The amplitude equations for the first instability of electro-convection in nematic liquid crystals in case of two unbounded space directions

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February 22, 2007

This paper is dedicated to Lorenz Kramer 1941–2005.

Abstract

Electro-convection in nematic liquid crystals is a paradigm for pattern formation in anisotropic systems. In this paper we discuss the amplitude equations obtained for this pattern forming system close to the first instability in case of two unbounded space directions. We prove error estimates showing the validity of these formal approximations for a regularized version of the weak electrolyte model (WEM). New mathematical aspects occur due to the possible instability mechanisms of the WEM and due to the external time-periodic forcing.

Contents

1 Introduction

In the experiments for electro-convection in nematic liquid crystals a thin layer of such a material is contained in between two spatially extended electrode plates, cf. fig. 1. When an alternating current is applied to the electrodes an electro-hydrodynamic instability occurs if the voltage is above a certain threshold. The trivial spatially homogeneous solution becomes unstable and bifurcates into a non-trivial pattern [Cha77, PB98]. In this paper we discuss the validity of the amplitude equations obtained for this pattern forming system close to the first instability in case of two unbounded space directions.

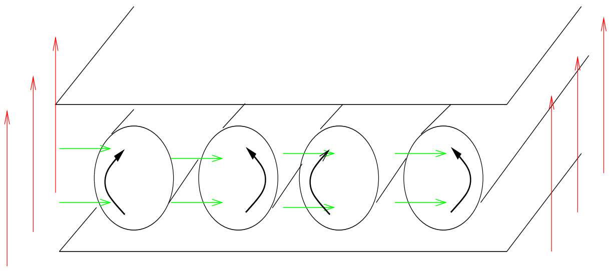

Figure 1: Roll solutions in nematic crystals. The director field of the nematic crystals is almost parallel to the plates. The external time-periodic electric field is perpendicular to the plates.

We consider a layer of nematic liquid crystals in between two infinitely extended horizontal plates of height π, i.e. in the following $(x, y, z) \in \Omega = \mathbb{R}^2 \times (0, \pi)$. There are essentially two models for the mathematical description of electro-convection in nematic liquid crystals. These are the standard model ([ZK85] and the references therein) and the weak electrolyte model (WEM). The latter more advanced model is considered here. It has been introduced by Kramer and Treiber in [Tre96, TK98] to overcome a number of insufficiencies of the standard model.

The WEM, which can be found in detail in Section A.1, is based on the continuum theory of Ericksen [Eri61] and Leslie [Les68]. In this theory, nematic liquid crystals are treated as incompressible fluids, in which the average molecular axis of the material is described locally by a director field n of unit vectors which satisfy the so called Leslie-Erickson equations. They are coupled with generalized Navier-Stokes equations for the fluid velocity v and the pressure p in the presence of an external time-periodic electric field $E_p(t) = E_0 \cos \omega_0 t$. The liquid crystal would be destroyed by electrolysis if ω_0 is too small, especially if $\omega_0 = 0$. As usual the pressure is eliminated with a projection Q into the space of divergence-free vector fields, see Lemma A.4. The second part of the WEM comes from a quasi-static approximation of Maxwell's equations describing the electromagnetic aspects of the experiment. The equations for n and v are then completed by two balance equations for the charge density ρ and the deviation σ of the local conductivity from 1. Since $n_1^2 + n_2^2 + n_3^2 = 1$, for our purposes it is sufficient to consider n_2 and n_3. Thus, the WEM can be written as an evolutionary system for the variables

$$V = (n_2, n_3, v_1, v_2, v_3, \rho, \sigma),$$

see (66)-(70) in Appendix A.1. It is abbreviated in the following by

$$\partial_t V = M(t)V + \tilde{N}(t, V) \tag{1}$$

where $M(t)V$ stands for the linear and $\tilde{N}(t, V)$ for the nonlinear terms with respect to V. The set of partial differential equations (1) is completed with the boundary conditions

$$n_2 = n_3 = v_1 = v_2 = v_3 = 0 \quad \text{at} \quad z = 0, \pi. \tag{2}$$

These are derived in case of ideal conducting plates, rigid anchoring for the director and finite viscosity, i.e. for (2) the coordinate system is chosen such that $n = (1, 0, 0)$ at the lower and upper plates. Due to the anisotropy in the boundary conditions (2) there is no rotational symmetry of the WEM. However, the WEM is invariant under arbitrary translations in x and y, and under the reflections

$$\mathcal{S}_1 : (x, n_2, n_3, v_1) \;\rightarrow\; -(x, n_2, n_3, v_1), \tag{3}$$
$$\mathcal{S}_2 : (y, n_2, v_2) \;\rightarrow\; -(y, n_2, v_2), \tag{4}$$
$$\mathcal{S}_3 : (z, n_3, v_3) \;\rightarrow\; -(z, n_3, v_3). \tag{5}$$

For (1) we have the trivial solution

$$V = (n_2, n_3, v_1, v_2, v_3, \rho, \sigma) = (0, 0, 0, 0, 0, 0, 0). \tag{6}$$

In order to analyze its stability we consider the linearized system

$$\partial_t V = M(t)V. \tag{7}$$

Due to the translation invariance and the time periodicity of the problem the solutions are given by Floquet-Fourier modes $V = \hat{\varphi}(k, l, z, t)e^{i(kx+ly)}e^{\lambda(k,l)t}$ with $k, l \in \mathbb{R}$ and

$$\hat{\varphi}(\cdot, \cdot, \cdot, t) = \hat{\varphi}(\cdot, \cdot, \cdot, t + 2\pi/\omega_0).$$

Since for fixed $k, l \in \mathbb{R}$ the operator $M(t)$ is elliptic on the compact cross section $[0, \pi]$ we have discrete spectrum for fixed $k, l \in \mathbb{R}$, hence the modes $\hat{\varphi}$ and multipliers λ come in families

$$\{\hat{\varphi}_m(k, l, z, t)e^{i(kx+ly)}e^{\lambda_m(k,l)t} : k, l \in \mathbb{R}, m \in \mathbb{N}\}. \tag{8}$$

The Floquet exponents λ_m form smooth surfaces as functions of the wave numbers $k, l \in \mathbb{R}$ as long as they are simple. Moreover, the spectrum depends smoothly on the control parameter E_0.

For $V = 0$ asymptotically stable we have for all $m \in \mathbb{N}$ and $k, l \in \mathbb{R}$ that the Floquet exponents λ_m satisfy

$$\text{Re}\, \lambda_m(k, l) < 0.$$

Experimental and numerical observations show that close to the threshold of instability of the trivial spatially homogenous solution there are essentially two different regions in the (ω, E_0)-plane separated by a frequency ω_L. For $\omega > \omega_L$ the instability occurs at some wave

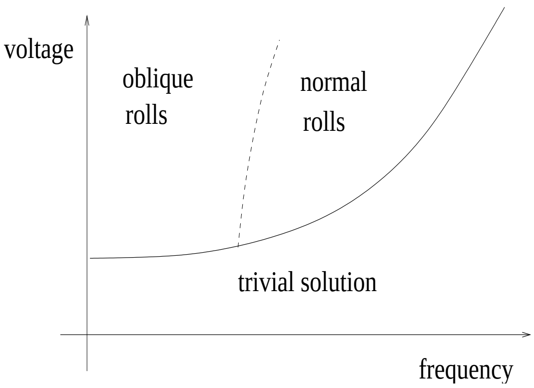

Figure 2: Schematic bifurcation diagram observed in experiments.

vector $(k_c, 0)$ and due to the fact that we have a real-valued problem also at $(-k_c, 0)$. This region is called normal rolls (NR). For $\omega < \omega_L$ the instability occurs at some wave vector (k_c, l_c) and due to the symmetries of the problem also at the wave vectors $(k_c, -l_c)$, $(-k_c, l_c)$ and $(-k_c, -l_c)$. This region is called oblique rolls (OR). See Figures 2 and 3. Experimentally, in (OR) a Turing–Hopf bifurcation is observed, while in (NR) both, Turing or Turing–Hopf bifurcations, may occur, cf. [Tre96].

The mathematical analysis of bifurcations over unbounded domains is based very often on the reduction of the governing partial differential equations to amplitude equations which are expected to capture the essential dynamics near the bifurcation. See [AK02, Mie02] for general introductions. The most famous amplitude equation occurring in a setup with two unbounded space directions is the Ginzburg-Landau equation (GLe)

$$\partial_T A = c_0 A + c_3 \partial_X^2 A + c_5 \partial_Y^2 A + c_6 A|A|^2 \tag{9}$$

with $A = A(X, Y, T) \in \mathbb{C}$ and coefficients $c_0, c_3, c_5, c_6 \in \mathbb{C}$, where the indices were chosen for later comparison with more complicated amplitude equations. The GLe (9) is derived by multiple scaling analysis and describes slow modulations in time T and space X, Y of the amplitude of the linearly most unstable modes.

In the present paper we discuss how for a given experiment, which fixes all other parameters of the system, the spectrum and the associated amplitude equations have to look like

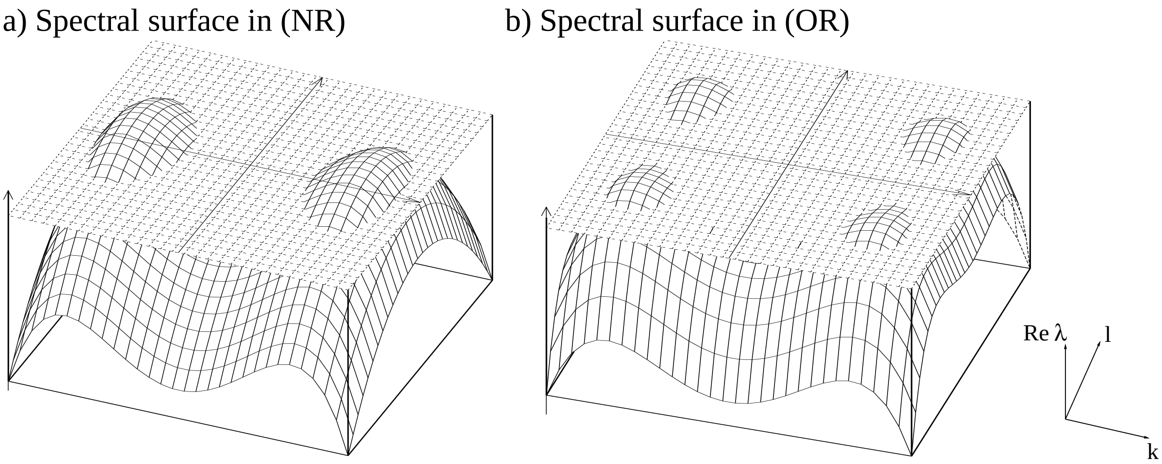

Figure 3: Schematic sketches of the surfaces of the real parts of the Floquet exponents with largest real part. a) In the parameter region (NR). There are neighborhoods of two wave vectors $(\pm k_c, 0)$ where we have positive real part. b) In the parameter region (OR). There are neighborhoods of four wave vectors $(\pm k_c, \pm l_c)$ where we have positive real part. The surface in a) is double in case II, due to the symmetries (3) and (4) and the non vanishing imaginary parts. The surface in b) is always double.

in order to have Fig. 2 as a robust situation. In other words, we discuss the scenarios which are robust under changes of ω, i.e. codimension one phenomena. We do not discuss scenarios which are only stable if a second parameter is changed simultaneously, i.e. codimension two phenomena. Hence, especially the transition from (NR) to (OR) has to be analysed. The only codimension two points which we will discuss in the following are the transition points $(E_0, \omega) = (E_{0,crit}, \omega_L)$ between (NR) and (OR) at the threshold of instability.

It turns out that there are two cases, in the following called Case I, with a Turing bifurcation in (NR), and Case II, with a Turing–Hopf bifurcation in (NR). There is no smooth transition for the critical spectral surfaces between (NR) and (OR) in Case I, but in Case II.

The single Ginzburg–Landau equation (9) also occurs in our problem, namely in Case I (NR), with $c_j \in \mathbb{R}$. In the other cases we obtain more complicated amplitude equations. They are systems of coupled equations of Ginzburg-Landau type and they still depend in a singular way on the small bifurcation parameter. However, for spatially localized solutions this singular dependence vanishes and, moreover, all amplitude equations decouple.

Additional to the analysis of the instability scenario we prove the validity of the associated amplitude equations. The validity of (9) in a situation as Case I is already covered by the analysis of [BSU06] where we discussed the validity of the Ginzburg-Landau approximation in pattern forming systems with external time-periodic forcing described by semilinear parabolic equations with one unbounded space direction. Hence in the following we will mainly concentrate on the other cases.

Notation. The Sobolev-space $H^m(\Omega)$, the space of m-times weakly differentiable functions $\Omega \to \mathbb{R}$, is equipped with the norm $\|u\|_{H^m(\Omega)} = \sum_{|j|=0}^{m} \|\partial_x^j u\|_{L^2(\Omega)}$. Throughout the paper we denote possibly different constants C with the same symbol if they can be chosen independent of the small bifurcation parameter $0 < \varepsilon \ll 1$.

Acknowledgments. The paper is partially supported by the Deutsche Forschungsgemeinschaft DFG under the grant Kr 690/18-1/2. The authors are grateful for discussions with Gerhard Dangelmayr, Lorenz Kramer, and Ian Melbourne.

2 Two different instability mechanisms

In order to have Figure 2 as a robust situation for a given experiment, i.e. for E_0 and ω_0 variable and all other parameters of the system fixed, the spectrum and the associated amplitude equations have to look as follows. We define regions (NR) and (OR) through

(NR) *There exists a m (w.l.o.g. $m = 1$), a $k = k_c \neq 0$ and a $E_0 = E_{0,crit}$, such that*

$$\mathrm{Re}\,\lambda_1(k_c, 0)\big|_{E_0=E_{0,crit}} = 0\;;$$

(OR) *There exists a m (w.l.o.g let $m = 1$), $(k, l) = (k_c, l_c)$ with $k_c \neq 0$, $l_c \neq 0$ and a $E_0 = E_{0,crit}$, such that*

$$\mathrm{Re}\,\lambda_1(k_c, l_c)\big|_{E_0=E_{0,crit}} = 0.$$

These assumptions have a number of consequences due to the fact that we have a real-valued problem and due to the symmetries (3) and (4). For the Floquet exponents λ_1 with largest real part of the linearized system we find, since we have a real valued problem,

$$\mathrm{Re}\lambda_1(k, l) = \mathrm{Re}\lambda_1(-k, -l), \qquad \mathrm{Im}\lambda_1(k, l) = -\mathrm{Im}\lambda_1(-k, -l).$$

Thus, we also have $\mathrm{Re}\,\lambda_1(-k_c, 0)\big|_{E_0=E_{0,crit}}=0$ for (NR) and $\mathrm{Re}\,\lambda_1(-k_c, -l_c)\big|_{E_0=E_{0,crit}} = 0$ for (OR). Next we find

$$\mathrm{Re}\lambda_1(k, l) = \mathrm{Re}\lambda_1(k, -l), \qquad \mathrm{Re}\lambda_1(k, l) = \mathrm{Re}\lambda_1(-k, l)$$

due to the reflection symmetries (3) and (4). Hence we also have $\mathrm{Re}\,\lambda_1(k_c, -l_c)\big|_{E_0=E_{0,crit}} = 0$ and $\mathrm{Re}\,\lambda_1(-k_c, l_c)\big|_{E_0=E_{0,crit}} = 0$ for (OR). The symmetries also yield

$$\mathrm{Im}\lambda_1(-k_c, l_c) = \mathrm{Im}\lambda_1(k_c, l_c) = \mathrm{Im}\lambda_1(k_c, -l_c) = \mathrm{Im}\lambda_1(-k_c, -l_c).$$

However, since at least in (OR) experimentally a Hopf bifurcation is observed there must be a second surface λ_2 with

$$\mathrm{Re}\lambda_1(k, l) = \mathrm{Re}\lambda_2(k, l), \qquad \mathrm{Im}\lambda_1(k, l) = -\mathrm{Im}\lambda_2(k, l) \neq 0$$

close to $(k, l) = (\pm k_c, \pm l_c)$ in (OR), and close to $(k, l) = (\pm k_c, 0)$ in case of a Hopf-bifurcation in (NR).

We assume that the associated eigenfunctions are invariant under the discrete symmetry $\mathcal{S}_3$ such that $\mathcal{S}_3$ is irrelevant for the following considerations.

We now discuss two generic cases, postponing the details of the derivation of the respective amplitude equations to sec. 3. We refer to [DO04] for a numerical investigation of the spectral situation at the bifurcation point in a slightly simplified model. There, for two different nematic crystal materials, the question which of the above bifurcations occur in which experiment is discussed, in particular the transition (NR) to (OR) in Case II.

We also mention that in [Tre96] some other, presumably more realistic, boundary conditions have also been studied. For these boundary conditions the WEM has a time-periodic, in x, y spatially homogeneous solution of the form

$$V_0(t) = V_0(t + 2\pi/\omega_0) = (0, 0, 0, 0, 0, \rho_0(z, t), \sigma_0(z, t)). \tag{10}$$

Qualitatively, this would not change our analysis, since the linearization around (10) again yields a system of the form (1). Moreover, according to [Tre96, p.38/39] the quantities $\rho_0(z, t)$ and $\sigma_0(z, t)$ are small except close to the boundaries. Therefore, the linear and weakly nonlinear analysis for (10) is also quantitatively very similar to the one for (6).

2.1 Case I

(NR): Due to the fact that we have a real-valued problem we also have $\mathrm{Re}\lambda_1(-k_c, 0) = 0$. We assume that for (k, l) close to $(k_c, 0)$ the surface $\mathrm{Re}\lambda_1$ is simple. Due to (3) and (4) this implies

$$\lambda_1(k, l) = \lambda_1(-k, l) = \lambda_1(k, -l) = \lambda_1(-k, -l)$$

and so $\mathrm{Im}\lambda_1(k_c, 0) = 0$ for these wave numbers. See Fig. 4. Moreover, we assume that except of λ_1 in a neighborhood of $(\pm k_c, 0)$ the spectrum has strictly negative real part, i.e. all other Floquet exponents have real parts less than $-\sigma_0$ for a $\sigma_0 > 0$. We introduce the bifurcation parameter ε by

$$\varepsilon^2 = E_0 - E_{0,crit}.$$

Thus we obtain

$$\lambda_1(\mathbf{k}_c + \varepsilon\mathbf{K}) = \varepsilon^2(c_0 - c_3\mathbf{K}_1^2 - c_5\mathbf{K}_2^2) + \mathcal{O}(\varepsilon^3), \tag{11}$$

with $\mathbf{K} = (\mathbf{K}_1, \mathbf{K}_2)$, $c_0 = \partial_\varepsilon^2\lambda_1(\mathbf{k}_c) \in \mathbb{R}$, $c_3 = -\frac{1}{2}\partial_k^2\lambda_1(\mathbf{k}_c) \in \mathbb{R}$ and $c_5 = -\frac{1}{2}\partial_l^2\lambda_1(\mathbf{k}_c) \in \mathbb{R}$, while

$$\partial_l\partial_k\lambda_1(\mathbf{k}_c) = 0 \quad \text{due to} \quad \lambda_1(k, -l) = \lambda_1(k, l). \tag{12}$$

The ansatz for the derivation of the Ginzburg-Landau equation in (NR) is

$$\varepsilon\psi_A(x, y, z, t) = \varepsilon A(X, Y, T)e^{ik_cx}\hat{\varphi}_1(k_c, 0, z, t) + \text{c.c.} + \mathcal{O}(\varepsilon^2), \tag{13}$$

where

$$X = \varepsilon x, \qquad Y = \varepsilon y, \qquad \text{and} \qquad T = \varepsilon^2 t.$$

Inserting (13) into (1) shows that A has to satisfy the Ginzburg-Landau equation (9), i.e.,

$$\partial_T A = c_0 A + c_3\partial_X^2 A + c_5\partial_Y^2 A + c_6 A|A|^2,$$

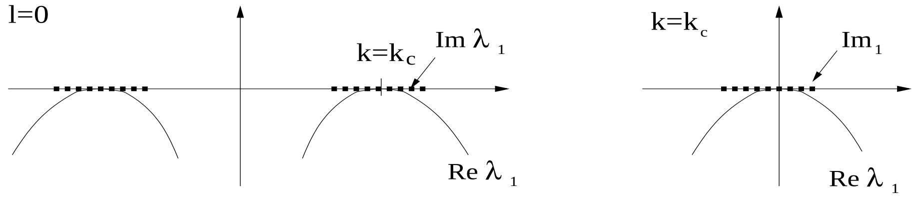

Figure 4: Case I (NR): The pictures show two cross sections through the surface λ_1, namely at $l = 0$ and $k = k_c$. The Floquet exponents λ_1 are simple and touch the axis $\mathrm{Re}\lambda = 0$ in $(\pm k_c, 0)$. The imaginary part of λ_1 vanishes in a neighborhood of these wave vectors.

with c_0, c_3, c_5 from (11), while c_6 is determined by the nonlinearity.
(OR): Due to the symmetries (3)-(4) and the fact that we experimentally obtain a Hopf bifurcation in (OR) we get the spectral situation sketched in fig. 5. Thus, up to $\mathcal{O}(\varepsilon^3)$ we have

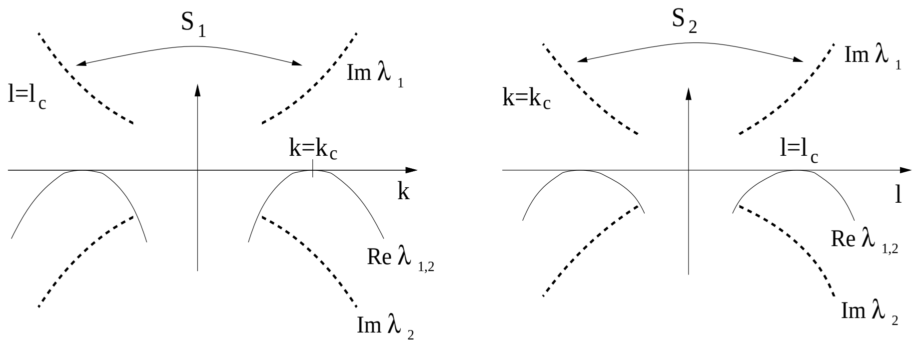

Figure 5: Case I (OR): The pictures show two cross sections through the surfaces λ_1 and λ_2, namely at $l = l_c$ and $k = k_c$. The Floquet exponents λ_1 and λ_2 are simple and touch the axis $\mathrm{Re}\lambda = 0$ in $(\pm k_c, \pm l_c)$. The imaginary parts of λ_1 and λ_2 are non zero in a neighborhood of these wave vectors.

$$
\begin{aligned}
\lambda_1(\mathbf{k}_c+\varepsilon\mathbf{K}) &= i\omega_H+\mathrm{i}\varepsilon(c_1\mathbf{K}_1+c_2\mathbf{K}_2)+\varepsilon^2 c_0 - \varepsilon^2(c_3\mathbf{K}_1^2+c_4\mathbf{K}_1\mathbf{K}_2+c_5\mathbf{K}_2^2),\\
\lambda_2(\mathbf{k}_c+\varepsilon\mathbf{K}) &= -i\omega_H - \mathrm{i}\varepsilon(c_1\mathbf{K}_1+c_2\mathbf{K}_2)+\varepsilon^2\overline{c}_0 - \varepsilon^2(\overline{c}_3\mathbf{K}_1^2+\overline{c}_4\mathbf{K}_1\mathbf{K}_2+\overline{c}_5\mathbf{K}_2^2),\\
\lambda_1((k_c,-l_c)+\varepsilon\mathbf{K}) &= i\omega_H + \mathrm{i}\varepsilon(c_1\mathbf{K}_1-c_2\mathbf{K}_2)+\varepsilon^2 c_0 - \varepsilon^2(c_3\mathbf{K}_1^2+\overline{c}_4\mathbf{K}_1\mathbf{K}_2+c_5\mathbf{K}_2^2),\\
\lambda_2((k_c,-l_c)+\varepsilon\mathbf{K}) &= -i\omega_H-\mathrm{i}\varepsilon(c_1\mathbf{K}_1-c_2\mathbf{K}_2)+\varepsilon^2\overline{c}_0 - \varepsilon^2(\overline{c}_3\mathbf{K}_1^2+c_4\mathbf{K}_1\mathbf{K}_2+\overline{c}_5\mathbf{K}_2^2).
\end{aligned}
$$

The ansatz for the derivation of the Ginzburg-Landau equations in (OR) is

$$
\begin{aligned}
\varepsilon\psi_A(x,y,z,t,\varepsilon) \;=\; & \varepsilon A_1(X,Y,T)e^{ik_cx+il_cy+i\omega_Ht}\hat{\varphi}_1(k_c,l_c,z,t)\\
&+\varepsilon A_2(X,Y,T)e^{ik_cx+il_cy-i\omega_Ht}\hat{\varphi}_2(k_c,l_c,z,t)\\
&+\varepsilon A_3(X,Y,T)e^{ik_cx-il_cy+i\omega_Ht}\hat{\varphi}_1(k_c,-l_c,z,t)\\
&+\varepsilon A_4(X,Y,T)e^{ik_cx-il_cy-i\omega_Ht}\hat{\varphi}_2(k_c,-l_c,z,t) + c.c. + \mathcal{O}(\varepsilon^2),
\end{aligned}
\tag{14}
$$

where $X = \varepsilon x, Y = \varepsilon y$ and $T = \varepsilon^2 t$, see fig.6 for an illustration of the distribution of these modes. Inserting (13) into (1) shows that the $A_1, \ldots, A_4$ have to satisfy the set of four

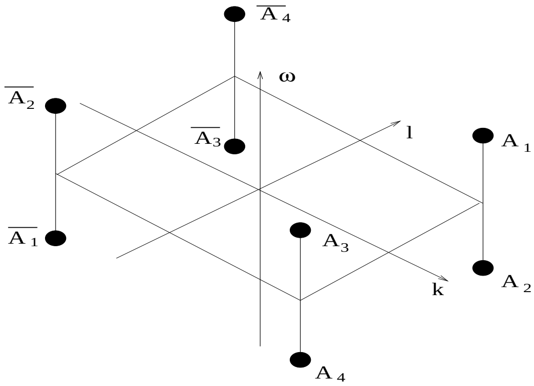

Figure 6: Mode distribution in the ansatz (14).

coupled Ginzburg-Landau equations

$$\partial_T A_1 = \frac{1}{\varepsilon} c_1 \partial_X A_1 + \frac{1}{\varepsilon} c_2 \partial_Y A_1 + c_0 A_1 + c_3 \partial_X^2 A_1 + c_4 \partial_X \partial_Y A_1 + c_5 \partial_Y^2 A_1 \\ + A_1(c_6|A_1|^2 + c_7|A_2|^2 + c_8|A_3|^2 + c_9|A_4|^2) + c_{10} A_2 \overline{A}_3 A_4, \tag{15}$$

$$\partial_T A_2 = -\frac{1}{\varepsilon} c_1 \partial_X A_2 - \frac{1}{\varepsilon} c_2 \partial_Y A_2 + \overline{c}_0 A_2 + \overline{c}_3 \partial_X^2 A_2 + \overline{c}_4 \partial_X \partial_Y A_2 + \overline{c}_5 \partial_Y^2 A_2 \\ + A_2(\overline{c}_6|A_2|^2 + \overline{c}_7|A_1|^2 + \overline{c}_8|A_3|^2 + \overline{c}_9|A_4|^2) + \overline{c}_{10} A_1 \overline{A}_3 A_4, \tag{16}$$

$$\partial_T A_3 = \frac{1}{\varepsilon} c_1 \partial_X A_3 - \frac{1}{\varepsilon} c_2 \partial_Y A_3 + c_0 A_3 + c_3 \partial_X^2 A_3 + \overline{c}_4 \partial_X \partial_Y A_3 + c_5 \partial_Y^2 A_3 \\ + A_3(c_6|A_3|^2 + c_7|A_2|^2 + c_8|A_4|^2 + c_9|A_1|^2) + c_{10} A_1 \overline{A}_2 A_4 \tag{17}$$

$$\partial_T A_4 = -\frac{1}{\varepsilon} c_1 \partial_X A_4 + \frac{1}{\varepsilon} c_2 \partial_Y A_4 + \overline{c}_0 A_4 + \overline{c}_3 \partial_X^2 A_4 + c_4 \partial_X \partial_Y A_4 + \overline{c}_5 \partial_Y^2 A_4 \\ + A_4(\overline{c}_6|A_4|^2 + \overline{c}_7|A_1|^2 + \overline{c}_8|A_3|^2 + \overline{c}_9|A_2|^2) + \overline{c}_{10} \overline{A}_1 A_2 A_3, \tag{18}$$

with $A_j = A_j(X, Y, T) \in \mathbb{C}$, $j = 1, \ldots, 4$, depending on $X, Y \in \mathbb{R}$ and $T \geq 0$ and with coefficients $c_1, c_2 \in \mathbb{R}$ and $c_0, c_3, \ldots, c_{10} \in \mathbb{C}$. The form of the nonlinearity again follows from equivariance under the two symmetries $k \mapsto -k$ and $l \mapsto -l$, see [DW99]. The appearance of, e.g., $c_{10} A_2 \overline{A}_3 A_4$ as the only purely mixed term in (15) follows from the fact that this is the only combination which yields $e^{ik_c x + il_c y + i\omega_H t}$, and similar in (16)–(18). These combinations can be read off from fig.6.

The amplitude equations (15)–(18) still depend in a singular way on the small perturbation parameter $0 < \varepsilon \ll 1$. Moreover, we have the four complex conjugate equations for the modes concentrated at $(-k_c, -l_c)$ and $(-k_c, l_c)$.

2.2 Case II

In Case II there are two surfaces λ_1 and λ_2 in a neighborhood of the critical wave numbers with

$$\mathrm{Re}\lambda_1(k, l) = \mathrm{Re}\lambda_2(k, l) \qquad \text{and} \qquad \mathrm{Im}\lambda_1(k, l) = -\,\mathrm{Im}\lambda_2(k, l).$$

Thus, generically we have $\nabla\lambda_1(k_c,0)\neq 0$ and $\nabla\lambda_1(k_c,l_c)\neq 0$. At the bifurcation point we have in (NR)

$$\mathrm{Re}\lambda_1(k_c,0)=0 \quad \text{and} \quad \mathrm{Im}\lambda_1(k_c,0)\neq 0,$$

see Fig. 7, while in (OR) we have the same situation as in Case I.

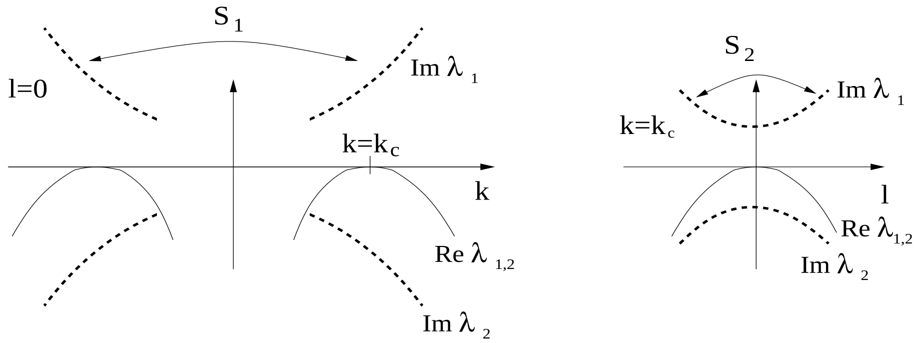

Figure 7: Case II (NR): The pictures show two cross sections through the surfaces λ_1 and λ_2, namely at $l=0$ and $k=k_c$. The Floquet exponents λ_1 and λ_2 are simple and touch the axis $\mathrm{Re}\lambda=0$ in $(\pm k_c,0)$. The imaginary parts of λ_1 and λ_2 are non zero in neighborhoods of these wave vectors.

(NR): Due to $\lambda_1(k,l)=\lambda_1(k,-l)$ we again have $\partial_l\lambda_1(k_c,0)=0$ and $\partial_k\partial_l\lambda_1(k_c,0)=0$. Therefore,

$$\begin{aligned}\lambda_1(\mathbf{k}_c{+}\varepsilon\mathbf{K}) &= i\omega_H{+}\mathrm{i}\varepsilon c_1\mathbf{K}_1{+}\varepsilon^2c_0-\varepsilon^2(c_3\mathbf{K}_1^2{+}c_5\mathbf{K}_2^2)+\mathcal{O}(\varepsilon^3),\\ \lambda_2(\mathbf{k}_c{+}\varepsilon\mathbf{K}) &= -i\omega_H-\mathrm{i}\varepsilon c_1\mathbf{K}_1{+}\varepsilon^2\overline{c}_0-\varepsilon^2(\overline{c}_3\mathbf{K}_1^2{+}\overline{c}_5\mathbf{K}_2^2)+\mathcal{O}(\varepsilon^3),\end{aligned} \tag{19}$$

with $c_1\in\mathbb{R}$ and $c_0,c_3,c_5,\in\mathbb{C}$. The ansatz

$$\begin{aligned}\varepsilon\psi_A(x,y,z,t,\varepsilon) \;=\; & \varepsilon A_1(X,Y,T)e^{ik_cx+i\omega_Ht}\hat{\varphi}_1(k_c,0,z,t)\\ & +\varepsilon A_2(X,Y,T)e^{ik_cx-i\omega_Ht}\hat{\varphi}_2(k_c,0,z,t)+c.c.+\mathcal{O}(\varepsilon^2)\end{aligned} \tag{20}$$

with $X=\varepsilon x$, $Y=\varepsilon y$ and $T=\varepsilon^2 t$ yields

$$\partial_T A_1 \;=\; \frac{c_1}{\varepsilon}\partial_X A_1+c_0A_1+c_3\partial_X^2A_1+c_5\partial_Y^2A_1+c_6A_1|A_1|^2+c_7A_1|A_2|^2, \tag{21}$$

$$\partial_T A_2 \;=\; -\frac{c_1}{\varepsilon}\partial_X A_2+\overline{c}_0A_2+\overline{c}_3\partial_X^2A_2+\overline{c}_5\partial_Y^2A_2+\overline{c}_6A_2|A_2|^2+\overline{c}_7A_2|A_1|^2\;, \tag{22}$$

with c_0,c_1,c_3,c_5 from (19) and $c_6,c_7\in\mathbb{C}$,

(OR): In (OR) with the ansatz (14) we again obtain the system (15)-(18).

2.3 The transition points

In this subsection let μ_1, respectively $\mu_{1,2}$, denote the critical surface(s) in (NR), and let again $\lambda_{1,2}$ denote those in (OR). Also, let $(\tilde{k}_c,0)$ denote the critical wave vector in (NR).

In Case I, μ_1 is not related to the surfaces λ_1 and λ_2 in (OR). Hence there is no transition between (NR) and (OR) on the linear level, see fig. 8a) for illustration. Thus, a weakly nonlinear analysis near the transition points yields a system of 5 coupled amplitude equations, namely (9) for A from (NR) and (15)–(18) for $A_1, \dots, A_4$ from (OR), with coupling between, e.g. A and A_1, of the form $c_{11}|A_1|A$ in (9) and $c_{12}|A|^2 A_1$ in (15). Near the treshhold of first instability the transition from (NR) to (OR) then essentially proceeds by changes of sign of the coefficients $\tilde{c}_0$ (from (9) (NR)) and c_0 (from (15)–(18), (OR)): on the (NR) side of ω_L we have $\tilde{c}_0 > 0$ and $\mathrm{Re} c_0 < 0$, while on the (OR) side of ω_L it is vice versa.

In Case II there are 2 subcases, IIa and IIb. If $\mu_{1,2}$ are not related to $\lambda_{1,2}$, Case IIa, see again fig. 8a), then we have a similar situation as in Case I. On the weakly nonlinear level we now obtain a system of six coupled amplitude equations, 2 from (NR) and 4 from (OR). Again on the (NR) side of ω_L we have $\mathrm{Re}\tilde{c}_0 > 0$ and $\mathrm{Re} c_0 < 0$, and vice versa on the (OR) side of ω_L.

The other subcase is Case IIb with $\lambda_j(k,l) = \mu_j(k,l)$ near ω_L and for (k,l) near $(\tilde{k}_c, 0)$, see fig. 8b). It follows that $(k_c, l_c) \to (\tilde{k}_c, 0)$ as we approach ω_L, and at $\omega = \omega_L$ we have $\mathrm{Re}\partial_l^2 \lambda_1(k_c, 0) = 0$. Due to $\lambda_1(k, -l) = \overline{\lambda}_1(k,l)$ we thus altogether have

$$\partial_l \lambda_1(k_c, 0) = \mathrm{Re}\partial_l^2 \lambda_1(k_c, 0) = \partial_l^3 \lambda_1(k_c, 0) = 0$$

at $\omega = \omega_L$. If we also had $\mathrm{Im}\partial_l^2 \lambda_1(k_c, 0) = 0$ then we could scale $Y = \varepsilon^{1/2} y$ in order to obtain two amplitude equations containing fourth order Y-derivatives. See [RD98] for an example where conditions equivalent to $\mathrm{Im}\lambda(k_c, 0) = \mathrm{Im}\partial_l^2 \lambda(k_c, 0) = 0$ hold due to reversibility. However, generically $\mathrm{Im}\partial_l^2 \lambda_1(k_c, 0) \neq 0$, fig. 8c), and therefore at the transition point with the same ansatz as in (NR) we again obtain (21)-(22) (with $\mathrm{Re} c_5 = 0$). A consistent expansion with $Y = \varepsilon^{1/2} y$ in order to obtain fourth order derivative terms is therefore not possible, and we again have to use the system of 6 coupled Ginzburg–Landau equations for a weakly nonlinear analysis.

a) Cases I and IIa b) Case IIb c) Case IIb, $k = k_c$ fixed.

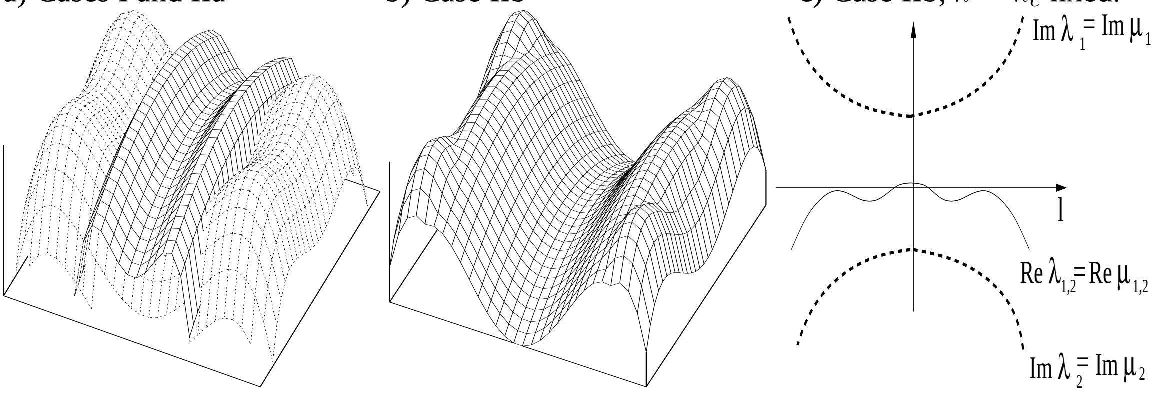

Figure 8: Sketches of $\mathrm{Re}\mu_{1,2}$ (from (NR)) and $\mathrm{Re}\lambda_{1,2}$ (from (OR)) for two possible scenarios at the transition between (NR) and (OR). In a) (Cases I and IIa) the surfaces μ_1 and $\lambda_{1,2}$, respectively $\mu_{1,2}$ and $\lambda_{1,2}$, are not related to each other. In b) we have $\mu_j = \lambda_j$ near $(\tilde{k}_c, 0)$ and $(k_c, l_c) \to (\tilde{k}_c, 0)$ as $\omega \to \omega_L$ (Case IIb). Consequently, $\partial_l^2 \mathrm{Re}\lambda_1(k_c, 0) = 0$ at $\omega = \omega_L$. However, in general still $\partial_l^2 \mathrm{Im}\lambda_1(k_c, 0) \neq 0$, as illustrated in the $l \mapsto \mu_j(k_c, l)$ section c).

3 Formal derivation of the amplitude equations

In order to keep the notational complexity on a reasonable level we concentrate on the Case II (NR). The Case I (NR) already has been handled in [BSU06]. Cases I-II (OR) are very similar to the subsequent lines.

3.1 Formal expansion in eigenfunctions

In Fourier space (1) yields

$$\partial_t \hat{V}(\mathbf{k},t) = \hat{M}(\mathbf{k},t)\hat{V}(\mathbf{k},t) + \hat{N}(\hat{V})(\mathbf{k},t), \tag{23}$$

with $\mathbf{k} \in \mathbb{R}^2$ and $\hat{V}(\mathbf{k},t)$ a vector-valued function of z. We derive the GLe from (23) under the assumptions from Case II (NR). For the subsequent analysis it is sufficient that the critical Floquet exponents λ_1 of $\hat{M}(t)$ are simple near $\mathbf{k}_c$, see Remark 4.1. However, in order to make things less abstract, i.e., to illustrate an algorithmic approach to the calculation of the coefficients of the nonlinearity here we assume the following: the linear operator $\hat{M}(\mathbf{k},t)$ with $\hat{M}(\mathbf{k},t) = \hat{M}(\mathbf{k},t+2\pi/\omega_0)$ has for every $\mathbf{k} \in \mathbb{R}^2$ and $t \in [0,2\pi/\omega_0)$ a Floquet Schauder basis $(\hat{\varphi}_j(\mathbf{k},t))_{j\in\mathbb{N}}$ of $L^2((-\pi/2,\pi/2),\mathbb{C}^7)$ of $2\pi/\omega_0$-periodic functions $\hat{\varphi}_j(\mathbf{k},t) = \hat{\varphi}_j(\mathbf{k},t+2\pi/\omega_0)$ solving

$$\partial_t \hat{\varphi}_j(\mathbf{k},t) = \hat{M}(\mathbf{k},t)\hat{\varphi}_j(\mathbf{k},t) - \lambda_j(\mathbf{k})\hat{\varphi}_j(\mathbf{k},t),$$

i.e. the Floquet functions $e^{\hat{\lambda}_j(\mathbf{k})t}\hat{\varphi}_j(\mathbf{k},t)$ are solution of $\partial_t \hat{V}(\mathbf{k},t) = \hat{M}(\mathbf{k},t)\hat{V}(\mathbf{k},t)$ and $\lambda_j(\mathbf{k})$ are the associated Floquet exponents. In other words, we assume for simplicity that there are no Jordan blocks in the monodromy operator for $\hat{M}(t)$. The functions $\hat{\varphi}_j$ are normalized by setting $\|\hat{\varphi}_j(\mathbf{k},0)\|_{L^2}=1$. For defining projections on the $\hat{\varphi}_j(\mathbf{k},t)$ we consider the adjoint problem $-\partial_t \hat{V}(\mathbf{k},t) = \hat{M}^*(\mathbf{k},t)\hat{V}(\mathbf{k},t)$. Consequently also this problem has for every $\mathbf{k} \in \mathbb{R}^2$ and $t \in [0,2\pi/\omega_0)$ a Floquet Schauder basis $(\hat{\varphi}_j^*(\mathbf{k},t))_{j\in\mathbb{N}}$ of $L^2((-\pi/2,\pi/2),\mathbb{C}^7)$ of $2\pi/\omega_0$-periodic functions $\hat{\varphi}_j^*(\mathbf{k},t) = \hat{\varphi}_j^*(\mathbf{k},t+2\pi/\omega_0)$ solving

$$-\partial_t \hat{\varphi}_j^*(\mathbf{k},t) = \hat{M}^*(\mathbf{k},t)\hat{\varphi}_j^*(\mathbf{k},t) - \overline{\lambda_j(\mathbf{k})}\hat{\varphi}_j^*(\mathbf{k},t),$$

and satisfying the orthogonality

$$\langle \hat{\varphi}_i^*, \hat{\varphi}_j \rangle = \delta_{ij}, \tag{24}$$

where $\langle \hat{u}, \hat{v} \rangle = \int_0^\pi \hat{u}(z)\overline{\hat{v}(z)}\, dz$. A solution $\hat{V}(\mathbf{k},t)$ of (23) is expanded in terms of the Floquet functions $\hat{\varphi}_j(\mathbf{k},t)$, i.e.

$$\hat{V}(\mathbf{k},t) = \sum_{j\in\mathbb{N}} \hat{a}_j(\mathbf{k},t)\hat{\varphi}_j(\mathbf{k},t) \quad \text{with} \quad \hat{a}_j(\mathbf{k},t) \in \mathbb{C}, \tag{25}$$

such that

$$\begin{aligned}\partial_t \left(\sum_{j\in\mathbb{N}} \hat{a}_j(\mathbf{k},t)\hat{\varphi}_j(\mathbf{k},t) \right) &= \sum_{j\in\mathbb{N}} ((\partial_t \hat{a}_j(\mathbf{k},t))\hat{\varphi}_j(\mathbf{k},t) + \hat{a}_j(\mathbf{k},t)\partial_t \hat{\varphi}_j(\mathbf{k},t)) \\ &= \sum_{j\in\mathbb{N}} \hat{a}_j(\mathbf{k},t)\hat{M}(\mathbf{k},t)\hat{\varphi}_j(\mathbf{k},t) + \hat{N}(\hat{V})(\mathbf{k},t).\end{aligned}$$

In order to find the equations for the coefficient functions $\hat{a}_j(\mathbf{k},t)$ we apply the adjoint eigenfunction $\hat{\varphi}_j^*(\mathbf{k},t)$ and find

$$\partial_t \hat{a}_j(\mathbf{k},t) = \hat{\lambda}_j(\mathbf{k})\hat{a}_j(\mathbf{k},t) + \langle \hat{\varphi}_j^*(\mathbf{k},t), \hat{N}(\mathbf{k},t)\rangle. \tag{26}$$

We used (24) and

$$-\langle \hat{\varphi}_j^*(\mathbf{k},t), \partial_t \hat{\varphi}_i(\mathbf{k},t)\rangle + \langle \hat{\varphi}_j^*(\mathbf{k},t), \hat{M}(\mathbf{k},t)\hat{\varphi}_i(\mathbf{k},t)\rangle = \langle \hat{\varphi}_j^*(\mathbf{k},t), \hat{\lambda}_j(\mathbf{k})\hat{\varphi}_i(\mathbf{k},t)\rangle = \hat{\lambda}_j(\mathbf{k})\delta_{ij}.$$

Our derivation of the GLe is now based on (26). For notational simplicity we avoid the explicit notation of the small parameter ε in the following. We make the ansatz

$$\begin{aligned}
a_1(\mathbf{x},t) &= \varepsilon A_{1,1}(\mathbf{X},T)e^{i\mathbf{k}_c\mathbf{x}+i\omega_H t} + \varepsilon^2 A_{2,2,1}(\mathbf{X},T)e^{2(i\mathbf{k}_c\mathbf{x}+i\omega_H t)} + \varepsilon^2 A_{2,0,1}(\mathbf{X},T)e^{2i\mathbf{k}_c\mathbf{x}} \\
&\quad + \varepsilon^2 A_{2,-2,1}(\mathbf{X},T)e^{2(i\mathbf{k}_c\mathbf{x}-i\omega_H t)} + \frac{\varepsilon^2}{2}A_{0,0,1}(\mathbf{X},T) + \varepsilon^2 A_{0,2,1}(\mathbf{X},T)e^{2i\omega_H t} + \text{c.c.}, \\
a_2(\mathbf{x},t) &= \varepsilon A_{1,-1}(\mathbf{X},T)e^{i\mathbf{k}_c\mathbf{x}-i\omega_H t} + \varepsilon^2 A_{2,2,2}(\mathbf{X},T)e^{2(i\mathbf{k}_c\mathbf{x}+i\omega_H t)} + \varepsilon^2 A_{2,0,2}(\mathbf{X},T)e^{2i\mathbf{k}_c\mathbf{x}} \\
&\quad + \varepsilon^2 A_{2,-2,2}(\mathbf{X},T)e^{2(i\mathbf{k}_c\mathbf{x}-i\omega_H t)} + \frac{\varepsilon^2}{2}A_{0,0,2}(\mathbf{X},T) + \varepsilon^2 A_{0,2,2}(\mathbf{X},T)e^{2i\omega_H t} + \text{c.c.}, \\
a_j(\mathbf{x},t) &= \varepsilon^2 A_{2,2,j}(\mathbf{X},T)e^{2(i\mathbf{k}_c\mathbf{x}+i\omega_H t)} + \varepsilon^2 A_{2,0,j}(\mathbf{X},T)e^{2i\mathbf{k}_c\mathbf{x}} \\
&\quad + \varepsilon^2 A_{2,-2,j}(\mathbf{X},T)e^{2(i\mathbf{k}_c\mathbf{x}-i\omega_H t)} + \frac{\varepsilon^2}{2}A_{0,0,j}(\mathbf{X},T) + \varepsilon^2 A_{0,2,j}(\mathbf{X},T)e^{2i\omega_H t} + \text{c.c.},
\end{aligned}$$

where $\omega_H = \lambda_1(\mathbf{k}_c)$, $j \in \mathbb{N} \setminus \{1,2\}$, $\mathbf{X} = (X,Y) = \varepsilon\mathbf{x} = \varepsilon(x,y) \in \mathbb{R}^2$, $\mathbf{k}_c = (k_c, 0)$, and $T = \varepsilon^2 t$. The idea of the notation is as follows: $A_{1,1}$ ($A_{1,-1}$) takes care of the critical modes

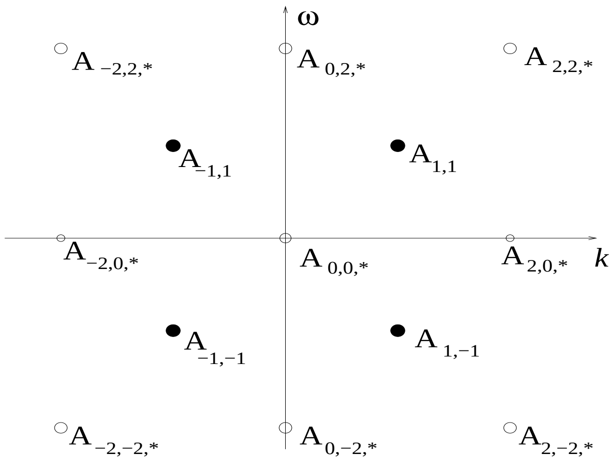

Figure 9: Mode distribution and notation in the extended ansatz.

concentrated at $\mathbf{k}_c$ in the first (second) equation; $A_{j_1,j_2,j}$ with $j_1, j_2 \in \{0, \pm 1, \pm 2\}$ and $j \in \mathbb{N}$ takes care of the noncritical modes in the j-th equation obtained by an interaction of A_{j_3,j_4} and A_{j_5,j_6} with $j_1 = j_3 + j_5$ and $j_2 = j_4 + j_6$, i.e., of the noncritical modes multiplying $\mathrm{e}^{ij_1\mathbf{k}_c + ij_2\omega t}$

in the j-th equation. See fig. 9. Since the a_j are real valued we have, e.g., $\overline{A}_{1,1}(\mathbf{X},T) = A_{-1,-1}(\mathbf{X},T)$.

With this ansatz we derive formally a GLe with time periodic coefficients. We write the nonlinearity of (1) in the form

$$N(V) = B(t,V,V) + C(t,V,V,V) + \mathcal{O}(V^4), \tag{27}$$

with bilinear and trilinear symmetric terms B and C, i.e., as in $f(u) = u^2 = b(u,u)$ and $g(u) = u^3 = c(u,u,u)$ with

$$b(u,v) = \frac{1}{2}(uv+vu) \quad \text{and} \quad c(u,v,w) = \frac{1}{6}(uvw+uwv+vuw+vwu+wuv+wvu).$$

Moreover, we introduce the abbreviations

$$\begin{aligned}\hat{B}_{j_1,j_2}(t,\mathbf{k},\mathbf{k}-\mathbf{m},\mathbf{m}) = \frac{1}{2}e^{-i\mathbf{kx}}\Big[&B(t,\hat{\varphi}_{j_1}(\mathbf{k}-\mathbf{m},t)e^{i(\mathbf{k}-\mathbf{m})\mathbf{x}},\hat{\varphi}_{j_2}(\mathbf{m},t)e^{i\mathbf{mx}}),\\ &+B(t,\hat{\varphi}_{j_2}(\mathbf{k}-\mathbf{m},t)e^{i(\mathbf{k}-\mathbf{m})\mathbf{x}},\hat{\varphi}_{j_1}(\mathbf{m},t)e^{i\mathbf{mx}})\Big],\end{aligned}$$

$$\begin{aligned}&\hat{C}_{j_1,j_2,j_3}(t,\mathbf{k},\mathbf{k}-\mathbf{l}_1,\mathbf{l}_1-\mathbf{l}_2,\mathbf{l}_2)\\ &= \frac{1}{6}e^{-i\mathbf{kx}}\Big[C(t,\hat{\varphi}_{j_1}(\mathbf{k}-\mathbf{l}_1,t)e^{i(\mathbf{k}-\mathbf{l}_1)\mathbf{x}},\hat{\varphi}_{j_2}(\mathbf{l}_1-\mathbf{l}_2,t)e^{i(\mathbf{l}_1-\mathbf{l}_2)\mathbf{x}},\hat{\varphi}_{j_3}(\mathbf{l}_2,t)e^{il_2\mathbf{x}})\\ &\qquad +C(t,\hat{\varphi}_{j_1}(\mathbf{k}-\mathbf{l}_1,t)e^{i(\mathbf{k}-\mathbf{l}_1)\mathbf{x}},\hat{\varphi}_{j_3}(\mathbf{l}_1-\mathbf{l}_2,t)e^{i(\mathbf{l}_1-\mathbf{l}_2)\mathbf{x}},\hat{\varphi}_{j_2}(\mathbf{l}_2,t)e^{il_2\mathbf{x}})\\ &\qquad +\ldots\\ &\qquad +C(t,\hat{\varphi}_{j_3}(\mathbf{k}-\mathbf{l}_1,t)e^{i(\mathbf{k}-\mathbf{l}_1)\mathbf{x}},\hat{\varphi}_{j_2}(\mathbf{l}_1-\mathbf{l}_2,t)e^{i(\mathbf{l}_1-\mathbf{l}_2)\mathbf{x}},\hat{\varphi}_{j_1}(\mathbf{l}_2,t)e^{il_2\mathbf{x}})\Big].\end{aligned}$$

For $\varepsilon^2 e^{0i\mathbf{x}}$ in the j-the equation we obtain

$$\begin{aligned}(\lambda_j(0,0)+2w_H)A_{0,2,j} &= -2\langle\hat{\varphi}_j^*,\hat{B}_{1,2}(t,0,\mathbf{k}_c,-\mathbf{k}_c)\rangle A_{1,1}A_{-1,1},\\ \lambda_j(0,0)A_{0,0,j} &= -2\langle\hat{\varphi}_j^*,\hat{B}_{1,1}(t,0,\mathbf{k}_c,-\mathbf{k}_c)\rangle |A_{1,1}|^2,\\ (\lambda_j(0,0)-2w_H)A_{0,-2,j} &= -2\langle\hat{\varphi}_j^*,\hat{B}_{1,2}(t,0,\mathbf{k}_c,-\mathbf{k}_c)\rangle A_{1,1}A_{-1,-1},\end{aligned} \tag{28}$$

where we omit the argument $(t,0)$ of $\hat{\varphi}_j^*$, and similar in the following. For $\varepsilon^2 e^{2i\mathbf{k}_c\mathbf{x}}$ in the j-th equation we obtain

$$\begin{aligned}(\lambda_j(2\mathbf{k}_c,0)+2w_H)A_{2,2,j} &= -2\langle\hat{\varphi}_j^*,\hat{B}_{1,1}(t,2\mathbf{k}_c,\mathbf{k}_c,\mathbf{k}_c)\rangle A_{1,1}^2,\\ \lambda_j(2\mathbf{k}_c,0)A_{2,0,j} &= -2\langle\hat{\varphi}_j^*,\hat{B}_{1,2}(t,2\mathbf{k}_c,\mathbf{k}_c,\mathbf{k}_c)\rangle A_{1,1}A_{1,-1},\\ (\lambda_j(2\mathbf{k}_c,0)-2w_H)A_{2,-2,j} &= -2\langle\hat{\varphi}_j^*,\hat{B}_{2,2}(t,2\mathbf{k}_c,\mathbf{k}_c,\mathbf{k}_c)\rangle A_{1,-1}^2.\end{aligned} \tag{29}$$

For $\varepsilon^3 e^{i\mathbf{k}_c\mathbf{x}}$ in the equation for $j=1$ we obtain

$$
\begin{aligned}
&\partial_T A_{1,1} = \lambda_1 A_{1,1} + 2\sum_{j\in\mathbb{N}} \left\langle \hat\varphi_1^*, \hat B_{1,j}(t,\mathbf{k}_c,\mathbf{k}_c,0)A_{1,1}A_{0,0,j} + \hat B_{2,j}(t,\mathbf{k}_c,\mathbf{k}_c,0)A_{1,-1}A_{0,2,j}\right\rangle \\
&+2\sum_{j\in\mathbb{N}} \left\langle \hat\varphi_1^*, \hat B_{1,j}(t,\mathbf{k}_c,-\mathbf{k}_c,2\mathbf{k}_c)A_{-1,-1}A_{2,2,j} + \hat B_{2,j}(t,\mathbf{k}_c,-\mathbf{k}_c,2\mathbf{k}_c)A_{-1,1}A_{2,0,j}\right\rangle \\
&+2e^{2i\omega_H t}\sum_{j\in\mathbb{N}} \left\langle \hat\varphi_1^*, \hat B_{1,j}(t,\mathbf{k}_c,\mathbf{k}_c,0)A_{1,1}A_{0,2,j} + \hat B_{2,j}(t,\mathbf{k}_c,-\mathbf{k}_c,2\mathbf{k}_c)A_{-1,1}A_{2,2,j}\right\rangle \\
&+2e^{-2i\omega_H t}\sum_{j\in\mathbb{N}} \left\langle \hat\varphi_1^*, \hat B_{1,j}(t,\mathbf{k}_c,\mathbf{k}_c,0)A_{1,1}A_{0,-2,j} + \hat B_{2,j}(t,\mathbf{k}_c,\mathbf{k}_c,0)A_{1,-1}A_{0,0,j}\right\rangle \\
&+2e^{-2i\omega_H t}\sum_{j\in\mathbb{N}} \left\langle \hat\varphi_1^*, \hat B_{1,j}(t,\mathbf{k}_c,-\mathbf{k}_c,2\mathbf{k}_c)A_{-1,-1}A_{2,0,j} + \hat B_{2,j}(t,\mathbf{k}_c,-\mathbf{k}_c,2\mathbf{k}_c)A_{-1,1}A_{2,-2,j}\right\rangle \\
&+2e^{-4i\omega_H t}\sum_{j\in\mathbb{N}} \left\langle \hat\varphi_1^*, \hat B_{2,j}(t,\mathbf{k}_c,\mathbf{k}_c,0)A_{-1,-1}A_{0,-2,j} + \hat B_{1,j}(t,\mathbf{k}_c,-\mathbf{k}_c,2\mathbf{k}_c)A_{-1,1}A_{2,-2,j}\right\rangle \\
&+3\left\langle \hat\varphi_1^*, C_{1,1,1}(\mathbf{k}_c,\mathbf{k}_c,\mathbf{k}_c,-\mathbf{k}_c)|A_{1,1}|^2A_{1,1} + 2C_{1,2,2}(\mathbf{k}_c,\mathbf{k}_c,\mathbf{k}_c,-\mathbf{k}_c)A_{1,1}|A_{1,-1}|^2\right\rangle \\
&+3e^{2\mathrm{i}\omega t}\left\langle \hat\varphi_1^*, C_{1,1,2}(\mathbf{k}_c,\mathbf{k}_c,\mathbf{k}_c,-\mathbf{k}_c)A_{1,1}^2A_{-1,1}\right\rangle \\
&+3e^{-2\mathrm{i}\omega t}\left\langle \hat\varphi_1^*, C_{2,2,2}(\mathbf{k}_c,\mathbf{k}_c,\mathbf{k}_c,-\mathbf{k}_c)A_{1,-1}^2A_{-1,-1} + 2C_{1,1,2}(\mathbf{k}_c,\mathbf{k}_c,\mathbf{k}_c,-\mathbf{k}_c)|A_{1,1}|^2A_{1,-1}^2\right\rangle \\
&+3e^{-4\mathrm{i}\omega t}\left\langle \hat\varphi_1^*, C_{2,2,1}(\mathbf{k}_c,\mathbf{k}_c,\mathbf{k}_c,-\mathbf{k}_c)A_{1,-1}^2A_{-1,-1}\right\rangle,
\end{aligned}
$$

and a similar equation for $\partial_T A_{1,-1}$, taking into account the symmetries of the problem. If we eliminate the A_{j_1,j_2,j_2} by the time dependent algebraic equations (28) and (29) we obtain a system of Ginzburg-Landau equations for $B_1 := A_{1,1}$ and $B_2 := A_{1,-1}$ alone, namely

$$
\begin{aligned}
\partial_T B_1 =& c_0 B_1 + c_1\varepsilon^{-1}\partial_X B_1 + c_3\partial_X^2 B_1 + c_5\partial_Y^2 B_1 \\
&+ d_6(t)B_1|B_1|^2 + d_7(t)B_1|B_2|^2 + d_8(t)B_2|B_2|^2 e^{-2i\omega_H t} \\
&+ d_9(t)B_2|B_1|e^{-2i\omega_H t} + d_{10}(t)B_2^2\overline{B}_1 e^{-4i\omega_H t} + d_{11}(t)B_1^2\overline{B}_2 e^{2i\omega_H t},
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
\partial_T B_2 =& \overline{c}_0 B_2 - c_1\varepsilon^{-1}\partial_X B_2 + \overline{c}_3\partial_X^2 B_2 + \overline{c}_5\partial_Y^2 B_2 \\
&+ \overline{d}_6(t)B_2|B_2|^2 + \overline{d}_7(t)B_2|B_1|^2 + \overline{d}_8(t)B_1|B_1|^2 e^{2i\omega_H t} \\
&+ \overline{d}_9(t)B_1|B_2|e^{2i\omega_H t} + \overline{d}_{10}(t)B_1^2\overline{B}_2 e^{4i\omega_H t} + \overline{d}_{11}(t)B_2^2\overline{B}_1 e^{-2i\omega_H t}
\end{aligned} \tag{31}
$$

with c_0, c_1, c_3, c_5 from (19), and with time-periodic coefficients $d_j(t)$, $j = 6,\ldots,11$. In the next step by some averaging argument we will eliminate the terms with the $e^{2im\omega_H t}$ factors and prove that only the mean values

$$
c_j = \int_0^{\omega_0} d_j(t)\,\mathrm{d}t, \quad j = 6,7,
$$

of the highly oscillating terms $d_j(t) = d_j(T/\varepsilon^2)$ play a role, while for $j = 8,\ldots,11$ we always have $\int_0^{\omega_0} d_j(t)\mathrm{e}^{\mathrm{i}m_j\omega t}\,\mathrm{d}t = \mathcal{O}(\varepsilon^2)$.

Thus, finally we consider (21),(22), which we repeat for convenience,

$$
\partial_T A_1 = c_0 A_1 + c_1\varepsilon^{-1}\partial_X A_1 + c_3\partial_X^2 A_1 + c_5\partial_Y^2 A_1 + c_6 A_1|A_1|^2 + c_7 A_1|A_2|^2, \tag{32}
$$

$$
\partial_T A_2 = \overline{c}_0 A_2 - c_1\varepsilon^{-1}\partial_X A_2 + \overline{c}_3\partial_X^2 A_2 + \overline{c}_5\partial_Y^2 A_2 + \overline{c}_6 A_2|A_2|^2 + \overline{c}_7 A_2|A_1|^2. \tag{33}
$$

These equations still depend on the small bifurcation parameter ε in a singular way. By going into a moving frame these terms can be eliminated. However A_1 and A_2 then depend on different variables, namely $(X + \varepsilon^{-1}c_1T, Y)$ respectively $(X - \varepsilon^{-1}c_1T, Y)$. Thus it can be expected that only the mean values of A_1 and A_2 over large intervals play a role and so (32),(33) can be transfered into so called mean field coupled Ginzburg-Landau equations, see [PW96]. In case that A_1 and A_2 are spatially localized they simplify further and decouple completely, cf. Remark 4.7 and [Sch97].

3.2 Comparison of the Ginzburg-Landau equations

The system (32),(33) of averaged Ginzburg-Landau equations approximates the nonaveraged system (30),(31) of Ginzburg-Landau equations in the following sense.

Theorem 3.1 *Let $m \geq 2$. Then for all $C_1 > 0$ and $T_0 > 0$ there exist $C_2 > 0$, $\varepsilon_0 > 0$ such that the following holds. For all $\varepsilon \in (0,1)$ let $(A_1, A_2) \in C([0,T_0], H^m \times H^m)$ be a solution of the averaged system (32),(33) satisfying*

$$\sup_{T\in[0,T_0]} \|A_j(\cdot,T)\|_{H^m} \leq C_1.$$

Then for all $\varepsilon \in (0,\varepsilon_0)$ the nonaveraged system (30),(31) has a solution (B_1, B_2) satisfying

$$\sup_{T\in[0,T_0]} \|A_j(\cdot,T) - B_j(\cdot,T)\|_{H^m} \leq C_2\varepsilon^2 .$$

Proof. We write (30),(31) and (32),(33) as

$$\partial_T \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 B_1 \\ \Lambda_2 B_2 \end{pmatrix} + \begin{pmatrix} \tilde{C}_1(t,B,B,B) \\ \tilde{C}_2(t,B,B,B) \end{pmatrix},$$

$$\partial_T \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 A_1 \\ \Lambda_2 A_2 \end{pmatrix} + \begin{pmatrix} C_1(A,A,A) \\ C_2(A,A,A) \end{pmatrix},$$

respectively, with linear parts Λ_j, symmetric autonomous cubic parts C_j, and symmetric nonautonomous cubic parts $\tilde{C}_j$, $j = 1,2$, where $t = T/\varepsilon^2$.

Let $B(T) = A(T) + \varepsilon^2 R(T)$. Then $R(T) = \varepsilon^{-2}(B(T) - A(T))$ fulfills

$$\partial_T R = \Lambda R + 3\tilde{C}(t,A,A,R) + 3\varepsilon^2\tilde{C}(t,A,R,R) + \varepsilon^4\tilde{C}(t,R,R,R) + \varepsilon^{-2}I(A), \tag{34}$$

where

$$I(A) = \tilde{C}(t,A,A,A) - C(A,A,A)$$

is an inhomogeneity. For $R(0) = 0$ the variation of constant formula yields

$$R(T) = \int_0^T \mathrm{e}^{(T-\tau)\Lambda} \Big[3\tilde{C}(t,A,A,R) + 3\varepsilon^2\tilde{C}(t,A,R,R) + \varepsilon^4\tilde{C}(t,R,R,R) + \varepsilon^{-2}I(A) \Big](\tau)\,\mathrm{d}\tau.$$

The crucial estimate is

$$\left\| \int_0^T \mathrm{e}^{(T-\tau)\Lambda} \varepsilon^{-2} I(A(\tau))\,\mathrm{d}\tau \right\|_{H^m} \leq C_A. \tag{35}$$

For instance, the term $\varepsilon^{-2} I_{11} := \varepsilon^{-2}(d_6(\tau/\varepsilon^2) - c_6)|A_1|^2 A_1$ in the first component $I_1(A)$ yields

$$\begin{aligned} \varepsilon^{-2} I_{11} :=& \left\| \int_0^T \mathrm{e}^{(T-\tau)\Lambda_1} |A_1|^2 A_1 \partial_\tau g_6(\tau)\,\mathrm{d}\tau \right\|_{H^m} \\ \leq& \left\| \left[\mathrm{e}^{(T-\tau)\Lambda_1} |A_1|^2 A_1 g_6(\tau) \right]_0^T \right\|_{H^m} \\ &+ \left\| \int_0^T \mathrm{e}^{(T-\tau)\Lambda_1} \big(\Lambda_1 |A_1|^2 A_1 - 2\partial_T A_1 |A_1|^2 - A_1^2 \partial_T \overline{A}_1 \big) g_6(\tau)\,\mathrm{d}\tau \right\|_{H^m}, \end{aligned} \tag{36}$$

where we set $\partial_\tau g_6 = \varepsilon^{-2}(d_6(\tau/\varepsilon^2) - c_6)$, hence

$$g_6(\tau) = \int_0^{\tau/\varepsilon^2} d_6(s) - c_6\,\mathrm{d}s,$$

which is $\mathcal{O}(1)$ bounded by definition of $c_6 = \int_0^{1/\omega_0} d_6(s)\,\mathrm{d}s$ since d_6 is $2\pi/\omega_0$ periodic. Next, replacing $\partial_T A_1$ and $\partial_T \overline{A}_1$ in (36) by the right hand side of (32) we find that $\varepsilon^{-2} I_{11} \leq C$. Similar estimates for the remaining terms yield (35), and the theorem now follows by a simple application of Gronwall's lemma. ■

Remark 3.2 It is easy to see that for every $m \geq 4$ we have $\|\partial_T B_j\|_{H^{m-2}} = \mathcal{O}(1)$, $j = 1, 2$ by expressing for instance $\partial_T B_1$ by the right hand side of (30), but $\partial_T^2 B_j = \mathcal{O}(\varepsilon^{-2})$. ⌋

3.3 Estimates for the residual

For the proof of the approximation result we need estimates for the residual, defined by

$$\mathrm{Res}(V) = -\partial_t V + M(t)V + \tilde{N}(t, V),$$

i.e. for those terms which do not cancel after inserting the approximation in (1). Since we loose ε^{-1} due to the scaling properties of the L^2-norm in $\mathbb{R}^2$, we extend the above approximation as in the autonomous case [Sch99a] by higher order terms. We refrain from writing down these terms and the lengthy calculation of the equations for the functions appearing in this extended ansatz. We only remark that the new amplitude functions in the ansatz satisfy linearized inhomogeneous Ginzburg-Landau equations and some inhomogeneous linear algebraic equations.

Next we split the critical modes from the noncritical modes, i.e. the modes with positive or slightly negative growth rates from the ones with strictly negative growth rates. In order to do so we define

$$E_c = \chi_{|\mathbf{k}-\mathbf{k}_c|<\delta} + \chi_{|\mathbf{k}+\mathbf{k}_c|<\delta},$$

and $E_s = 1 - E_c$ for a small fixed $\delta > 0$ independent of ε.

Remark 3.3 Due to the disjoint supports of E_c and $B(E_cV_1, E_cV_2)$ in Fourier space we have $E_cB(E_cV_1, E_cV_2) = 0$. ⌋

Let $\tilde{\psi}_A(\cdot, t)$ be the approximation defined through the extended ansatz and

$$\varepsilon\tilde{\psi}_A = \varepsilon\psi_c + \varepsilon^2\psi_s, \tag{37}$$

with $E_s\psi_c = 0,\ E_c\psi_s = 0$. Like in the autonomous case we have the following lemma.

Lemma 3.4 *Fix $C_1 > 0$. For all $\varepsilon \in (0,1)$ let $(A_1, A_2) \in C([0, T_0], H^8(\mathbb{R}^2, \mathbb{C}))$ be a family of solutions of (32,33) with $\sup\limits_{\varepsilon\in[0,1]} \sup\limits_{T\in[0,T_0]} \|A_j(\cdot, T)\|_{H^8} \le C_1$. Then there exists a $C_2 > 0$ such that, $\forall\varepsilon \in [0, 1]$,*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\psi_A(\cdot,t) - \tilde{\psi}_A(\cdot,t)\|_{H^4} \le C_2\varepsilon^2, \qquad \sup_{t\in[0,T_0/\varepsilon^2]} (\|\psi_s(\cdot,t)\|_{H^4} + \|\psi_c(\cdot,t)\|_{H^4}) \le C_2,$$

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|E_s(\mathrm{Res}(\varepsilon\tilde{\psi}_A(\cdot,t)))\|_{H^4} \le C_2\varepsilon^3, \qquad \sup_{t\in[0,T_0/\varepsilon^2]} \|E_c(\mathrm{Res}(\varepsilon\tilde{\psi}_{A_1}(\cdot,t)))\|_{H^4} \le C_2\varepsilon^4.$$

4 The approximation results

System (1) for $(n_2, n_3, v, \rho, \sigma)$ is fully nonlinear and a mixture of different types of PDEs, like quasilinear parabolic equations and balance laws. Thus, a local existence and uniqueness result for (1), which is fundamental for any approximation result, is highly non-trivial, and we are not aware of one in the literature. Therefore we consider a regularized version of the WEM. In order to obtain a semilinear system, i.e., for purely mathematical reasons, we add artificially a regularizing differential operator

$$\Lambda V = (-\beta\Delta^2 n_2, -\beta\Delta^2 n_3, -\beta Q\Delta^2 v, -\beta\Delta^2\rho, -\beta\Delta^2\sigma)$$

with small $\beta > 0$ to the right hand side of (1). Thus we consider

$$\partial_t V = \Lambda V + M(t)V + \tilde{N}(t, V) \tag{38}$$

equipped with the boundary conditions from the non-regularized system (2), and additional artificial boundary conditions due to the regularization, namely

$$\partial_z^2 n_2 = \partial_z^2 n_3 = \partial_z^2 v_1 = \partial_z^2 v_2 = \partial_z^2 v_2 = \partial_z\sigma = \partial_z^3\sigma = \rho = \partial_z^2\rho = 0. \tag{39}$$

For small $\beta > 0$ the regularized system and the original system show qualitatively the same bifurcation behavior. In particular, all calculations from Sections 2 and 3 also apply to (38).

Remark 4.1 Setting

$$\begin{aligned} V(\mathbf{x}, z, t) =& \varepsilon A_1(\mathbf{X}, T)e^{ik_cx}\hat{\varphi}_1(k_c, 0, z, t) + \varepsilon A_2(\mathbf{X}, T)e^{ik_cx}\hat{\varphi}_2(k_c, 0, z, t) \\ &+ \text{c.c.} + \varepsilon^2 W_s(\mathbf{x}, z, t), \end{aligned}$$

there exists a $2\pi/\omega_0$ periodic bounded invertible transform $Q(t) : L^2(\mathbb{R}^2 \times (-\pi/2, \pi/2)) \to L^2(\mathbb{R}^2 \times (-\pi/2, \pi/2))$ such that $W_s(t) = Q(t)Z(t)$ and $Z(t)$ fulfills

$$\partial_t Z(t) = \Lambda_s Z(t) + \varepsilon^{-2}\tilde{N}(V, t)$$

with $\Lambda_s^{-1} : L^2(\mathbb{R}^2 \times (-\pi/2, \pi/2)) \to H^4(\mathbb{R}^2 \times (-\pi/2, \pi/2))$ bounded, see, e.g. [Hen81, Theorem 7.2.3], which can be applied to our regularized system. Thus, the contribution of the quadratic terms to the cubic coefficients $d_6(t), \ldots, d_{11}(t)$ via coupling with stable modes is obtained via Λ_s^{-1} instead of (28) and (29). On the other hand, in practical calculations only a finite number of stable Floquet solutions $\hat{\varphi}_j(\mathbf{k}, t)$ at $(k, l) = (0, 0)$ and $(k, l) = (2k_c, 0)$ are calculated and the inversion is done via (28) and (29). This is why in sec.3 w.l.o.g. we also used this algorithmic approach. ⌋

We have two kinds of approximation results. In Case I (NR), where the amplitude equations are independent of the small parameter ε, the result is as follows, see [BSU06].

Theorem 4.2 *Let $m \geq 8$ and $A = A(X, Y, T)$ be a solution of the GLe (9) for $T \in [0, T_0]$, satisfying*

$$\sup_{T\in[0,T_0]} \|A(T)\|_{H^m} < \infty.$$

Then there are $\varepsilon_0 > 0$ and $C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions V of (38) satisfying

$$\sup_{t\in[0,T_0/\varepsilon^2]} \ \sup_{(x,y,z)\in\mathbb{R}^2\times(-\pi/2,\pi/2)} |V(x, y, z, t) - \varepsilon\psi_A(x, y, z, t)| \leq C\varepsilon^2.$$

In case that the amplitude equations still depend on the small bifurcation parameter, i.e. in the Cases I (OR) and II, the result is as follows, here formulated for the amplitude equations (32), (33), i.e., Case II (NR).

Theorem 4.3 *Let $m \geq 8$ and $(A_1, A_2) = (A_1, A_2)(X, Y, T, \varepsilon)$ be a family of solutions of the coupled Ginzburg-Landau equations (32),(33), satisfying*

$$\sup_{\varepsilon\in(0,1)} \sup_{T\in[0,T_0]} (\|A_1(T)\|_{H^m} + \|A_2(T)\|_{H^m}) < \infty.$$

Then there are $\varepsilon_0 > 0$ and $C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions V of (38) satisfying

$$\sup_{t\in[0,T_0/\varepsilon^2]} \ \sup_{(x,y,z)\in\mathbb{R}^2\times(-\pi/2,\pi/2)} |V(x, y, z, t) - \varepsilon\psi_A(x, y, z, t, \varepsilon)| \leq C\varepsilon^2.$$

Remark 4.4 As a consequence of Theorems 4.2 and 4.3 the dynamics known for (9) and (15)-(18) can be found approximately in system (38), too. The error of order $\mathcal{O}(\varepsilon^2)$ is much smaller than the approximation $\varepsilon\psi_A$ and the solution V which are both of order $\mathcal{O}(\varepsilon)$ for all $T \in [0, T_0]$ or $t \in [0, T_0/\varepsilon^2]$, respectively. This fact should not be taken for granted: there are modulation equations [Sch95b] which, although derived by a formal perturbation analysis, do not reflect the true dynamics of the original system. The proof of Theorem 4.2 is not trivial since solutions of order $\mathcal{O}(\varepsilon)$ have to be bounded on a time interval of length $\mathcal{O}(1/\varepsilon^2)$. ⌋

Remark 4.5 Ginzburg–Landau equations have been derived for example for reaction-diffusion systems and hydrodynamical stability problems, as the Bénard and the Taylor-Couette problem. For these examples these GLe have been justified as amplitude equation by a number of mathematical results: so called approximation and attractivity theorems have been established by a several authors for model problems, but also for general systems including the Navier-Stokes equation, cf. [CE90, vH91, Eck93, Sch94c, Sch94a, Sch95a, TBD$^+$96]. Nowadays the theory is a well established mathematical tool which can be used to prove stability results [Uec01, SU03], upper semi-continuity of attractors [MS95, Sch99b] and global existence results [Sch94b, Sch99a]. As a consequence of our approximation results, this mathematical theory can be transfered almost one to one in case of systems with external time periodic forcing described by semilinear parabolic equations, see [BSU06] for discussion. Hence, the Ginzburg-Landau equation really gives a proper description of autonomous and time-periodic systems near the bifurcation point. ⌋

Remark 4.6 Theorem 4.2 can be improved in a number of directions. The error can be made smaller by adding higher order terms to the approximation. However the time scale cannot be extended [vH91]. By a more involved analysis [Sch94b] less regularity for the solutions of the Ginzburg-Landau equation is needed. In the y-independent case the space $H^m(\mathbb{R} \times (0,\pi))$ can be replaced by the larger space $H^m_{l,u}(\mathbb{R} \times (0,\pi))$ equipped with the norm $\|u\|_{H^m_{l,u}} = \sup_{x\in\mathbb{R}} \|u\|_{H^m((x,x+1)\times(0,\pi)}$ which contains constants, periodic functions, or fronts in contrast to H^m. The difficulties in $\mathbb{R}^2 \times (0,\pi)$ are due to the non smoothness of the symbol of the inverse Stokes operator or of the projection Q in case of two unbounded space directions. See, e.g., the proof of Lemma A.4 and Remark A.5. ⌋

Remark 4.7 For spatially localized solutions all amplitude equations decouple. For instance, assume that $A_1, A_2 \in H^m(2)$ where $H^m(n) = \{u \in H^m : \|u\rho^n\|_{H^m} < \infty\}$, $\rho(X) = (1+X^2)^{1/2}$. Then introducing $X_1 = X + \frac{c_1}{\varepsilon}T$ and $X_2 = X - \frac{c_1}{\varepsilon}T$ the system (21),(22) reduces to

$$\begin{aligned} \partial_T A_1 &= c_0 A_1 + c_3 \partial^2_{X_1} A_1 + c_5 \partial^2_Y A_1 + c_6 A_1 |A_1|^2 && (40)\\ \partial_T A_2 &= \overline{c}_0 A_2 + \overline{c}_3 \partial^2_{X_2} A_2 + \overline{c}_5 \partial^2_Y A_2 + \overline{c}_6 A_2 |A_2|^2 \,. && (41) \end{aligned}$$

The terms $c_7 A_1 |A_2|^2$ and $\overline{c}_7 A_2 |A_1|^2$ from (21),(22) no longer occur since their influence on the dynamics can be estimated to be of order $\mathcal{O}(\varepsilon)$. If A_1 and A_2 are spatially localized the interaction time of these terms is $\mathcal{O}(\varepsilon)$ due to the fact that they move with a relative velocity of order $\mathcal{O}(1/\varepsilon)$ through each other, cf. [Sch97].

Also note that the singular terms in the amplitude equations, e.g., $c_1\varepsilon^{-1}\partial_X A_1$ and $-c_1\varepsilon^{-1}\partial_X A_2$ in (32) and (33) are no problem for the validity result, which starts with a given family of solutions of (32) and (33). The singular terms do not occur in the error equations, e.g., (44) below. ⌋

Remark 4.8 For non small values of ε, i.e. away from the bifurcation point other amplitude equations take the role of the Ginzburg-Landau equation. In general the locally preferred patterns do not fit together globally, and so there will be some phase shifts in the pattern which will be transported or transformed by dispersion and diffusion. For the description of the evolution of the local wavenumber q of these pattern amplitude equations can be derived, such as phase diffusion equations, conservation laws, and the Burgers equation. Recently, approximation results in the above sense have been proved for this reduction, see [MS04b, MS04a, DSSS05]. For the modulation of the associated solutions in the two-dimensional real Ginzburg-Landau equation (9) the results from [MS04b] transfer almost line for line. The rescaled phase diffusion system for the evolution of the local wave numbers $q = (q_x, q_y)$ is given by

$$\partial_\tau q = \Delta q + \nabla(\nabla \cdot f(q)) \tag{42}$$

with coefficients $c_1, c_2 \in \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ a smooth mapping. Combining the transfered approximation result from [MS04b] for this equation with Theorem 4.2 shows that the dynamics of (42) can be found approximately in the regularized WEM, too. ⌋

Finally, we state the approximation result in case (OR).

Theorem 4.9 *Let $m \geq 8$, and let (A_1, A_2, A_3, A_4) be a family of solutions of the set coupled Ginzburg-Landau equations (15)–(18), satisfying*

$$\sup_{\varepsilon\in(0,1)} \sup_{T\in[0,T_0]} (\|A_1(T)\|_{H^m} + \|A_2(T)\|_{H^m} + \|A_3(T)\|_{H^m} + \|A_4(T)\|_{H^m}) < \infty.$$

Then there are $\varepsilon_0 > 0$ and $C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions V of (38) satisfying

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{(x,y,z)\in\mathbb{R}^2\times(-\pi/2,\pi/2)} |V(x, y, z, t) - \varepsilon\psi_A(x, y, z, t, \varepsilon)| \leq C\varepsilon^2.$$

5 Local existence and uniqueness

For the local existence and uniqueness of the solutions of the semilinear parabolic system (38) we follow [Hen81]. The regularizing term Λ is a sectorial operator in the space

$$\mathcal{X} = L^2(\mathbb{R}^2 \times [-\pi/2, \pi/2], \mathbb{R}^7) \cap \{Qu = u\}$$

with domain of definition

$$\mathcal{X}^1 = \{U \in H^4 \mid U \text{ satisfies the boundary conditions } (2) \text{ and } (39)\} \cap \{Qu = u\}.$$

Therefore Λ generates an analytic semigroup in the space $\mathcal{X}$. It is a lengthy but straightforward calculation (see Remark A.1) to prove that the remaining terms $M(t)V + \tilde{N}(t, V)$ on the right hand side of (38) are smooth mappings from H^3 into $\mathcal{X} \subset L^2$.

The interpolation space $\mathcal{X}^\alpha$ can be embedded into H^3 for $\alpha > 3/4$. Hence the term N_{rem} is a locally Lipschitz-continuous mapping from $\mathcal{X}^\alpha$ into $\mathcal{X}$ for $\alpha > 3/4$. Therefore, all assumptions of [Hen81, Theorem 3.3.3] are satisfied, which yields the following result.

Theorem 5.1 *Fix $\alpha \in (3/4, 1)$ and let $V_0 \in \mathcal{X}^\alpha$. Then there exists a $t_0 > 0$ and a unique solution $V \in C([0,t_0], \mathcal{X}^\alpha)$ of (38) with $V(0) = V_0$.*

Remark 5.2 The existence of solutions to (38) and hence also to the error equations (44) below is guaranteed as long as the solutions in $\mathcal{X}^\alpha$ are bounded. Thus it is sufficient to bound the $\mathcal{X}^\alpha$-norm of the error in the following. Since $\mathcal{X}^\alpha$ can be embedded into H^3 for $\alpha > 3/4$ and H^3 into C_b^0 in three space dimensions, the estimate in Theorem 4.2 follows from the associated estimate for the $\mathcal{X}^\alpha$-norm. $\rfloor$

6 The proof of the error estimates

As a major step of the proof of Theorem 4.3 we show that the solutions of (38) can be approximated via the solutions of the non averaged Ginzburg-Landau equations.

Theorem 6.1 *Let $C_1 > 0$. Let $(B_1, B_2) = (B_1, B_2)(X, T; \varepsilon) \in C([0, T_0], H^8 \times H^8)$, be a family of solutions of the non averaged GLe (30),(31) with $\sup_{\varepsilon\in[0,1]} \sup_{T\in[0,T_0]} (\|B_1(\cdot, T)\|_{H^8} + \|B_1(\cdot, T)\|_{H^8}) \leq C_1$. Then there are $\varepsilon_0 > 0$ and $C_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions V of (1) with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|V(t) - \varepsilon\psi_B(t)\|_{\mathcal{X}^\alpha} \leq C_2\varepsilon^2.$$

Proof. We write (38) as

$$\partial_t V = \tilde{M}(t)V + B(t, V, V) + C(t, V, V, V) + \mathcal{O}(\|V\|^4_{\mathcal{X}^\alpha}), \tag{43}$$

where $\tilde{M}(t) = \Lambda + M(t)$, and where B and C contain the quadratic and cubic terms, respectively, cf. (27). Inserting

$$V = \varepsilon\psi_c + \varepsilon^2\psi_s + \varepsilon^2 R_c + \varepsilon^3 R_s \ ,$$

with $R_c = E_c R_c$, $R_s = E_s R_s$, $\psi_c = E_c\psi_c$, and $\psi_s = E_s\psi_s$ gives

$$\begin{aligned} \partial_t R_c &= \tilde{M}(t)R_c + \varepsilon^2 L_c(R) + \varepsilon^3 N_c(R) + \varepsilon^2 \mathrm{Res}_c \ , \\ \partial_t R_s &= \tilde{M}(t)R_s + L_s(R_c) + \varepsilon N_s(R) + \mathrm{Res}_s \ , \end{aligned} \tag{44}$$

where

$$\mathrm{Res}_c = \varepsilon^{-4} E_c(\mathrm{Res}(\varepsilon\psi_B)) \ , \qquad \mathrm{Res}_s = \varepsilon^{-3} E_s(\mathrm{Res}(\varepsilon\psi_B)) \ ,$$
$$L_c(R) = 2E_c(B(R_s, \psi_c) + B(R_c, \psi_s)) \ , \qquad L_s(R_c) = 2E_s B(R_c, \psi_c) \ ,$$

and where $N_c(R)$ and $N_s(R)$ satisfy

$$\|N_c(R)\|_{\mathcal{X}} \leq C(D_c, D_s)(\|R_c\|_{\mathcal{X}^\alpha} + \|R_s\|_{\mathcal{X}^\alpha})^2, \tag{45}$$

$$\|N_s(R)\|_{\mathcal{X}} \leq C\|R_s\|_{\mathcal{X}^\alpha} + C(D_c, D_s)(\|R_c\|_{\mathcal{X}^\alpha} + \|R_s\|_{\mathcal{X}^\alpha})^2, \tag{46}$$

as long as

$$\|R_c\|_{\mathcal{X}^\alpha} \leq D_c \quad \text{and} \quad \|R_s\|_{\mathcal{X}^\alpha} \leq D_s \ . \tag{47}$$

Here $C(D_c, D_s)$ is a constant depending on D_c and D_s independent of $0 \leq \varepsilon \ll 1$. The constants D_c and D_s will be chosen later on independent of ε. System (44) is solved with initial datum $(R_c(0), R_s(0)) = (0, 0)$. The solutions of

$$\partial_t R = \tilde{M}(t) R, \quad R|_{t=\tau} = R_0$$

define via $R(t) = \mathcal{K}(t,\tau)R_0$ a linear evolution operator $\mathcal{K}(t,\tau)$ which satisfies $\mathcal{K}(t,\tau) = \mathcal{K}(t + 2\pi/\omega, \tau + 2\pi/\omega)$ and whose properties are summarized in the following lemma.

Lemma 6.2 *There exist $C, \sigma > 0$ independent of $0 < \varepsilon \ll 1$ such that for the stable part we have*

$$\|\mathcal{K}(t,\tau)E_s\|_{\mathcal{X}\to\mathcal{X}^\alpha} \leq C \ \max(1, (t-\tau)^{-\alpha})e^{-\sigma(t-\tau)},$$

and for the critical part we have

$$\|\mathcal{K}(t,\tau)E_c\|_{\mathcal{X}\to\mathcal{X}^\alpha} \leq C \max(1, (t-\tau)^{-\alpha})e^{C\varepsilon^2(t-\tau)}.$$

Proof. The operator $M(t)$ is a relatively bounded perturbation of the sectorial operator Λ. Thus $M(t)$ generates an evolution operator whose growth properties are fixed by the location of the Floquet spectrum, see [Hen81, Theorem 7.1.3 and Exercise 1 on p.197]. This spectrum already has been discussed in Section 2 and yields the above growth rates. The constant C can be chosen independent of ε due to the fact that the critical eigenvalues for fixed $\mathbf{k}$ near $\mathbf{k}_c$ are semisimple. ■

To conclude the proof of Theorem 6.1 we apply the variation of constant formula to (44) and obtain

$$\begin{aligned} R_c(t) &= \int_0^t \mathcal{K}(t,\tau)E_c^h(\varepsilon^2 L_c(R) + \varepsilon^3 N_c(R) + \varepsilon^2 \mathrm{Res}_c)(\tau)d\tau \ , \\ R_s(t) &= \int_0^t \mathcal{K}(t,\tau)E_s^h(L_s(R_c) + \varepsilon N_s(R) + \mathrm{Res}_s)(\tau)d\tau \ . \end{aligned}$$

Let $S_i(s) := \sup_{0\leq t\leq s} \|R_i(t)\|_{\mathcal{X}^\alpha}$, $(i = s, c)$. Using Lemma 3.4, (45) and

$$\Big(\int_0^t C \ \max(1, \tau^{-\alpha})e^{-\sigma\tau}d\tau\Big) = \mathcal{O}(1)$$

for all $t > 0$, we obtain that

$$\begin{aligned} S_s(t) &\leq CS_c(t) + \varepsilon(CS_s(t) + C_s(D_c, D_s)(S_c(t) + S_s(t))^2) + C_{\mathrm{Res}}, \\ &\leq CS_c(t) + 1 + C_{\mathrm{Res}}, \end{aligned} \tag{48}$$

provided that

$$\varepsilon(CD_s + C_s(D_c, D_s)(D_c + D_s)^2) \leq 1. \tag{49}$$

Similarly, we find

$$
\begin{aligned}
S_c(t) \leq & \varepsilon^2 \int_0^t C \max(1,(t-\tau)^{-\alpha})(S_c(\tau)+S_s(\tau)) \\
& \qquad + \varepsilon C_s(D_c,D_s)(S_c(\tau)+S_s(\tau))^2 + C_{\mathrm{Res}} d\tau, \\
\leq & \varepsilon^2 \int_0^t C \max(1,(t-\tau)^{-\alpha})(S_c(\tau)+S_s(\tau)) + 1 + C_{\mathrm{Res}} d\tau,
\end{aligned}
$$

provided that

$$
\varepsilon C_s(D_c,D_s)(D_c+D_s)^2 \leq 1. \tag{50}
$$

Thus, (48) yields $S_c(t) \leq \varepsilon^2 \int_0^t C \max(1,(t-\tau)^{-\alpha})(S_c(\tau)+1+C_{\mathrm{Res}})d\tau$. Rescaling time, i.e. $T=\varepsilon^2 t$ and applying Gronwall's inequality [Hen81, Lemma 7.1.1] yields

$$
S_c(t) \leq C(1+C_{\mathrm{Res}})T_0 e^{CT_0} =: D_c
$$

for all $t \in [0, T_0/\varepsilon^2]$. Then $S_s(t) \leq CD_c + 1 + C_{\mathrm{Res}} =: D_s$ by (48). Thus, Theorem 6.1 follows by choosing $\varepsilon_0 > 0$ so small that for all $\varepsilon \in (0,\varepsilon_0)$ the conditions (49) and (50) are satisfied. ■

It remains to conclude Theorem 4.3 from Theorem 6.1 and Theorem 3.1. Let ψ_A be the approximation constructed via the solution $A=(A_1,A_2)$ of the averaged GLe (32),(33), and let ψ_B be the approximation constructed via the solution $B=(B_1,B_2)$ of the non-averaged GLe (30),(31). Moreover, let V be a solution from Theorem 6.1. Due to the embedding $\mathcal{X}^\alpha \subset C_b^0$ we have

$$
\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi_B(\cdot,t,\varepsilon) - V(\cdot,t)\|_{C_b^0} = \mathcal{O}(\varepsilon^2).
$$

From Theorem 3.1 and $H^2 \subset C_b^0$ we have $\sup_{T\in[0,T_0]} \|A(\cdot,T)-B(\cdot,T)\|_{C_b^0} = \mathcal{O}(\varepsilon^2)$ which implies $\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi_B(\cdot,t,\varepsilon) - \varepsilon\psi_A(\cdot,t,\varepsilon)\|_{C_b^0} = \mathcal{O}(\varepsilon^2)$. Hence, by the triangle inequality we have

$$
\begin{aligned}
\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi_A(\cdot,t,\varepsilon) - V(\cdot,t)\|_{C_b^0} & \leq C \sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi_A(\cdot,t,\varepsilon) - V(\cdot,t)\|_{C_b^0} \\
& \leq C\Big(\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi_A(\cdot,t,\varepsilon) - \varepsilon\psi_B(\cdot,t,\varepsilon)\|_{C_b^0} + \sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi_B(\cdot,t,\varepsilon) - V(\cdot,t)\|_{C_b^0}\Big) \\
& = \mathcal{O}(\varepsilon^2).
\end{aligned}
$$

The proof of Theorem 4.3 is complete. ■

7 Discussion

The electro-hydrodynamic instabilities of nematic liquid crystals may lead to complicated patterns. Here we analyzed three generic cases, namely Case I normal rolls (NR) (single real Ginzburg–Landau equation (9)) and Case II NR (2 coupled complex GL equations (21),(22)) and Case II OR (oblique rolls) (4 coupled complex GL equations (15)–(18)). In the latter two cases the amplitude equations still depend in a singular way on the small bifurcation parameter ε, which however can be removed by going into (separately) comoving frames, yielding nonlocal amplitude equations. For spatially localized solutions these decouple completely.

Moreover, for a regularized model we showed the validity of these amplitude equations. This puts studying the dynamics of the WEM using the respective amplitude equations on firm mathematical grounds. Thus, as a next step one may study in detail the dynamics of (21),(22) resp. (15)–(18). See [DO04] and the references therein for some first results, which show that these dynamics are very rich. A further open problem is to remove the artificial reguarization of the WEM. This will be subject of further research.

A Appendix

A.1 Description of the WEM

The following presentation and non-dimensionalization of the WEM follows [Tre96] and [DO04]. The director field n of unit vectors, the fluid velocity v and the pressure p in the presence of an electric field E satisfy

$$\begin{aligned} (\partial_t + v\cdot\nabla)n &= \omega\times n + \delta^{\perp}(\lambda A n - h)\,, && (51)\\ P_2(\partial_t + v\cdot\nabla)v &= -\nabla p - \nabla\cdot(T^{visc}+\Pi) + \pi^2\rho E\,, && (52)\\ \nabla\cdot v &= 0\,, && (53) \end{aligned}$$

for $(x,y,z)\in\Omega=\mathbb{R}^2\times(0,\pi)$. Herein,

$$\omega = (\nabla\times v)/2 \qquad (54)$$

is the vorticity. The molecular field h is given by

$$h = 2\left(\frac{\partial f}{\partial n} - \nabla\cdot\frac{\partial f}{\partial\nabla n}\right) - \varepsilon_a\pi^2(n\cdot E)E \qquad (55)$$

where

$$2f = (\nabla\cdot n)^2 + K_2[n\times(\nabla\times n)]^2 + K_3[n\cdot(\nabla\times n)]^2\,, \qquad (56)$$

is the elastic energy density describing splay, twist (K_2), and bend (K_3) deformations. We refer to [DO04] for a physical interpretation of the constants P_2, λ, K_2, K_3, and ε_a. The electric field $E = E(x,y,z,t)\in\mathbb{R}^3$ is considered to be quasistationary, i.e. rot $E = 0$. It is split into an external forcing and some potential part, i.e.

$$E = E_p(t)(0,0,1)^T - \nabla\phi\,, \quad \text{where} \quad E_p(t) = E_0\,\cos\omega_0 t. \qquad (57)$$

The tensors A, and T^{visc} are, respectively, the shear flow tensor

$$A_{ij} = (\partial_i v_j + \partial_j v_i)/2 \qquad (58)$$

and the viscous stress tensor

$$-T_{ij}^{visc} = \sum_{k=1}^{3}(\alpha_1 n_i n_j n_k \sum_{l=1}^{3}(n_l A_{kl}) + \alpha_2 n_j m_i + \alpha_3 n_i m_j + \alpha_4 A_{ij} + \alpha_5 n_j n_k A_{ki} + \alpha_6 n_i n_k A_{kj}) \tag{59}$$

with coefficients $\alpha_1, \ldots, \alpha_6$, and where

$$m = \delta^{\perp}(\lambda A n - h) \ . \tag{60}$$

The tensor Π is the nonlinear Ericksen stress tensor

$$\Pi_{ij} = \sum_{k=1}^{3} \frac{\partial f}{\partial n_{k,j}} n_{k,i}. \tag{61}$$

The projection tensor

$$\delta_{ij}^{\perp} = \delta_{ij} - n_i n_j \tag{62}$$

in (51) guarantees that $|n| = 1$ as long as the solution exists. The charge density ρ and the deviation of the local conductivity σ from 1 satisfy

$$P_1(\partial_t + v \cdot \nabla)\rho = -\nabla \cdot (\mu E \sigma) \ , \tag{63}$$

$$(\partial_t + v \cdot \nabla)\sigma = -\alpha^2\pi^2\nabla \cdot (\mu E \rho) - \frac{r}{2}(\sigma^2 - 1 - P_1\pi^2\alpha\rho^2) \ . \tag{64}$$

Finally the system is closed by Poisson's law

$$\rho = \nabla \cdot (\varepsilon E) \ . \tag{65}$$

The dielectric tensor ε and conductivity tensor μ are given by $\varepsilon_{ij} = \delta_{ij} + \varepsilon_a n_i n_j$ and $\mu_{ij} = \delta_{ij} + \sigma_a n_i n_j$, respectively. The parameters P_1 and P_2 are Prandtl–type time scale ratios. Again we refer [DO04] for a physical interpretation of the constants P_1, σ_a, α, and r.

Using Poisson's law E, respectively ϕ can be expressed in terms of ρ and so (51)-(53) and (63)-(64) can be rewritten as a system of dynamical equations for n, v, ρ, σ.

Summary: Since $n_1^2 + n_2^2 + n_3^2 = 1$ for our purposes it is sufficient to consider n_2 and n_3. Hence we finally consider

$$\partial_t n_2 = \langle e_2, -(v \cdot \nabla)n + \omega \times n + \delta^{\perp}(\lambda A n - h)\rangle \ , \tag{66}$$

$$\partial_t n_3 = \langle e_3, -(v \cdot \nabla)n + \omega \times n + \delta^{\perp}(\lambda A n - h)\rangle \ , \tag{67}$$

$$\partial_t v = P_2^{-1} Q(-(v \cdot \nabla)v - \nabla \cdot (T^{visc} + \Pi) + \pi^2 \rho E) \ , \tag{68}$$

$$\partial_t \rho = -v \cdot \nabla\rho - P_1^{-1}\nabla \cdot (\mu E \sigma), \tag{69}$$

$$\partial_t \sigma = -v \cdot \nabla\sigma - \alpha^2\pi^2\nabla \cdot (\mu E \rho) - \frac{r}{2}(2\sigma + \sigma^2 - P_1\pi^2\alpha\rho^2), \tag{70}$$

under the boundary conditions

$$n_2 = n_3 = v_1 = v_2 = v_3 = \phi = 0,$$

where Q is the projection on the divergence-free vector fields $\{v \mid \nabla \cdot v = 0\}$, see Sec. A.3, and where $E = E(n, \rho, E_p)$ is defined through (57) and (65) under the boundary conditions $\phi|_{z=0,\pi} = 0$, see Sec. A.2. As already said the WEM equations are invariant under arbitrary translations in x and y and under the reflections $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ defined in (3), (4), and (5).

Remark A.1 The right hand side of the non regularized WEM is a smooth mapping from H^3 into L^2. In order to see this let $\phi, n, v, \sigma \in H^3$. Then we have $\omega \in H^2$ by (54), $f \in H^2$ by (56), $E \in H^2$ by (57), $h \in H^1$ by (55), $\delta^\perp \in H^3$ by (62), $A \in H^2$ by (58), $m \in H^1$ by (60), $T^{visc} \in H^1$ by (59), and $\Pi \in H^2$ by (61). Hence the right hand side of (51) is in H^1 and the right hand side of (52) is in H^0. We have $\varepsilon \in H^3$ and $\mu \in H^3$. Then $\rho \in H^1$ and so the right hand side of (64) is in H^0. ⌋

A.2 The definition of $E = E(n, \rho, E_p)$

To express E, respectively ϕ, in terms of ρ we have to solve

$$\rho = \sum_{k=1}^{3} \partial_k(\varepsilon_{km} E_m) = \sum_{k=1}^{3}\sum_{m=1}^{3} \partial_k((\varepsilon_\perp \delta_{km} + \varepsilon_a n_k n_m)(E_p \delta_{m3} - \partial_m \phi))$$

with respect to ϕ under the boundary conditions $\phi|_{z=0,\pi} = 0$. We find

$$(M + G)\phi = F(n, \rho, E_p)$$

where

$$\begin{aligned} F(n, \rho, E_p) &= \rho - \sum_{k=1}^{3}\sum_{m=1}^{3} \partial_k((\varepsilon_\perp \delta_{km} + \varepsilon_a n_k n_m) E_p \delta_{m3}) , \\ M\phi &= \varepsilon_\perp \Delta\phi + \varepsilon_a \partial_1\partial_1\phi, \qquad G\phi = \varepsilon_a \sum_{k=1}^{3}\sum_{m=1}^{3} \partial_k(n_k n_m \partial_m \phi) - \varepsilon_a \partial_1\partial_1\phi. \end{aligned}$$

Lemma A.2 *The linear operator M^{-1} is bounded from H^s into H^{s+2}.*

Proof. We have to prove the invertibility of the operator M with the boundary conditions $\phi\big|_{z=\pm\frac{d}{2}} = 0$. Thus, to solve $M\phi(x, y, z) = f(x, y, z)$ we use Fourier series

$$\begin{aligned} \phi(x, y, z) &= \int\int (\sum_{m\in\mathbb{N}} \hat{\phi}(k, l, m) e^{ikx+ily} \sin(mz))\, dkdl, \\ f(x, y, z) &= \int\int (\sum_{m\in\mathbb{N}} \hat{f}(k, l, m) e^{ikx+ily} \sin(mz))\, dkdl. \end{aligned}$$

This yields $(-\varepsilon_a k^2 - \varepsilon_\perp(k^2 + l^2 + m^2))\hat{\phi}(k, l, m) = \hat{f}(k, l, m)$, or equivalently

$$\hat{\phi}(k, l, m) = -\frac{\hat{f}(k, l, m)}{(\varepsilon_a k^2 + \varepsilon_\perp(k^2 + l^2 + m^2))}.$$

We use that the H^s-norm of a function $v(x) = \sum_{m=0}^{\infty} v_m(x)e^{imx}$ is equivalent to the $\ell^2(s)$-norm of the Fourier coefficients, i.e. $\|(v_m)_{m\in\mathbb{N}_0}\|^2_{\ell^2(s)} = \sum_{m=0}^{\infty} |v_m|^2(1+m^2)^s$, such that

$$\begin{aligned}
\|\phi\|^2_{H^{s+2}} &= \int\int\sum_{m\in\mathbb{N}} |\hat{\phi}|^2\varepsilon_\perp(1+k^2+l^2+m^2)^{s+2}\,dkdl \\
&= \int\int\sum_{m\in\mathbb{N}} |\hat{f}|^2\frac{(1+k^2+l^2+m^2)^{s+2}}{(\varepsilon_a k^2+\varepsilon_\perp(k^2+l^2+m^2))^s}\,dkdl \\
&\le \sup_{k,l,m}\Big|\frac{1+k^2+l^2+m^2}{\varepsilon_a k^2+\varepsilon_\perp(k^2+l^2+m^2)}\Big|^2\int\int\sum_{m\in\mathbb{N}} |\hat{f}|^2(1+k^2+l^2+m^2)^s\,dkdl \\
&\le C\int\int\sum_{m\in\mathbb{N}} |\hat{f}|^2(1+k^2+l^2+m^2)^s\,dkdl = C\|f\|^2_{H^s}. \qquad \blacksquare
\end{aligned}$$

Hence the electric potential ϕ satisfies

$$(1+GM^{-1})M\phi = F(n,\phi,\rho,E_p)\ ,$$

where GM^{-1} is small for $\tilde{n} = n-(1,0,0)^T$ small. By using Neumann's series we finally obtain

$$\phi = M^{-1}(1+GM^{-1})^{-1}F(v,\rho,E_p). \tag{71}$$

Lemma A.3 *Let $\|V\|_{H^2} > 0$ be sufficiently small. Then the operator $M^{-1}(1+GM^{-1})^{-1}$ is bounded from L^2 into L^2.*

Proof. The operators $M^{-1}{:}L^2 \to H^2$ and $G{:}H^2 \to L^2$ are bounded. Moreover, $\|GM^{-1}\|_{L^2\to L^2}$ is small if $\|V\|_{H^2} > 0$ is small. Neumann's series gives the boundedness of $(1+GM^{-1})^{-1} : L^2 \to L^2$, but then also $M^{-1}(1+GM^{-1})^{-1} : L^2 \to L^2$ is bounded. $\blacksquare$

A.3 The projection onto divergence free vector fields

In the following we restrict ourselves to the hydrodynamic part of (1). We define the projection Q onto divergence free vector fields by $v = Qf$, where v solves

$$v - \nabla p \;=\; f, \quad \nabla\cdot v = 0, \quad v_3|_{z=0,\pi} = 0. \tag{72}$$

Lemma A.4 *The projection Q is continuous from H^m onto $\{v \in H^m\ :\ \nabla{\cdot}v{=}0,\ v_3|_{z=\pm\pi/2}{=}0\}$.*

Proof. In order to solve (72) we consider the Fourier transformed system

$$v_1 - ikp = f_1, \quad v_2 - ilp = f_2, \quad v_3 - \partial_z p = f_3, \quad ikv_1 + ilv_2 + \partial_z v_3 = 0,$$

together with the boundary conditions. This can be solved by the ansatz

$$v_1 = \sum_{m=0}^{\infty} v_{1,m}\cos(mz), \;\; v_2 = \sum_{m=0}^{\infty} v_{2,m}\cos(mz), \;\; v_3 = \sum_{m=0}^{\infty} v_{3,m}\sin(mz),$$

$$f_1 = \sum_{m=0}^{\infty} f_{1,m}\cos(mz), \;\; f_2 = \sum_{m=0}^{\infty} f_{2,m}\cos(mz), \;\; f_3 = \sum_{m=0}^{\infty} f_{3,m}\sin(mz),$$

$$p = \sum_{m=0}^{\infty} p_m\cos(mz).$$

We obtain

$$
\begin{aligned}
&v_{1,m} - ikp_m = f_{1,m}, \quad v_{2,m} - ilp_m = f_{2,m}, \quad v_{3,m} - mp_m = f_{3,m},\\
&ikv_{1,m} + ilv_{2,m} - mv_{3,m} = 0,
\end{aligned}
$$

which is solved for $m \neq 0$ by

$$
\begin{aligned}
\begin{pmatrix} v_{1,m} \\ v_{2,m} \\ v_{3,m} \end{pmatrix} &= A_m(k,l) \begin{pmatrix} f_{1,m} \\ f_{2,m} \\ f_{3,m} \end{pmatrix} \\
&= \frac{1}{m^2+k^2+l^2} \begin{pmatrix} m^2+l^2 & -lk & -ikm \\ -lk & m^2+k^2 & -ilm \\ ikm & ilm & k^2+l^2 \end{pmatrix} \begin{pmatrix} f_{1,m} \\ f_{2,m} \\ f_{3,m} \end{pmatrix}.
\end{aligned}
$$

The entries of the matrices $A_m(k,l)$ are bounded uniformly with respect to m, k and l, i.e. there exists a C such that for all m, k, and l $|v_{j,m}(k,l)| \leq C \sum_{j=1}^{3} |f_{j,m}(k,l)|$. For $m = 0$ we obtain

$$
\begin{pmatrix} v_{1,0} \\ v_{2,0} \\ v_{3,0} \end{pmatrix} = \frac{1}{k^2+l^2} \begin{pmatrix} l^2 & -lk & 0 \\ -lk & k^2 & 0 \\ 0 & 0 & k^2+l^2 \end{pmatrix} \begin{pmatrix} f_{1,0} \\ f_{2,0} \\ f_{3,0} \end{pmatrix}.
$$

Again the entries of the matrices $A_0(k,l)$ are bounded uniformly with respect to k and l, i.e. there exists a C such that for all k, and l

$$
|v_{j,0}(k,l)| \leq C \sum_{j=1}^{3} |f_{j,0}(k,l)|.
$$

The solution is extended to k=l=0 by $v_{1,m} = f_{1,m}$, $v_{2,m} = f_{2,m}$ and $v_{3,m} = f_{3,m}$. The assertion follows by using that the H^s-norm of a function $v(x) = \sum_{m=0}^{\infty} v_m(x) e^{imx}$ is equivalent to the $\ell^2(s)$-norm $\|(v_m)_{m\in\mathbb{N}_0}\|^2_{\ell^2(s)} = \sum_{m=0}^{\infty} |v_m|^2 (1+m^2)^s$ of the Fourier coefficients. ■

Remark A.5 The extension to k=0 can be made smoothly in $\mathbb{R} \times (0,\pi)$ such that multiplier theory in $H^m_{l,u}$ spaces can be applied in order to extend these results from the smaller H^m spaces to the larger $H^m_{l,u}$ spaces. However, the extension is not smooth in case of two unbounded directions. Nevertheless, in this case $Q\nabla : H^{m+1}_{l,u} \to H^m_{l,u}$ is still a smooth operation, cf. [SS01], such that the Navier-Stokes equations itself in $\mathbb{R}^2$ can be solved in $H^m_{l,u}$ spaces, cf. [GMS01] for a result in C^0_b-spaces. However, the term ρE in (68) cannot be expressed as a derivative, i.e., with the ∇-operator in front. Thus, this idea does not apply to the equations of the weak electrolyte model (66)-(70). ⌋

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