf{b} = 0, \quad \operatorname{div}\mathbf{b} = 0 \qquad \text{in } \mathcal{V}, \tag{23}$$

$$\mathbf{n}\cdot\mathbf{b}_{P,V} = \mathbf{B}_{P,V}\cdot\nabla\xi + \xi\,\mathbf{n}\cdot\operatorname{curl}(\mathbf{n}\times\mathbf{B}_{P,V}) \qquad \text{on } \mathcal{I}, \tag{24}$$

$$\mathbf{n}\cdot\mathbf{b} = 0 \qquad \text{on } \mathcal{W}, \tag{25}$$

as well as the flux conditions

$$\psi_P^{(t)} = \oint_{\mathcal{C}_s} \mathbf{dl} \cdot (\xi\, \mathbf{n} \times \mathbf{B}_P), \tag{26}$$

$$\psi_V^{(t)} = -\oint_{\mathcal{C}_s} \mathbf{dl} \cdot (\xi\, \mathbf{n} \times \mathbf{B}_V), \tag{27}$$

$$\psi_V^{(p)} = \oint_{\mathcal{C}_l} \mathbf{dl} \cdot (\xi\, \mathbf{n} \times \mathbf{B}_V), \tag{28}$$

which are due to (8). In (26)–(28), $\mathcal{C}_s$ and $\mathcal{C}_l$ are loops in $\mathcal{I}$ the short and the long way around the torus, oriented such that $\mathbf{n}$, a vector along $\mathcal{C}_s$, and a vector along $\mathcal{C}_l$, form a right-handed system, $\psi_P^{(t)}$ is the flux of $\mathbf{b}$ through a disk-like surface enclosed by $\mathcal{C}_s$, $\psi_V^{(t)}$ is the flux through an annular surface enclosed by $\mathcal{C}_s$ and a similar loop in $\mathcal{W}$, and $\psi_V^{(p)}$ is the flux through an annular surface enclosed by $\mathcal{C}_l$ and a similar loop in $\mathcal{W}$.

The equilibrium is now a relaxed state if the reduced functional (21) is positive for all normal displacements ξ of the interface and corresponding magnetic fields $\mathbf{b}$ determined by (22)–(28).

The case without pressure as discussed in [7] is obtained by simply putting $P = 0$ in the preceeding formulas. Taylor's theory of plasma relaxation is obtained by dropping the vacuum part in the energy functional (3) and keeping fixed the interface, which is now the wall. The only conserved flux is now $\Psi_P^{(t)}$. The equilibrium equations reduce to (4) with $P = 0$ and (7), the second variation reduces to the plasma contribution (10), and the eigenvalue problem reduces to (13) and (17).

3 Cylinder equilibria

An axially periodic circular cylinder is an approximation of a genuine torus which is simple enough to allow explicit calculations. In comparison to a torus the cylinder retains only part of the curvature effects; in contrast to a still rougher approximation, a plane slab (cf. the appendix), the cylinder retains, however, the magnetic axis. Moreover, the cylinder results can be compared with Taylor's original theory of plasma relaxation without vacuum region [1, 2].

In the following we consider solutions of the equilibrium equations (4)–(7) with cylindrical symmetry, i.e., using cylindrical coordinates (r, θ, z) all scalar quantities depend only on r. The plasma vacuum interface $\mathcal{I}$ and the wall $\mathcal{W}$ are then spherical cylinders with (normalized) radius $r = 1$ and $r = l > 1$, respectively. In order to simulate the topology of a torus all physical quantities are assumed to be periodic in the z-direction with period L. The equilibrium magnetic field has no radial component while the other two components are given

by

$$B_\theta = \begin{cases} \sqrt{1-\beta}\, J_1(\mu r) & 0 \le r \le 1\,, \\ (J_1(\mu)\cos\delta + J_0(\mu)\sin\delta)/r & 1 \le r \le l\,, \end{cases} \tag{29}$$

$$B_z = \begin{cases} \sqrt{1-\beta}\, J_0(\mu r) & 0 \le r \le 1\,, \\ J_0(\mu)\cos\delta - J_1(\mu)\sin\delta & 1 \le r \le l\,. \end{cases} \tag{30}$$

Here, $J_{0/1}$ denote Bessel functions of the first kind of order 0/1, δ measures the jump of the pitch angle of the magnetic field across the interface and β is related to the pressure constant P by

$$\beta = \frac{2\,P}{B^2|_{r=1+}} = \frac{2\,P}{J_0^2(\mu) + J_1^2(\mu)}\,. \tag{31}$$

Observe that the magnitude of the vacuum magnetic field at the interface $B^2|_{r=1+}$ does never vanish. The parameter β may vary in the interval $[0,1]$, where the boundaries describe a force-free field ($\beta = 0$) and a field-free plasma ($\beta = 1$). For later use we note the relation

$$\begin{aligned} \langle B\,\partial_r B\rangle\Big|_{r=1} &= (J_1^2(\mu) - J_0^2(\mu))\sin^2\delta - 2\,J_1(\mu)J_0(\mu)\sin\delta\cos\delta - \beta\,J_1^2(\mu) \\ &= -(J_0(\mu)\sin\delta + J_1(\mu)\cos\delta)^2 + (1-\beta)J_1^2(\mu). \end{aligned} \tag{32}$$

Up to an overall normalization, (29)–(31) is the only cylinder symmetric solution of the equilibrium equations (4)–(7).

Computing the conserved quantities we obtain for the helicity

$$H = 2\pi L\,(1-\beta)\left(\frac{1}{\mu}\,(J_0^2(\mu) + J_1^2(\mu)) - \frac{2}{\mu^2}\,J_0(\mu)J_1(\mu)\right),$$

for the total mass

$$M = \pi L\Big(\frac{1}{2}\,\beta(J_0^2(\mu) + J_1^2(\mu)\Big)^{1/\gamma}, \tag{33}$$

and for the conserved fluxes

$$\begin{aligned} \Psi_P^{(t)} &= 2\pi\sqrt{1-\beta}\,J_1(\mu)/\mu, \quad \Psi_V^{(t)}{=}2\pi(J_0(\mu)\cos\delta - J_1(\mu)\sin\delta)(l^2-1)/2, \\ \Psi_V^{(p)} &= L(J_1(\mu)\cos\delta + J_0(\mu)\sin\delta)\ln l, \end{aligned}$$

where $\Psi_P^{(t)}$ and $\Psi_V^{(t)}$ denote the toroidal flux (z-direction) in the plasma and the vacuum region, respectively, and $\Psi_V^{(p)}$ the poloidal flux in the vacuum region. Introducing the nondimensional conserved quantities (L and l have already been made dimensionless with the plasma radius)

$$\tilde{H} := \frac{2\pi H}{L\Psi_P^{(t)}} = \mu\Big(\frac{J_0(\mu)}{J_1(\mu)}\Big)^2 - 2\,\frac{J_0(\mu)}{J_1(\mu)} + \mu, \tag{34}$$

$$\tilde{\Psi}^{(t)} := \frac{\Psi_V^{(t)}}{\Psi_P^{(t)}} = \frac{1}{\sqrt{1-\beta}}\,\frac{\mu}{J_1(\mu)}(J_0(\mu)\cos\delta - J_1(\mu)\sin\delta)(l^2-1)/2, \tag{35}$$

$$\tilde{\Psi}^{(p)} := \frac{2\pi\,\Psi_V^{(p)}}{L\,\Psi_P^{(t)}} = \frac{1}{\sqrt{1-\beta}}\,\frac{\mu}{J_1(\mu)}(J_1(\mu)\cos\delta + J_0(\mu)\sin\delta)\ln l, \tag{36}$$

the four nondimensional equilibrium parameters μ, δ, l and β can equivalently be expressed by $\tilde{H}$, $\tilde{\Psi}^{(t)}$, $\tilde{\Psi}^{(p)}$ and $\tilde{M} = M/\pi L$. In fact, for $|\mu| < \mu_T \approx 3.112$, which is necessary for stability (see below), the right-hand side of (34) is monotonous in μ. Thus, (34) can be uniquely solved for μ. Using (33), β can then be expressed by $\tilde{M}$ and $\tilde{H}$, and δ and l are determined from (35) and (36).

4 Relaxed states in the cylinder

Whether the equilibrium solution discussed in the preceeding section is a relaxed state or not depends on the second variation $\delta^2 W$ of the functional (3). In the following we use its reduced form (21). Since the whole problem is homogeneous in the variables θ and z, it suffices to consider test functions of the form

$$\xi(\theta, z) = X\,e^{i(m\theta+kz)}, \quad m \in \mathbb{Z}, \quad k \in \frac{2\pi}{L}\,\mathbb{Z}, \quad X \in \mathbb{C}. \tag{37}$$

Note that ξ needs only to be defined on the interface $r = 1$. The magnetic field $\mathbf{b}$ is then determined by (22)–(28) and allows the ansatz

$$\mathbf{b}(\mathbf{x}) = \tilde{\mathbf{b}}(r)\,e^{i(m\theta+kz)}. \tag{38}$$

Inserting (38) into (22) yields for fixed m and k the following system of ordinary differential equations in $\mathcal{P}$, i.e. for $r \le 1$:

$$i\,\frac{m}{r}\,\tilde{b}_z - i\,k\,\tilde{b}_\theta = \mu\,\tilde{b}_r\,, \tag{39}$$

$$i\,k\,\tilde{b}_r - \frac{d}{dr}\,\tilde{b}_z = \mu\,\tilde{b}_\theta\,, \tag{40}$$

$$\frac{1}{r}\,\frac{d}{dr}\Big(r\,\tilde{b}_\theta\Big) - i\,\frac{m}{r}\,\tilde{b}_r = \mu\,\tilde{b}_z\,. \tag{41}$$

Inserting (38) into (23) amounts to (39)–(41) with a vanishing right-hand side and the independent equation

$$\frac{1}{r}\,\frac{d}{dr}\Big(r\,\tilde{b}_r\Big) + i\,\frac{m}{r}\,\tilde{b}_\theta + i\,k\,\tilde{b}_z = 0\,, \tag{42}$$

valid in the vacuum region $\mathcal{V}$, i.e. for $1 \le r \le l$.

By substitution one obtains from (39)–(41) a single ordinary differential equation for $\tilde{b}_z$,

$$\Big[\frac{d}{dr} + \frac{1}{r}\,\frac{d}{dr} - \Big(K^2 + \Big(\frac{m}{r}\Big)^2\Big)\Big]\tilde{b}_z = 0, \tag{43}$$

where $K^2 := k^2 - \mu^2$. Once $\tilde{b}_z$ is determined, $\tilde{b}_r$ and $\tilde{b}_\theta$ are given by the algebraic equations

$$\tilde{b}_r = -\frac{i}{K^2}\Big(\mu\,\frac{m}{r}\,\tilde{b}_z + k\,\frac{d}{dr}\,\tilde{b}_z\Big), \quad \tilde{b}_\theta = \frac{1}{K^2}\Big(k\,\frac{m}{r}\,\tilde{b}_z + \mu\,\frac{d}{dr}\,\tilde{b}_z\Big). \tag{44}$$

Two linearly independent solutions of (43) are the modified Bessel functions $I_m(x)$ and $K_m(x)$ with $x{:=}Kr$. Note that $I_m(x)$ is regular at $x{=}0$ but $K_m(x)$ is not [11].

In the plasma region $\mathcal{P}$ eqs. (43), (44) are equivalent to (39)–(41) as long as $K \neq 0$; thus for $K \neq 0$ the eigenfunctions, regular at $r = 0$, in $\mathcal{P}$ take the form

$$\tilde{b}_r^P = -iN_P\Big(\frac{\mu\,m}{Kx}\,I_m(x) + \frac{k}{K}\,I_m'(x)\Big), \tag{45}$$

$$\tilde{b}_\theta^P = N_P\Big(\frac{k\,m}{Kx}\,I_m(x) + \frac{\mu}{K}\,I_m'(x)\Big), \tag{46}$$

$$\tilde{b}_z^P = N_P\,I_m(x)\,. \tag{47}$$

The prime means differentiation with respect to the argument and N_P is a normalization constant yet to be determined. For $K = 0$ the cases $m \neq 0$ and $m = 0$ have to be distinguished. For $m \neq 0$, one obtains after some algebra

$$\tilde{b}_r^P = -iN_P\,k\Big(\frac{\mu^2}{|m|+1}\,r^{|m|+1} + |m|\,r^{|m|-1}\Big), \tag{48}$$

$$\tilde{b}_\theta^P = N_P\,\mu\Big(\frac{\mu^2}{|m|+1}\,r^{|m|+1} - |m|\,r^{|m|-1}\Big), \tag{49}$$

$$\tilde{b}_z^P = 2\,N_P\,\mu^2\,r^{|m|}, \tag{50}$$

and in the case $m = 0$

$$\tilde{b}_r^P = -iN_P\,k\,r, \qquad \tilde{b}_\theta^P = N_P\,\mu\,r, \qquad \tilde{b}_z^P = 2\,N_P. \tag{51}$$

It is not hard to verify that for $k \neq 0$ eqs. (43), (44) with $\mu = 0$ describe solutions in the vacuum region $\mathcal{V}$ as well. For $1 \leq r \leq l$ we obtain

$$\tilde{b}_r^V = -iN_V\Big(I_m'(x) + c_m K_m'(x)\Big), \tag{52}$$

$$\tilde{b}_\theta^V = N_V\frac{m}{x}\Big(I_m(x) + c_m K_m(x)\Big), \tag{53}$$

$$\tilde{b}_z^V = N_V\Big(I_m(x) + c_m K_m(x)\Big), \tag{54}$$

with $x{=}kr$ and N_V a normalization constant. The coefficient $c_m := I_m'(kl)/K_m'(kl)$ has been determined from the boundary condition (25). The special case $k = 0$ is directly determined from (39)–(41) and (25). For $m \neq 0$ this yields

$$\tilde{b}_r^V = -i\frac{N_V}{r}\Big((r/l)^m - (r/l)^{-m}\Big), \tag{55}$$

$$\tilde{b}_\theta^V = \frac{N_V}{r}\Big((r/l)^m + (r/l)^{-m}\Big), \tag{56}$$

$$\tilde{b}_z^V = 0\,, \tag{57}$$

and for $m = 0$

$$\tilde{b}_r^V = 0, \qquad \tilde{b}_\theta^V = N_V^\theta / r, \qquad \tilde{b}_z^V = N_V^z. \tag{58}$$

Equation (24) can be used to relate the normalization constants $N_{P/V}$ to the amplitude X of the displacement ξ of the interface, cf.(37). Observing that the second term on the right-hand side of (24) always vanishes, and using (29), (30) and (45), (52), we calculate

$$N_P = -X \, \frac{\sqrt{1-\beta}\, K^2 \, u_k^m(\mu)}{\mu \, m I_m(K) + k K I_m'(K)} \, , \tag{59}$$

$$N_V = -X \, \frac{u_k^m(\mu) \cos\delta + v_k^m(\mu) \sin\delta}{I_m'(k) + c_m K_m'(k)} \, , \tag{60}$$

where

$$u_k^m(\mu) := k \, J_0(\mu) + m \, J_1(\mu), \qquad v_k^m(\mu) := m \, J_0(\mu) - k \, J_1(\mu). \tag{61}$$

Similar relations are obtained in the special cases $K = 0$ and $k = 0$, $m \neq 0$. Note, however, that (24) does not provide any information in the case $k = 0$, $m = 0$. In that case the normalization constants are determined by the flux conditions (26)–(28),

$$N_P = -X \sqrt{1-\beta} \, \mu \, \frac{J_0(\mu)}{J_1(\mu)} \, , \tag{62}$$

$$N_V^\theta = \frac{X}{\ln l} \Big(J_1(\mu) \cos\delta + J_0(\mu) \sin\delta \Big), \tag{63}$$

$$N_V^z = \frac{2X}{l^2 - 1} \Big(J_0(\mu) \cos\delta - J_1(\mu) \sin\delta \Big). \tag{64}$$

In all other cases these conditions contain no additional information.

If (46), (47), (53), (54) together with (59), (60) and the equilibrium quantities (29), (30) and (32) are inserted in (21), $\delta^2 W$ takes the form

$$\begin{aligned} \delta^2 W &= \delta^2 W_k^m(\mu, \delta, l, \beta) \\ &= 2\pi \, L \, |X|^2 \Big\{ F_k^m(\mu, \delta, l) + (1-\beta)[G_k^m(\mu) + J_1^2(\mu)] + H(\mu, \delta) \Big\} \end{aligned} \tag{65}$$

with

$$F_k^m := \frac{(u_k^m(\mu) \cos\delta + v_k^m(\mu) \sin\delta)^2}{f_m(k) \, g_m(k, lk)} \, , \tag{66}$$

$$G_k^m := \frac{u_k^m(\mu)(k \, u_k^m(\mu) - \mu \, w_k^m(\mu))}{k \, f_m(K) + \mu \, m} \, , \tag{67}$$

$$H := -(J_0(\mu) \sin\delta + J_1(\mu) \cos\delta)^2, \tag{68}$$

and

$$f_m(x) := x \, \frac{I_m'(x)}{I_m(x)} \, , \tag{69}$$

$$g_m(x,y) := \Big(\frac{I_m'(y)\,K_m'(x)}{I_m'(x)\,K_m'(y)} - 1\Big)\Big(1 - \frac{I_m'(y)\,K_m(x)}{I_m(x)\,K_m'(y)}\Big)^{-1}, \tag{70}$$

as well as

$$w_k^m(\mu) := \mu\, J_0(\mu) - f_m(K)\, J_1(\mu)\,. \tag{71}$$

The quantities $u_k^m(\mu)$ and $v_k^m(\mu)$ are defined in (61) and K is the square root with positive real part of $K^2 = k^2 - \mu^2$. The special cases $K = 0$ and $k = 0$ (each with the further distinction $m = 0$ and $m \neq 0$) need not be considered separately. As it turns out all these cases are already contained in (65) if one takes the appropriate limits $K \to 0$ and $k \to 0$. Thus, the equilibrium solution (29), (30) is a relaxed state if and only if for all $m \in \mathbb{Z}$ and all $k \in \frac{2\pi}{L}\mathbb{Z}$ we have $\delta^2 W > 0$.

Due to the complexity of $\delta^2 W$ the evaluation of this criterion has to be done numerically. However, a number of simplifications and general statements can be made beforehand. Let us begin with noting some properties of the functions f_m and g_m following from those of the modified Bessel functions I_m and K_m [11]: f_m is an even, monotonically increasing function on the positive real axis with $f_m(0) = m$ and $f_m(x) \sim x$ for $x \to \infty$. Especially for $m = 0$, we have $f_0(x) \sim x^2/2$ for $x \to 0$. For fixed $l > 1$, $g_m(x, lx)$ is a positive and even function of x with the limits $g_m \to 1$ for $x \to \infty$ and $g_m \to (l^{2m} - 1)/(l^{2m} + 1)$ if $m \neq 0$ and $g_0 \to l^2 - 1$ for $x \to 0$. These properties imply, in particular, that F_k^m is a nonnegative and hence stabilizing term. H is obviously nonpositive and G_k^m can take either sign.

A first simplification is that $\delta^2 W_k^m$ need not be considered for all $m \in \mathbb{Z}$; $m = 0$ and $m = 1$ are sufficient. This follows from a different representation of $\delta^2 W$ which allows the conclusion that the minimum of $\delta^2 W$ with respect to $m \in \mathbb{Z} \setminus \{0\}$ is obtained for $m = 1$. This result is contained in Appendix B of [12] and need not be repeated here. In fact, in [12] a slightly different functional is considered; the modifications, however, do not affect the conclusion.

A second simplification arises if k is allowed to vary continuously over the real line. Such a criterion corresponds to a cylinder of infinite length which is the the large aspect ratio limit of an axisymmetric torus. For cylinders of finite length the condition $\delta^2 W_k^m > 0$ for $m = 0, 1$ and all $k \in \mathbb{R}$ still represents a sufficient condition for stability.

Next we consider the symmetries of $\delta^2 W$ with respect to μ, δ and k: $\delta^2 W$ is invariant under the substitution

$$\delta \to \delta + \pi \tag{72}$$

as well as under the simultaneous substitutions

$$\mu \to -\mu, \qquad \delta \to -\delta, \qquad k \to -k\,. \tag{73}$$

It is thus sufficient to investigate $\delta^2 W$ in the range $\mu \geq 0$, $0 \leq \delta \leq \pi$, $k \in \mathbb{R}$. A further restriction on μ is well known from Taylor's theory: consider the zeros of

the denominator of G_k^1. An obvious zero is obtained for $k = -\mu$. Another one may appear for $|k| < \mu$. With $x := \sqrt{\mu^2 - k^2}$ the condition for a zero in this k-range takes the real form

$$k\, x\, J_0(x) + (\mu - k)\, J_1(x) = 0, \tag{74}$$

which is precisely Taylor's dispersion relation (cf. [2]). There are no zeros for $\mu < \mu_T \approx 3.112$ and there is always at least one zero k_μ for any $\mu \geq \mu_T$. The obvious zero happens to be a zero of the numerator of G_k^1, too, and G_k^1 remains, in fact, bounded. For μ_T and $k_{\mu_T} =: k_T \approx 1.234$, however, the numerator takes the negative value $(u_{k_T}^1(\mu_T))^2(k_T^2 - \mu_T^2)/k_T$, whereas the denominator approaches zero from the positive side. Since in this limit all other terms in $\delta^2 W$ take finite values we have for $\beta \neq 1$

$$\lim_{k \to k_T} \delta^2 W_k^1(\mu_T, \delta, l, \beta) = \lim_{k \to k_T} G_k^1(\mu_T) = -\infty, \tag{75}$$

which implies instability. It is easy to see that, in fact, instability prevails for all $\mu \geq \mu_T$. The field-free case $\beta = 1$ is considered below. It turns out to be at best marginally stable.

It was a major finding in [7] that pressureless equilibria with a smooth interface and ergodic field lines on that surface are relaxed states only if the pitch angle δ of the magnetic field jumps across the interface. This property holds for the present equilibrium with pressure as well. For $\delta = 0$ we find

$$\begin{aligned} \delta^2 W_k^1(\mu, 0, l, \beta) \leq 2\pi\, L\, |X|^2 \Bigg\{ & \frac{(u_k^1(\mu))^2}{f_1(k)\, g_1(k, lk)} + (1-\beta) \frac{k\, (u_k^1(\mu))^2}{k\, f_1(K) + \mu} \\ & - (1-\beta)\, u_k^1(\mu)\, \frac{\mu\, w_k^1(\mu)}{k\, f_1(K) + \mu} \Bigg\}. \end{aligned} \tag{76}$$

For $\beta \neq 1$, $\mu \neq \mu_0 \approx 2.405$, the lowest zero of J_0, and k close to $k_0 := -J_1(\mu)/J_0(\mu)$, $\delta^2 W_k^1$ is dominated by the third term on the right-hand side of (76), which takes both signs close to k_0. Thus, with the above restrictions $\delta = 0$ implies instability. For $\delta = 0$ and $\mu = \mu_0$ marginal stability can be obtained at best.

In order to obtain a more complete description of the stability region in the space of the parameters μ, δ, l and β we have evaluated $\delta^2 W$ numerically. More precisely, $\delta^2 W_k^m(\mu, \delta, l, \beta)$ is minimized with respect to $k \in \mathbb{R}$ for fixed values of μ, δ, l and β, and for $m = 0$ and 1. The result is denoted by $\delta^2 W_{\min}^m$. Note that $\delta^2 W$ simplifies considerably for $m = 0$, i.e.,

$$\begin{aligned} \delta^2 W_k^0(\mu, \delta, l, \beta) = & \frac{k^2}{f_0(k)\, g_0(k, lk)} \left(J_0(\mu) \cos\delta - J_1(\mu) \sin\delta\right)^2 \\ & + (1-\beta) \left(J_1^2(\mu) \frac{K^2}{f_0(K)} + J_0(\mu) J_1(\mu) + J_1^2(\mu) \right) \\ & - (J_0(\mu) \sin\delta + J_1(\mu) \cos\delta)^2. \end{aligned} \tag{77}$$

The minimization procedure evaluates $\delta^2 W_k^m$ in the K-range $-K \leq k \leq K$, starting with $-K$ and using the constant k-step $\triangle k$. The procedure stops if a negative value is found, otherwise the equilibrium is classified stable (with respect to m). Typically we used $K = 5$, $\triangle k = 0.005$ for $m = 0$ and $K = 20$, $\triangle k = 0.002$ for $m = 1$. The minimizing k-value $k_{\min}$ always turned out to be of order one (with one exception, see below). Larger k-ranges or smaller k-steps did not change the sign of $\delta^2 W_{\min}^m$.

We present our results by indicating stable regions or lines of marginal stability in the δ–μ–plane for fixed values of l and β, and separately for $m = 0$ and 1. We first discuss pressureless equilibria ($\beta = 0$), which turn out to be the most unstable.

5 Pressureless relaxed states

Figure 1a) presents the stable region (unhatched) in the δ–μ–plane with nondimensional wall radius $l = 1.5$. The unstable region due to $m = 1$–modes (i.e.

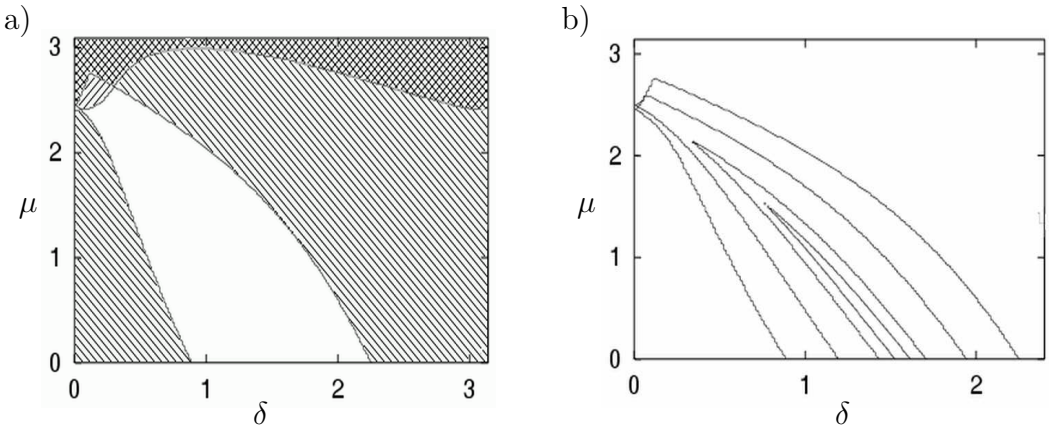

Figure 1: a) Stable region (unhatched) in the δ–μ–plane with $l = 1.5$ and $\beta = 0$. Regions hatched from top left to bottom right are ($m=1$)–unstable, regions hatched the other way round are ($m=0$)–unstable. The figure has been produced by evaluating $\delta^2 W$ for 200×200 points in the δ–μ–plane. In the minimization procedure with respect to k we used $K = 5$, $\triangle k = 0.005$ for $m = 0$ and $K = 20$, $\triangle k = 0.002$ for $m = 1$. b) Zero level lines of $\delta^2 W_{\min}^1$ (corresponding to lines of marginal stability for $\mu < \mu_0$) with $\beta = 0$ and $l = 1.5, 2, 3, 4$, increasing inwards.

$\delta^2 W_{\min}^1 < 0$) is hatched from top left to bottom right, that due to axisymmetric modes ($m = 0$) is hatched from bottom left to top right. Obviously, the stability region is predominantly bounded by $m = 1$; only close to the point ($\delta = 0$, $\mu = \mu_0$), $m = 0$ adds a further restriction. In fact, $\delta^2 W_k^0$ is always stable for $\beta = 0$ and $0 \leq \mu < \mu_0$: with $2f_0(x) \leq x^2$ and $J_0(\mu) > 0$ for $0 \leq \mu < \mu_0$ the

second and third term in (77) allow the estimate

$$\begin{aligned}
&J_0^2(\mu)\tfrac{K^2}{f_0(K)} + \mu J_0(\mu)J_1(\mu) + J_1^2(\mu) - (J_0(\mu)\sin\delta + J_1(\mu)\cos\delta)^2 \\
&\quad \geq J_0^2(\mu)+J_0^2(\mu)(1-\sin^2\delta)-J_0(\mu)J_1(\mu)(2\sin\delta\cos\delta-\mu)+J_1^2(\mu)(1-\cos^2\delta) \\
&\quad \geq J_0^2(\mu) + (J_0(\mu)\cos\delta - J_1(\mu)\sin\delta)^2 > 0\,,
\end{aligned}$$

and hence $\delta^2 W_k^0 > 0$.

The point ($\delta = 0$, $\mu = \mu_0$) has another special feature. Numerically we find minimizing values $k_{\min}$ for k of order 1 except at ($\delta = 0$, $\mu = \mu_0$), where $k_{\min}$ becomes large. This corresponds to the fact that $\delta^2 W_k^m \sim |k|$ for large $|k|$ except at ($\delta = 0$, $\mu = \mu_0$), as can be seen from (65)–(67).

Figure 1b) presents a series of zero level lines of $\delta^2 W_{\min}^1$ with increasing l. Below μ_0 these lines are lines of marginal stability as follows from the above remark. With increasing l the stable region shrinks to a "most stable point" in the δ–μ–plane. From the symmetries (72), (73) follows that the stable region in the δ–μ–plane is symmetric with respect to ($\delta = \pi/2$, $\mu = 0$). The "most stable point" is thus ($\delta = \pi/2$, $\mu = 0$). It is reached for $l = l_c \approx 4.983$. Cylinder-symmetric relaxed states with nondimensional wall radius greater than l_c do not exist.

Figure 2 illustrates the case of decreasing l. The most remarkable finding,

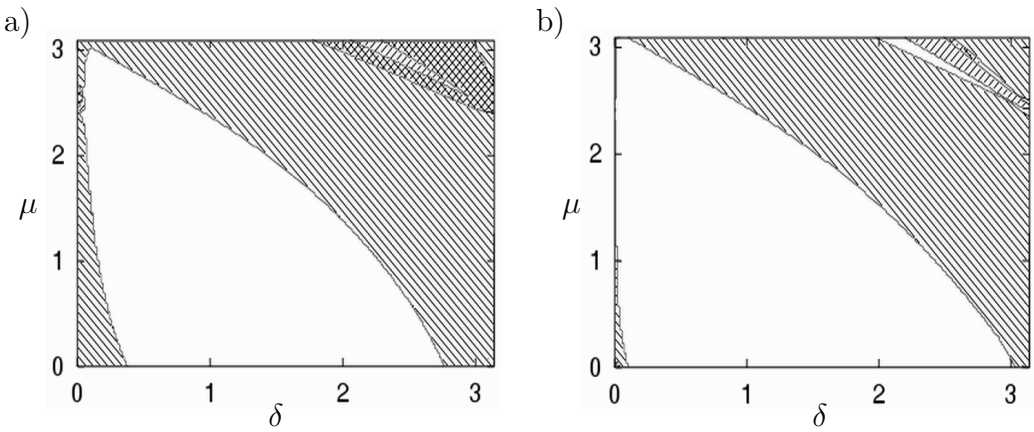

Figure 2: Same as fig.1a) with a) $l = 1.1$ and b) $l = 1.01$.

which has no counterpart in plane geometry (cf. [7]), is an additional wedge-shaped stability region in the upper-right corner of the stability diagram. With decreasing l a ($m = 1$)–stable wedge "opens" at $l \approx 1.2$. This wedge is, however, still ($m=0$)–unstable. With further decrease of l the ($m=1$)–stable wedge increases, whereas the ($m = 0$)–unstable region decreases. At $l \approx 1.03$ a completely stable wedge appears. In the limit of vanishing vacuum region ($l \to 1$) the ($m = 0$)–unstable region disappears and the stable wedge takes a limit position marked by the points ($\delta \approx 1.94, \mu = \mu_T$), ($\delta \approx 2.52, \mu = \mu_T$) and ($\delta = \pi, \mu = \mu_0$).

6 Relaxed states with pressure

The general case $\beta \neq 0$ has not been as thoroughly investigated as the pressureless case. The overall impression is that the stability region always shrinks with increasing β. This has been confirmed for a number of equilibria. However, it is not obvious from the analytic form of $\delta^2 W$ and there is no simple relationship between β and the other parameters, as we have in plane geometry (see the appendix). Only the field-free limit case $\beta = 1$ is easy to treat. Minimization with respect to m and k yields

$$\delta^2 W_0^0(\mu, \delta, l, 1) = -(J_0(\mu)\sin\delta + J_1(\mu)\cos\delta)^2. \tag{78}$$

Thus, there is instability for $\tan\delta \neq -J_1(\mu)/J_0(\mu)$ and marginal stability otherwise.

There is a remarkable difference between equilibria with and without pressure concerning the connectivity of the main stable region. For $\beta \neq 0$ the $(m=0)$–unstable region is no longer restricted to the range $\mu_0 \leq \mu \leq \mu_T$ (see fig. 3). In

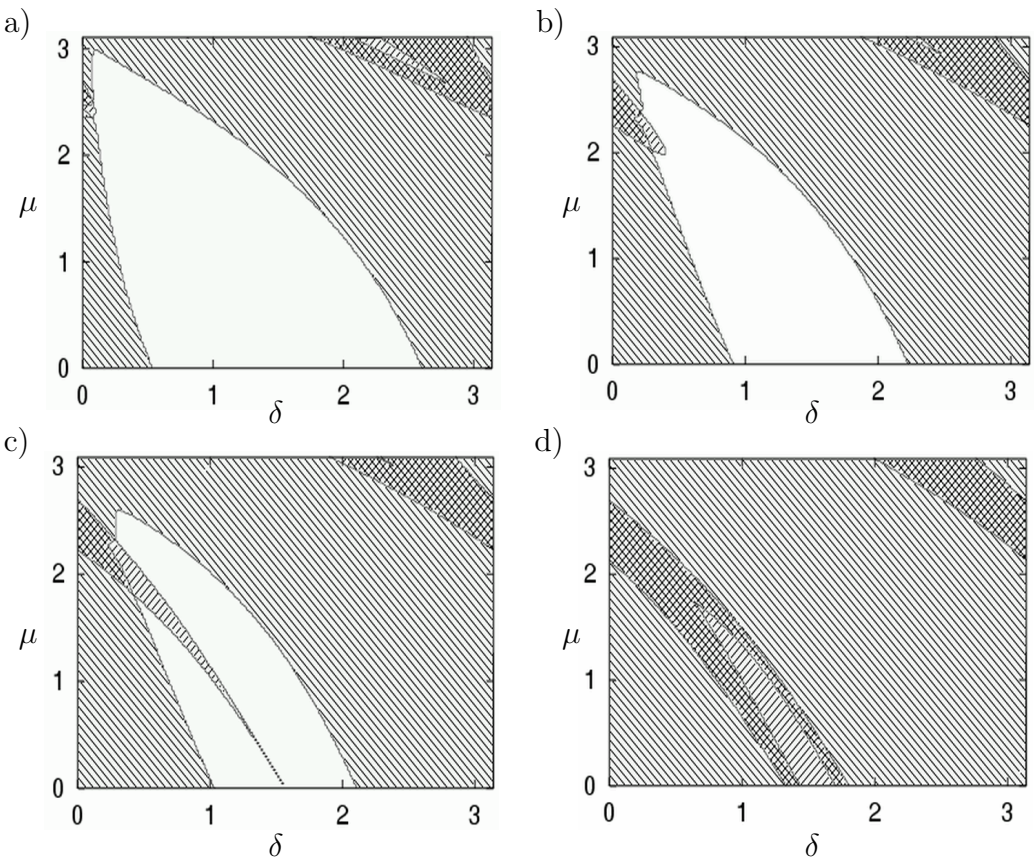

Figure 3: Same as fig.1a) with $l = 1.1$ and a) $\beta = 0.1$, b) $\beta = 0.4$, c) $\beta = 0.5$, d) $\beta = 0.8$.

fact, with increasing β the $(m=0)$–unstable region extends to lower values of μ,

finally reaching the point ($\delta = \pi/2$, $\mu = 0$) for $\beta = 1/2$. The ($m=1$)–unstable region shrinks with increasing β qualitatively in the same way as with increasing l to the "($m=1$)–most stable point" ($\delta = \pi/2$, $\mu = 0$). This can lead to a doubly connected stable region for $\beta \geq 1/2$ as demonstrated in fig.3c). With further increase of β the equilibrium becomes unstable even before the ($m=1$)–unstable region has shrunk to ($\delta = \pi/2$, $\mu = 0$).

Let us, finally, consider the critical nondimensional wall radius l_c beyond which no cylinder-symmetric relaxed state exists. It depends now on β. For $\beta \leq 1/2$ the "($m=1$)–most stable point" ($\delta = \pi/2$, $\mu = 0$) determines l_c (as in the case $\beta = 0$). Evaluating $\delta^2 W$ at that point yields

$$\delta^2 W^0_{\min}(0, \frac{\pi}{2}, l, \beta) = (1-\beta) \min_k \frac{k^2}{f_0(k)} - 1 = 1 - 2\,\beta\,, \tag{79}$$

$$\delta^2 W^1_{\min}(0, \frac{\pi}{2}, l, \beta) = \min_k \Big(\frac{1}{f_1(k)\, g_1(k, lk)} + (1-\beta)\, \frac{k^2}{f_1(k)} \Big) - 1\,. \tag{80}$$

Equation (79) demonstrates that ($\delta = \pi/2$, $\mu = 0$) is for $\beta < 1/2$ always ($m=0$)–stable, independently of l. Setting (80) to zero thus yields the critical curve $l_c(\beta)$ (see fig.4). For $\beta \geq 1/2$ the graph in fig. 4 still gives an upper bound on $l_c(\beta)$.

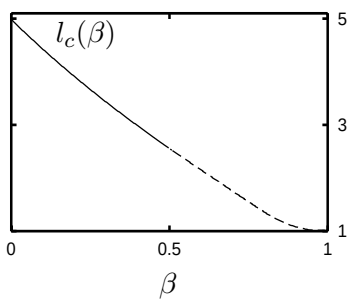

Figure 4: Critical nondimensional wall radius l_c (solid line) versus plasma pressure β for $0 < \beta \leq 1/2$, and an upper bound on l_c (dashed line) for $1/2 < \beta < 1$.

As expected $l_c(\beta)$ is a monotonically decreasing function of β, with $l_c(\beta) \to 1$ as $\beta \to 1$, and with a global maximum given by $l_c(0) = l_c \approx 4.983$ (cf. section 5).

7 Conclusions

In this study the variational principle for relaxed plasma vacuum systems with pressure has been applied to axially periodic circular cylinders. The emphasis of the work is less on providing realistic stability boundaries for toroidal pinch experiments than on a thorough exploration of the consequences of the variational principle in a nontrivial system. Systems with cylindrical symmetry are simple enough to allow explicit calculations. Moreover, the cylinder results can

be compared with Taylor's original theory of plasma relaxation without vacuum region. Still simpler systems, e. g. a plane slab as considered in [7], have serious drawbacks. The magnetic axis is missing, and artificial boundary conditions have to be used. Moreover, the dependence of stability boundaries on the equilibrium parameters degenerates in an unrealistic way.

The over-all impression of our analysis is that relaxed states form a complicated set in the space of all force-free plasma-vacuum states with cylindrical symmetry. Nevertheless, a number of general features can be observed. Most important, there is no relaxed state without tangential discontinuity at the plasma-vacuum interface. This result is in accordance with general ideas about the relaxation process. However, as far as we know, it has not yet been demonstrated in any system in a rigorous way.

If compared to the case without vacuum region the critical pinch parameter of a plasma-vacuum relaxed state lies in an interval which is allways smaller than that known from Taylor's theory. This interval is shrinking with increasing vacuum region or increasing plasma pressure. In particular, there is a critical ratio of wall radius over interface radius (depending on the plasma pressure) beyond which no relaxed states exist.

A Appendix

In the simplest possible geometry, the vessel is a topologically toroidal plane slab. This situation, where all curvature effects are eliminated, has been discussed in [7] for the case of vanishing pressure. We report here the modifications due to a nonzero constant plasma pressure.

We use Cartesian coordinates (x, y, z), where x denotes the radial coordinate, y the poloidal one and z the toroidal one. Physical quantities are assumed to be periodic in y and z with periods L_y and L_z. The vessel is the domain $0 \leq x \leq L_x$ with an inner wall at the plane $x = 0$ (simulating the magnetic axis) and the outer wall at $x = L_x$. We assume at $x = 0$ the same boundary condition $B_x = 0$ as at $x = L_x$. Additionally we require $\int_0^{L_y} B_y \, dy = 0$ at $x = 0$ in order to eliminate the additional Neumann field present in a toroidal layer in comparison to a solid torus.

The equilibrium to be considered depends only on the radial coordinate x. The magnetic field has no radial component ($B_x \equiv 0$), and its poloidal component vanishes at the "magnetic axis" ($B_y = 0$ at $x = 0$). If the plasma vacuum interface is the plane $x = L$, then

$$B_y = B_P \sin \mu x, \qquad B_z = B_P \cos \mu x, \tag{81}$$

for $0 \leq x \leq L$ (force-free field), and

$$B_y = B_V \sin(\mu L + \delta), \qquad B_z = B_V \cos(\mu L + \delta), \tag{82}$$

for $L \leq x \leq L_x$ (vacuum field). Here, B_P and B_V are nonnegative constants related by $B_V^2 - B_P^2 = 2P$ with P denoting the constant plasma pressure, and δ is an arbitrary angle which specifies the jump of the pitch angle of the magnetic field across the interface. If the vessel (i.e. the three lengths L_x, L_y and L_z) is given, then there are four nondimensional equilibrium parameters. These can be chosen as the pinch ratio μL, the β-factor $\beta = 2P/B_V^2$, the jump of the pitch angle δ and the ratio $l = (L_x - L)/L$ of vacuum over plasma layer thickness.

The evaluation of the stability criterion $\delta^2 W > 0$ for the equilibrium (81), (82) is completely analogous to the case without pressure (cf. [7]) and need not be repeated here. In the limit of a thin slab the dispersion relation characterizing the stability region in the parameter space takes the same form as in the case without pressure with the only difference that the lengths ratio l is replaced with $\tilde{l} := l(1-\beta)$, i.e. the equilibrium (81), (82) is stable if and only if the pinch ratio μL is in the interval

$$\mu_-(\delta, \tilde{l})L < \mu L < \mu_+(\delta, \tilde{l})L,$$

where $\mu_+(\delta, \tilde{l})L$ is the smallest positive root of the equation

$$\tilde{l}\mu L \sin \mu L - 4 \sin \delta \sin(\mu L + \delta) = 0$$

and $\mu_-(\delta, \tilde{l})L := -\mu_+(-\delta, \tilde{l})L$. Thus, the most remarkable finding in [7], i.e., that stability implies a nonzero jump δ of the pitch angle, is preserved if pressure is added. Nonzero pressure corresponds simply to an enlarged lengths ratio l, which implies a shrinking stability region in the δ–μ–plane (cf. Fig.1 in [7]). The limit case of a field-free plasma region ($\beta = 1$) corresponds to an unbounded vacuum region ($l = \infty$), a situation which is known to be unstable.

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