

Well-posedness of some initial-boundary-value problems for dynamo-generated poloidal magnetic fields

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Abstract

Given a bounded domain $G \subset \mathbb{R}^d$, $d \geq 3$, we study smooth solutions of a linear parabolic equation with non-constant coefficients in G , which at the boundary have to C^1 -match with some harmonic function in $\mathbb{R}^d \setminus \overline{G}$ vanishing at spatial infinity.

This problem arises in the framework of magnetohydrodynamics if certain dynamo-generated magnetic fields are considered: For example, in the case of axisymmetry or for non-radial flow fields the poloidal scalar of the magnetic field solves the above problem.

We first investigate the Poisson problem in G with the above described boundary condition as well as the associated eigenvalue problem and prove the existence of smooth solutions. As a by-product we obtain the completeness of the well-known poloidal “free decay modes” in \mathbb{R}^3 if G is a ball. Smooth solutions of the evolution problem are then obtained by Galerkin approximation based on these eigenfunctions.

Key Words: Magnetohydrodynamics, dynamo theory, poloidal field, harmonic field.

1 Introduction

We are concerned in this paper with the following initial-boundary-value problem:

$$\partial_t u - a\Delta u = b \cdot \nabla u + c u \quad \text{in } G \times \mathbb{R}_+, \quad (1.1a)$$

$$\Delta u = 0 \quad \text{in } \widehat{G} \times \mathbb{R}_+, \quad (1.1b)$$

$$u \text{ and } \nabla u \text{ continuous} \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+, \quad (1.1c)$$

$$u(x, t) \rightarrow u_\infty(t) \quad \text{for } |x| \rightarrow \infty, t \in \mathbb{R}_+, \quad (1.1d)$$

$$u(\cdot, 0) = u_0 \quad \text{on } G \times \{t = 0\}. \quad (1.1e)$$

Here, $G \subset \mathbb{R}^d$, $d \geq 3$ is a bounded domain with (sufficiently) smooth boundary ∂G , and $\widehat{G} := \mathbb{R}^d \setminus \overline{G}$. The scalar-valued coefficients a and c , and the vector-valued coefficient b are sufficiently smooth functions of $x \in G$ and $t \in \mathbb{R}_+$; a is, moreover, bounded from below by $a_0 > 0$. The asymptotic behaviour of solutions at spatial infinity is described by the (given) function $u_\infty : \mathbb{R}_+ \rightarrow \mathbb{R}$, and the initial-value u_0 is prescribed on G only.

Problem (1.1) arises in the context of magnetohydrodynamic dynamo theory: The generation of a magnetic field B by motion of a liquid conductor (of conductivity $\eta > 0$) according to some prescribed flow field v is described by the induction equation (cf. [13])

$$\partial_t B = \nabla \times (v \times B) - \nabla \times (\eta \nabla \times B), \quad \nabla \cdot B = 0. \quad (1.2)$$

Equation (1.2)₁ constitutes a system of parabolic equations for the magnetic field components. In general, the flow field couples these components in a nontrivial way which makes the

question for “dynamo solutions”, i.e. solutions which do not decay in time, difficult to answer. Only in special situations a field component or a related scalar quantity decouples, and a general decay result, a so-called antidynamo theorem, may be obtained. For instance, if the conductor fills a ball $B_R \subset \mathbb{R}^3$ with radius R , if the conductivity is radially symmetric, and if the flow field has no radial component the quantity $P := B \cdot x$ satisfies the scalar problem (cf. Appendix A or [11]):

$$\partial_t P - \eta \Delta P = -\nabla \cdot (vP) \quad \text{in } B_R \times \mathbb{R}_+, \quad (1.3a)$$

$$\Delta P = 0 \quad \text{in } \widehat{B}_R \times \mathbb{R}_+, \quad (1.3b)$$

$$P \text{ and } \nabla P \text{ continuous} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \quad (1.3c)$$

$$P(x, t) = O(|x|^{-2}) \quad \text{for } |x| \rightarrow \infty, t \in \mathbb{R}_+, \quad (1.3d)$$

$$P(\cdot, 0) = P_0, \quad \langle P_0 \rangle = 0 \quad \text{on } B_R \times \{t = 0\}. \quad (1.3e)$$

Note that conditions (1.1b) and (1.1d) with $u_\infty \equiv 0$ imply the spatial decay condition $P(x, t) = O(|x|^{-1})$ (see Appendix C). The stronger condition (1.3d) is a consequence of the additional zero-spherical-mean condition $\langle P \rangle := \frac{1}{4\pi r^2} \int_{S_r} P_0 ds = 0$, $r \in (0, R)$ on the initial value. This condition is preserved by eqs. (1.3a,b) and holds, consequently, for P on $\mathbb{R}^3 \times \mathbb{R}_+$. Equation (1.3b) describes a vacuum field outside the conductor and condition (1.3c) guarantees a continuous magnetic field throughout space.

Another instance is the axisymmetric dynamo problem in ordinary space \mathbb{R}^3 . Again, a scalar quantity describing the poloidal part of the magnetic field decouples and, if reformulated in \mathbb{R}^5 , is precisely a solution of problem (1.1) (see Appendix B). Note that the conductor is here assumed to be axisymmetric but need not be a ball. It is this application which motivates the investigation of problem (1.1) in more than 3 dimensions and in domains more general than balls.

The focus of dynamo theory is less on existence theorems than on decay results for the magnetic field under certain restrictions on the magnetic field and/or the flow field, thus excluding dynamo action under these restrictions. However, proving decay results requires sometimes the solution of an auxiliary problem. For instance, in proving a “non-radial velocity theorem” for solutions of problem (1.3) one needs positive solutions of an auxiliary problem of type (1.1); and it is this application which requires a non-zero asymptotic condition like (1.1d) (cf. [11]). Similarly in the axisymmetric problem, Backus makes use of solutions of an auxiliary problem (cf. [3]). He made the existence of such solutions plausible by physical arguments but could not establish them rigorously. It is the aim of the present paper to prove rigorously the existence of smooth solutions of problem (1.1).

A problem closely related to (1.1) has been treated in [16]: It is inspired by the dynamo problem with plane symmetry (which means $d = 2$) and differs from (1.1) in that $c \equiv 0$ and by a different asymptotic condition at spatial infinity. The authors treat this problem and two related ones in arbitrary dimension and prove existence of solutions and, moreover, some decay results. However, this problem does not precisely meet the requirements of the non-radial problem (1.3) nor those of the axisymmetric problem. Moreover, only weak solutions are established, whose behavior at the boundary remains open.

The basic idea of our treatment is to consider (1.1) as parabolic problem in a bounded domain with non-local boundary condition, and to carry over the well-established methods for linear parabolic equations with standard boundary conditions such as Dirichlet’s or Neumann’s boundary condition to our situation (cf. e.g. [7]): In Section 2 we solve weakly a Poisson-type problem related to (1.1) in all space. The regularity of the weak solution fol-

lows with standard arguments in G and \widehat{G} , only at ∂G we need special considerations. In Section 3 we treat the corresponding eigenvalue problem, introduce the associated Fourier series, and characterize the elements of various function spaces which are useful in the following by the behaviour of their Fourier coefficients. In Section 4 problem (1.1) is solved by a Galerkin procedure and for sufficiently smooth data the smoothness of the obtained solution is established. In two appendices the relation between problem (1.1) and the non-radial-flow as well as the axisymmetric problem is elucidated. A third appendix collects some facts about harmonic exterior functions, and clarifies the relation between different kinds of spatial decay conditions – a topic about which there has been some debate in the literature. In a fourth appendix the completeness of the so-called poloidal free decay modes is proved, a fact of common belief which to our knowledge, however, has never been proved. Finally, in a fifth appendix the evolution problem with time-independent principal coefficient is considered, a case which can be treated much more easily than the general case.

2 A Poisson problem

We establish in this section smooth solutions of the following Poisson problem in all space with suitable right-hand side f :

$$-\Delta u = f \quad \text{in } G, \quad (2.1a)$$

$$\Delta u = 0 \quad \text{in } \widehat{G}, \quad (2.1b)$$

$$u \text{ and } \nabla u \text{ continuous} \quad \text{in } \mathbb{R}^d, \quad (2.1c)$$

$$u(x) \rightarrow 0 \quad \text{for } |x| \rightarrow \infty. \quad (2.1d)$$

To obtain a weak formulation of problem (2.1) let us multiply (2.1a) and (2.1b) with a test function $v \in C_0^\infty(\mathbb{R}^d)$ and integrate over G and $B_R \setminus \overline{G}$, $\overline{G} \subset B_R$, respectively. Assuming $\partial G \in C^1$ one obtains after integration by parts:

$$\int_G (\nabla u \cdot \nabla v - f v) dx - \int_{\partial G} n \cdot \nabla u v ds = 0, \quad (2.2a)$$

$$\int_{B_R \setminus \overline{G}} \nabla u \cdot \nabla v dx + \int_{\partial G} n \cdot \nabla u v ds - \int_{S_R} \frac{x}{|x|} \cdot \nabla u v ds = 0. \quad (2.2b)$$

Here S_R denotes a sphere with radius R and n is the exterior unit normal at ∂G . Adding up (2.2a) and (2.2b) one finds in the limit $R \rightarrow \infty$:

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla v dx = \int_G f v dx \quad (2.3)$$

In view of (2.3) it is reasonable to consider functions satisfying the “finite energy condition” (cf. Remark A.1)

$$\int_{\mathbb{R}^d} |\nabla v|^2 dx =: \|v\|_{\mathcal{H}}^2 < \infty. \quad (2.4)$$

Condition (2.4) together with condition (2.1d) motivate the definition of the real Hilbert space

$$\mathcal{H}_0 := \text{clos}\{C_0^\infty(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}}\}$$

with scalar product $(v, w)_{\mathcal{H}} := \int_{\mathbb{R}^d} \nabla v \cdot \nabla w dx$. Observe that $v \in \mathcal{H}_0$ is locally square-integrable. In fact, the Gagliardo-Nirenberg-Sobolev-inequality (cf. [7, p. 263]) implies

$$\|v\|_{L^p(\mathbb{R}^d)} \leq C \|v\|_{\mathcal{H}}, \quad p = \frac{2d}{d-2} \quad (2.5)$$

for any $v \in \mathcal{H}_0$ and a constant C depending only on d . Combining (2.5) with Hölder's inequality yields for any $K \in \mathbb{R}^d$:

$$\|v\|_{L^2(K)} \leq |K|^{1/d} \|v\|_{L^p(K)} \leq C|K|^{1/d} \|v\|_{\mathcal{H}}. \quad (2.6)$$

So, defining (for later use)

$$\mathcal{H} := \{v \in H_{loc}^1(\mathbb{R}^d) : \|v\|_{\mathcal{H}} < \infty\},$$

there holds clearly $\mathcal{H}_0 \subset \mathcal{H}$. Inequality (2.6) implies, in particular,

$$\|v\|_{L^2(G)} \leq C_G \|v\|_{\mathcal{H}}, \quad v \in \mathcal{H}_0 \quad (2.7)$$

with $C_G := C|G|^{1/d}$; thus, $v \in \mathcal{H}_0$ yields in eq. (2.3) with $f \in L^2(G)$ a finite right-hand side as well.

A function $u \in \mathcal{H}_0$ is now called a weak solution of problem (2.1) with $f \in L^2(G)$ iff (2.3) holds for all $v \in \mathcal{H}_0$. Rewriting (2.3) in the form

$$(u, v)_{\mathcal{H}} = (f, v)_{L^2(G)} \quad \text{for any } v \in \mathcal{H}_0 \quad (2.8)$$

and noting that $(f, \cdot)_{L^2(G)}$ defines a linear functional on \mathcal{H}_0 (due to (2.7)) the existence of a unique weak solution follows immediately from the Riesz representation theorem.

Concerning regularity of the weak solution let us define $\hat{f} : \mathbb{R}^d \rightarrow \mathbb{R}$ by $\hat{f} = f$ on G and $\hat{f} = 0$ on \hat{G} . It is a standard result about interior regularity (cf. e.g. [7, p309f]) that $\hat{f} \in H^k(\mathbb{R}^d)$ implies $u \in H_{loc}^{k+2}(\mathbb{R}^d)$, $k \in \mathbb{N}_0$. This result means in particular:

$$u \in H_{loc}^2(\mathbb{R}^d), \quad (2.9a)$$

$$f \in H^k(G) \Rightarrow u \in H_{loc}^{k+2}(G), \quad k \in \mathbb{N}_0, \quad (2.9b)$$

$$u \in C^\infty(\hat{G}). \quad (2.9c)$$

With the usual Sobolev embeddings (2.9b) implies $u \in C^2(G)$ if $f \in H^k(G)$, $k > d/2$. So, choosing suitable test functions in (2.3) we find u satisfying (2.1b) in \hat{G} and, for sufficiently regular f , (2.1a) in G . At ∂G , however, we need some finer considerations.

As usual we flatten ∂G locally by means of a diffeomorphism $\Phi : U \rightarrow W$, $x \mapsto y$ with $\Phi(x_0) = 0$, $x_0 \in U \cap \partial G$, $|D\Phi| = 1$ (cf. e.g. [7, p626f]). Setting $v(y) := u(\Phi^{-1}(y))$ and $g(y) := f(\Phi^{-1}(y))$, Φ transforms eqs. (2.1) localized on U into

$$-Lv = g \quad \text{in } W^-, \quad (2.10a)$$

$$Lv = 0 \quad \text{in } W^+, \quad (2.10b)$$

$$v \text{ and } \nabla v \text{ continuous} \quad \text{in } W. \quad (2.10c)$$

Here, $L(\cdot) = \sum_{i,j=1}^d \partial_{y_i} (a_{ij} \partial_{y_j} \cdot)$ is a uniformly elliptic operator with coefficients determined by Φ , $W^- := W \cap \{y_d < 0\} = \Phi(U \cap G)$, $W^+ := W \cap \{y_d > 0\} = \Phi(U \cap \hat{G})$, and $W^0 := W \cap \{y_d = 0\} = \Phi(U \cap \partial G)$. A function $v \in H^1(W)$ is called a weak solution of problem (2.10) with $g \in L^2(W^-)$ iff

$$\sum_{i,j=1}^d \int_W a_{ij} \partial_{y_j} v \partial_{y_i} w \, dy = \int_{W^-} g w \, dy$$

for any $w \in H_0^1(W)$. Defining “non-isotropic” Sobolev spaces $H_{tan}^m(W)$, $m \in \mathbb{N}_0$ by

$$H_{tan}^m(W) := \{v : W \rightarrow \mathbb{R} \mid D^\alpha v \in L^2(W) \text{ for any multiindex } \alpha \text{ with } |\alpha| \leq m, \alpha_d = 0\}$$

with norm

$$\|v\|_{H_{tan}^k(W)}^2 := \sum_{\substack{|\alpha| \leq k \\ \alpha_d = 0}} \|D^\alpha v\|_{L^2(W)}^2,$$

the regularity of v over W^0 is characterized by

Lemma 2.1 *Let $g \in H_{tan}^k(W^-)$ with $k > (d+1)/2$, let v be the weak solution of (2.10), and let r be so small that the cylinder $V := \{y = (y', y_d) \in \mathbb{R}^d : |y'| < r, |y_d| < 2r\} \Subset W$. Then $v \in C^1(V_{1/2})$ with $V_{1/2} := V \cap \{|y_d| < r\}$ and we have the estimate*

$$\|v\|_{C^1(\bar{V}_{1/2})} \leq C(\|g\|_{H_{tan}^k(W^-)} + \|v\|_{L^2(W)}) \quad (2.11)$$

with C depending on W , V , k , and d .

PROOF: We prove first the estimate

$$\|v\|_{C^1(\bar{V}_{1/2})}^2 \leq C \sum_{i=0,1,2} \sum_{\substack{|\alpha| \leq k \\ \alpha_d = 0}} \|\partial_{y_d}^i D^\alpha v\|_{L^2(V)}^2 \quad (2.12)$$

for functions $v \in C^{k+2}(\bar{V})$ by applying Sobolev embeddings separately for y' and y_d . Let us start with a 1-dimensional estimate on the interval $(-r, r)$. For functions $f \in C^1((-2r, 2r))$ we have

$$|f(y)| \leq \frac{1}{2r} \int_{y-r}^{y+r} |f(y) - f(x)| dx + \frac{1}{2r} \int_{y-r}^{y+r} |f(x)| dx \leq \int_{-2r}^{2r} |f'(x)| dx + \frac{1}{2r} \int_{-2r}^{2r} |f(x)| dx.$$

This implies

$$\max_{|y| \leq r} |f|^2 \leq C \left(\int_{-2r}^{2r} |f'(x)|^2 dx + \int_{-2r}^{2r} |f(x)|^2 dx \right) \quad (2.13)$$

for some constant $C = C(r)$. On the other hand, Sobolev embeddings imply for $k > (d+1)/2$

$$\|f\|_{C^1(\bar{B}'_r)}^2 \leq C \|f\|_{H^k(B'_r)}^2 = C \sum_{|\alpha| \leq k} \int_{B'_r} |D^\alpha f(y')|^2 dy' \quad (2.14)$$

for some constant $C = C(k, d, r)$ and $B'_r := \{y' \in \mathbb{R}^{d-1} : |y'| < r\}$. Combining (2.13) and (2.14) we obtain

$$\begin{aligned} \max_{|y_d| \leq r} \|v(\cdot, y_d)\|_{C^1(\bar{B}'_r)}^2 &\leq C \sum_{i=0,1} \sum_{\substack{|\alpha| \leq k \\ \alpha_d = 0}} \int_{-2r}^{2r} \int_{B'_r} |\partial_{y_d}^i D^\alpha v(y', y_d)|^2 dy' dy_d \\ &= C \sum_{i=0,1} \sum_{\substack{|\alpha| \leq k \\ \alpha_d = 0}} \|\partial_{y_d}^i D^\alpha v\|_{L^2(V)}^2, \end{aligned} \quad (2.15)$$

and analogously

$$\max_{|y_d| \leq r} \|\partial_{y_d} v(\cdot, y_d)\|_{C^0(\bar{B}'_r)}^2 = C \sum_{i=1,2} \sum_{\substack{|\alpha| \leq k \\ \alpha_d = 0}} \|\partial_{y_d}^i D^\alpha v\|_{L^2(V)}^2.$$

This is (2.12). Let now $v \in L^2(V)$ with bounded right-hand side in (2.12). It follows with standard arguments that v has a representative in $C^1(\bar{V}_{1/2})$.

Next, let \widehat{g} be the trivial extension of g on W . $g \in H_{tan}^k(W^-)$ implies then $\widehat{g} \in H_{tan}^k(W)$, i.e. $D^\alpha \widehat{g} \in L^2(W)$ for $|\alpha| \leq k$, $\alpha_d = 0$. From interior regularity for weak solutions of (2.10) follows $D^\beta v \in L^2(V)$ for $|\beta| \leq k+2$, $\beta_d \leq 2$ and the estimate

$$\begin{aligned} \sum_{\substack{|\beta| \leq k+2 \\ \beta_d \leq 2}} \int_V |D^\beta v|^2 dy &\leq C \left(\sum_{\substack{|\alpha| \leq k \\ \alpha_d = 0}} \int_W |D^\alpha \widehat{g}|^2 dy + \|v\|_{L^2(W)} \right) \\ &= C \left(\|g\|_{H_{tan}^k(W^-)} + \|v\|_{L^2(W)} \right) \end{aligned} \quad (2.16)$$

with C depending on W , V , k , and d . Together with (2.12) this proves (2.11). \square

Remark 2.2 Observe for later use that Lemma 2.1 applies especially to the case $g := \lambda v|_{W^-}$ with $\lambda \in \mathbb{R}$. In fact, given any $k \in \mathbb{N}$ we find $v|_{W^-} \in H_{tan}^{2k}(\widetilde{W}^-)$ for some $V \Subset \widetilde{W} \Subset W$ just by iterating the interior regularity argument. So, Lemma 2.1 applies to this eigenvalue problem as well and yields C^1 -smoothness of solutions over the boundary.

Concerning the original problem (2.1), Lemma 2.1 implies for $\partial G \in C^{k+2}$ (i.e. $\Phi \in C^{k+2}$) and $f \in H^k(G)$, $k > (d+1)/2$ that $u \in C^1$ over ∂G . Collecting the foregoing results we have

Theorem 2.3 (Solution of the Poisson problem) *Let $G \subset \mathbb{R}^d$, $d \geq 3$ be a bounded domain with C^1 -boundary ∂G and $f \in L^2(G)$. The Poisson problem (2.1) has then a unique weak (in the sense of eq. (2.3)) solution $u \in H_{loc}^1(\mathbb{R}^d)$. Moreover, if $\partial G \in C^{k+2}$ and $f \in H^k(G)$ with $k > (d+1)/2$, then u is a classical solution, i.e. $u \in C^1(\mathbb{R}^d) \cap C^2(G \cup \widehat{G})$, satisfying pointwise eqs. (2.1).*

Remark 2.4 The condition $d \geq 3$ is crucial for Theorem 2.3. As an example for (2.1) in \mathbb{R} consider $-u'' = 1$ in $G = (-1, 1)$, $u'' = 0$ in \widehat{G} , $\lim_{|x| \rightarrow \infty} u(x) = 0$. The only continuous solution is $u(x) = \frac{1}{2} - \frac{1}{2}x^2$ in G and $u \equiv 0$ in \widehat{G} , which is not C^1 over ∂G . Similarly, in \mathbb{R}^2 , nontrivial harmonic functions show harmonic growths at infinity and therefore the only continuous solution of $-\Delta u = 1$ in $G = B_1(0)$, $\Delta u = 0$ in \widehat{G} , $\lim_{|x| \rightarrow \infty} u(x) = 0$ is $u(x) = \frac{1}{4} - \frac{1}{4}|x|^2$ in G , $u \equiv 0$ in \widehat{G} . However, in \mathbb{R}^3 the analogous problem has the C^1 solution $u(x) = \frac{1}{4} - \frac{1}{4}|x|^2$ in $G = B_1$, $u(x) = \frac{1}{3}|x|^{-1}$ in \widehat{G} .

Concerning the general asymptotic behaviour of u for large x we refer to well-known facts about exterior harmonic functions (cf. Appendix C). In particular, condition (2.1d) implies the representation (C.1), which yields the asymptotics

$$u = Y_0 |x|^{2-d} + O(|x|^{1-d}), \quad Y_0 \in \mathbb{R} \quad \text{for } |x| \rightarrow \infty. \quad (2.17)$$

The decay is faster (viz. $O(|x|^{1-d})$), if f has vanishing mean over G . In fact, one obtains from (2.1), (2.17), and Gauss's theorem:

$$|G| \bar{f} = - \int_G \Delta u dx = - \int_{\partial G} n \cdot \nabla u ds = - \int_{S_R} \frac{x}{|x|} \cdot \nabla u ds = (d-2) |S_1| Y_0. \quad (2.18)$$

Here, $\bar{f} := \frac{1}{|G|} \int_G f dx$ and $S_R \subset \widehat{G}$; the last equality arises in the limit $R \rightarrow \infty$. So, $\bar{f} = 0$ implies $Y_0 = 0$.

Let us note by the way that in general $\bar{u} = 0$ can only be achieved at the expense of relaxing (2.1d) to $u(x) = O(1)$ for $|x| \rightarrow \infty$. In this case the function $u - \bar{u}$ with u being a solution of (2.1) is obviously a zero-mean solution (see also Remark 2.5).

Remark 2.5 If G is a ball B_R it makes sense to consider spherically symmetric solutions of problem (2.1), i.e. $\langle u \rangle(r) = 0$ for any $r > 0$. Obviously, $\langle u \rangle = 0$ implies $\langle f \rangle = 0$; on the other hand, any solution of (2.1) with spherically symmetric f is spherically symmetric. This follows from the unique solvability of the sub-problem for the spherical mean arising from (2.1):¹

$$\begin{aligned} -(\langle u \rangle'' + (d-1)/r \langle u \rangle') &= \langle f \rangle, & 0 \leq r \leq R, \\ \langle u \rangle(R) &= Y_0/R^{d-2}, & \langle u \rangle'(R) = (2-d)Y_0/R^{d-1}. \end{aligned}$$

Observe here the representation (C.1), which implies $\langle u \rangle = Y_0 r^{2-d}$ for $r > R$, and (2.18), which implies $Y_0 = 0$ if $\langle f \rangle = 0$.

Note, finally, that spherically symmetric solutions decay at least like $|x|^{1-d}$ for large x .

3 The Eigenvalue problem

We treat in this section the eigenvalue problem corresponding to the Poisson problem (2.1) of the last section:

$$-\Delta u = \lambda u \quad \text{in } G, \quad (3.1a)$$

$$\Delta u = 0 \quad \text{in } \widehat{G}, \quad (3.1b)$$

$$u \text{ and } \nabla u \text{ continuous} \quad \text{in } \mathbb{R}^d, \quad (3.1c)$$

$$u(x) \rightarrow 0 \quad \text{for } |x| \rightarrow \infty. \quad (3.1d)$$

According to Theorem 2.2 we have for any $f \in L^2(G)$ a unique weak solution $u \in \mathcal{H}_0$ of (2.1), defining thus a Green's operator

$$\widetilde{\mathcal{G}} : L^2(G) \rightarrow \mathcal{H}_0, \quad f \mapsto u.$$

Thus (2.7), (2.8) yield ²

$$\|\widetilde{\mathcal{G}}f\|_{\mathcal{H}}^2 = (u, u)_{\mathcal{H}} = (f, u)_{L^2} \leq \|f\|_{L^2} \|\widetilde{\mathcal{G}}f\|_{L^2} \leq C_G \|\widetilde{\mathcal{G}}f\|_{\mathcal{H}} \|f\|_{L^2}$$

and

$$(f, \widetilde{\mathcal{G}}f)_{L^2} = (f, u)_{L^2} = (u, u)_{\mathcal{H}} = \|\widetilde{\mathcal{G}}f\|_{\mathcal{H}}^2 \geq 0.$$

Therefore, $\widetilde{\mathcal{G}}$ is a bounded linear operator between Hilbert spaces, which is, furthermore, positive and hence symmetric. Restricting u on G one obtains the operator

$$\mathcal{G} : L^2(G) \rightarrow L^2(G), \quad f \mapsto u|_G$$

which is likewise bounded and symmetric, and, moreover, compact due to the Rellich-Kondrachov Theorem and the observation $\{u|_G : u \in \mathcal{H}_0\} = \{u|_G : u \in \mathcal{H}\} = H^1(G)$. The spectral theorem for symmetric compact operators in Hilbert spaces establishes now a complete (in $L^2(G)$) orthonormal system $\{v_n : n \in \mathbb{N}\}$ of eigenvectors of \mathcal{G} :

$$\mathcal{G} v_n = \lambda_n^{-1} v_n, \quad n \in \mathbb{N}, \quad (3.2)$$

¹Prime means differentiation with respect to r .

²The symbol L^2 without specified domain means always $L^2(G)$.

with real, positive eigenvalues λ_n^{-1} of finite multiplicity and $\lim_{n \rightarrow \infty} \lambda_n^{-1} = 0$. In order to solve the original problem let us define the “harmonic extension” $\tilde{v}_n := \lambda_n \tilde{\mathcal{G}} v_n$ of v_n on \mathbb{R}^d .³ By definition the \tilde{v}_n are weak solutions of the Poisson problem (2.1) with $f := v_n$ and $u = \lambda^{-1} \tilde{v}_n$; thus, eq. (2.8) takes now the form

$$(\tilde{v}_n, v)_{\mathcal{H}} = \lambda_n (v_n, v)_{L^2} \quad \text{for any } v \in \mathcal{H}_0, \quad (3.3)$$

i.e. the pair (\tilde{v}_n, λ_n) is a weak solution of the eigenvalue problem (3.1). Thus, the uniqueness of weak solutions implies the uniqueness of the harmonic extensions in \mathcal{H}_0 .

Concerning regularity we have $\tilde{v}_n \in C^\infty(G \cup \hat{G}) \cap C^1(\mathbb{R}^d)$, which follows from (2.9c), iterating (2.9b), and Remark 2.2. So, (\tilde{v}_n, λ_n) is a classical solution of problem (3.1) as well. We summarize these results in

Theorem 3.1 (Solution of the eigenvalue problem) *Let $G \subset \mathbb{R}^d$, $d \geq 3$ be a bounded domain with C^k -boundary, $k > (d+5)/2$. The eigenvalue problem (3.1) has then a countable set of eigensolutions $\{(\tilde{v}_n, \lambda_n) : n \in \mathbb{N}\}$ satisfying (3.3), and their restrictions $\{v_n : n \in \mathbb{N}\}$ constitute an orthonormal basis of $L^2(G)$.*

Powers of the inverse Green’s operator and their domains of definition turn out to provide the right setting for the solution of the evolution problem in Section 4. The elements of these spaces can be characterized by the decay behaviour of their Fourier coefficients when expanded in the above eigenfunctions. This motivates the

Definition 3.2 *Let $\{v_n : n \in \mathbb{N}\}$ be the complete orthonormal system with associated eigenvalues λ_n according to Theorem 3.1, and $\alpha \in \mathbb{R}$. We define then the space of “formal series”*

$$\mathcal{S} := \left\{ \sum_{n=1}^{\infty} c_n v_n : c_n \in \mathbb{R} \right\},$$

with non-negative functional

$$\|\cdot\|_\alpha : \mathcal{S} \rightarrow [0, \infty], \quad \sum_{n=1}^{\infty} c_n v_n \mapsto \left(\sum_{n=1}^{\infty} \lambda_n^{2\alpha} |c_n|^2 \right)^{1/2},$$

linear mapping

$$\mathcal{A}^\alpha : \mathcal{S} \rightarrow \mathcal{S}, \quad \sum_{n=1}^{\infty} c_n v_n \mapsto \sum_{n=1}^{\infty} \lambda_n^\alpha c_n v_n,$$

and subspaces

$$D_\alpha := D(\mathcal{A}^\alpha) = \{v \in \mathcal{S} : \|v\|_\alpha < \infty\} \subset \mathcal{S}.$$

Obviously, \mathcal{A}^α maps D_α into $L^2(G)$. Furthermore, there is $D_\alpha \subset D_\beta$ if $\alpha \geq \beta$ and $D_0 = L^2(G)$. Thus, if $\alpha \geq 0$, there is $D_\alpha \subset L^2(G)$ and $v \in D_\alpha$ has the representation

$$v = \sum_{n=1}^{\infty} (v_n, v)_{L^2} v_n. \quad (3.4)$$

With the pairing $\langle w, v \rangle := \sum_{n=1}^{\infty} d_n c_n$ for $w = \sum_{n=1}^{\infty} d_n v_n \in D_{-\alpha}$ and $v = \sum_{n=1}^{\infty} c_n v_n \in D_\alpha$, $D_{-\alpha}$ is the dual space of D_α .

Applying \mathcal{A} on (finite) linear combinations of v_n we find $\mathcal{A} = \mathcal{G}^{-1}$, and $D(\mathcal{A})$ turns out to be the maximal domain of definition of \mathcal{G}^{-1} . Similarly, $D_{1/2}$ is related to \mathcal{H}_0 . More precisely we have

³In this section a quantity with tilde means always the “harmonic extension” on \mathbb{R}^d of a quantity defined on G .

Theorem 3.3 *Let $\{v_n : n \in \mathbb{N}\}$ be the complete orthonormal system defined by the eigenvalue problem (3.1), \mathcal{A} and D_α as defined in Definition 3.2, and \mathcal{G} the Green's operator associated to the Poisson problem (2.1). Then,*

$$D_{k/2} = H^k(G), \quad k \in \mathbb{N}_0, \quad (3.5)$$

and we have the equivalence of norms:

$$\|\cdot\|_{k/2} \sim \|\cdot\|_{H^k}, \quad k \in \mathbb{N}_0. \quad (3.6)$$

Futhermore, there holds

$$D_{1/2} = \{v|_G : v \in \mathcal{H}_0 \text{ and } v|_{\widehat{G}} \text{ is harmonic}\}, \quad (3.7)$$

i.e. any $v \in D_{1/2}$ has a unique harmonic extension $\tilde{v} \in \mathcal{H}_0$, and

$$D_1 = D(\mathcal{A}) = \mathcal{G}(L^2).$$

PROOF: We start with (3.7). Let $v \in D_{1/2}$ be decomposed as in (3.4), i.e., $v = \sum_{n=1}^{\infty} (v_n, v)_{L^2} v_n$ with $\sum_{n=1}^{\infty} \lambda_n |(v_n, v)_{L^2}|^2 < \infty$. We define

$$\tilde{v} := \sum_{n=1}^{\infty} (v_n, v)_{L^2} \tilde{v}_n$$

with $\tilde{v}_n \in \mathcal{H}_0$ being the unique harmonic extension of v_n . Computing

$$\|\tilde{v}\|_{\mathcal{H}}^2 = \sum_{n,m=1}^{\infty} (v_m, v)_{L^2} (v, v_n)_{L^2} (\tilde{v}_m, \tilde{v}_n)_{\mathcal{H}} = \sum_{n=1}^{\infty} \lambda_n |(v_n, v)_{L^2}|^2, \quad (3.8)$$

where we used (3.3), we find $\tilde{v} \in \mathcal{H}_0$. In order to prove that $\tilde{v}|_{\widehat{G}}$ is harmonic it suffices to show $(\tilde{v}, \hat{v})_{\mathcal{H}} = 0$ for any $\hat{v} \in C_0^\infty(G)$ with $\text{supp } \hat{v} \subset \widehat{G}$; this follows immediately with (3.3):

$$(\tilde{v}, \hat{v})_{\mathcal{H}} = \sum_{n=1}^{\infty} (v_n, v)_{L^2} (\tilde{v}_n, \hat{v})_{\mathcal{H}} = \sum_{n=1}^{\infty} \lambda_n (v_n, v)_{L^2} (v_n, \hat{v})_{L^2} = 0,$$

since $(v_n, \hat{v})_{L^2} = \int_G v_n \hat{v} dx = 0$. Therefore, $v \in \{v|_G : v \in \mathcal{H}_0 \text{ and } v|_{\widehat{G}} \text{ is harmonic}\}$. The opposite inclusion follows again with (3.8).

The inclusion $\{v|_G : v \in \mathcal{H}_0 \text{ and } v|_{\widehat{G}} \text{ is harmonic}\} \subset H^1(G)$ is obvious; the opposite inclusion follows with Theorem C.1: Let $v \in H^1(G)$, v_0 an H^1 -extension of v on \mathbb{R}^d with bounded support, $w := v_0|_{\widehat{G}} \in \widehat{\mathcal{H}}$ and \hat{u} the harmonic exterior solution of (C.6). The function u defined by v in G and \hat{u} in \widehat{G} is then the sought-after function $\in \mathcal{H}_0$. Together with (3.7) this proves (3.5) for $k = 1$.

To estimate the 1/2-norm we supply the last construction with bounds: Let $\text{supp } v_0 \subset K$, then

$$\|v_0\|_{H^1(\mathbb{R}^d)} \leq C(G, K) \|v\|_{H^1(G)}.$$

Thus, using the minimizing property of solutions of (C.6),

$$\|\hat{u}\|_{\widehat{\mathcal{H}}} \leq \|w\|_{\widehat{\mathcal{H}}} \leq \|v_0\|_{\mathcal{H}} \leq C(G, K) \|v\|_{H^1(G)},$$

and with (3.8):

$$\|v\|_{1/2}^2 = \|\tilde{v}\|_{\mathcal{H}}^2 = \|\nabla v\|_{L^2(G)}^2 + \|\hat{u}\|_{\widehat{\mathcal{H}}}^2 \leq C \|v\|_{H^1(G)}^2,$$

which is one half of (3.6) for $k = 1$. The other half follows with (2.7).

We show next $D(\mathcal{A}) = \mathcal{G}(L^2)$. Let $u \in \mathcal{G}(L^2)$ and $f = \sum_{n=1}^{\infty} (v_n, f)_{L^2} v_n \in L^2$ such that $u = \mathcal{G}f$. Computing the coefficients of u we find with (3.2):

$$(v_m, u)_{L^2} = (v_m, \mathcal{G}f)_{L^2} = \sum_{n=1}^{\infty} (v_n, f)_{L^2} (v_m, \mathcal{G}v_n)_{L^2} = \lambda_m^{-1} (v_m, f)_{L^2}.$$

Thus,

$$\sum_{m=1}^{\infty} \lambda_m^2 |(v_m, u)_{L^2}|^2 = \sum_{m=1}^{\infty} |(v_m, f)_{L^2}|^2 < \infty,$$

hence $u \in D(\mathcal{A})$. If, on the other hand, $u = \sum_{n=1}^{\infty} (v_n, u)_{L^2} v_n$ and $\sum_{n=1}^{\infty} \lambda_n^2 |(v_n, u)_{L^2}|^2 < \infty$, then $f := \sum_{n=1}^{\infty} \lambda_n (v_n, u)_{L^2} v_n \in L^2$ is well-defined and we find $\mathcal{G}f = u$.

The inclusion $\mathcal{G}(L^2) \subset H^2(G)$ is an immediate consequence of the H^2 -regularity of weak solutions. To prove the opposite inclusion let $w \in H^2(G)$. Defining $f := -\Delta w \in L^2$ the Poisson problem (2.1) yields a solution $u \in H_{loc}^2$. So, u satisfies pointwise a.e. $\Delta u = \Delta w$ in G and $\Delta u = 0$ in \widehat{G} . Denoting the harmonic extension of w by $\tilde{w} \in \mathcal{H}_0$ and applying standard regularity arguments we find $u - \tilde{w} \in \mathcal{H}_0 \cap C^\infty(\mathbb{R}^d)$ and $\Delta(u - \tilde{w}) = 0$ in \mathbb{R}^d ; thus $u = \tilde{w}$ (with Liouville's theorem) and, in particular, $w = \mathcal{G}f$. This proves (3.5) for $k = 2$.

To estimate the 1-norm of $v \in D(\mathcal{A})$ observe that we have its harmonic extension $\tilde{v} \in \mathcal{H}_0 \cap H_{loc}^2(\mathbb{R}^d)$, so we are free to integrate by parts in \mathbb{R}^d . One obtains with (3.1):

$$-(\lambda_n v_n, v)_{L^2(G)} = (\Delta v_n, v)_{L^2(G)} = (\Delta \tilde{v}_n, \tilde{v})_{L^2(\mathbb{R}^d)} = (\tilde{v}_n, \Delta \tilde{v})_{L^2(\mathbb{R}^d)} = (v_n, \Delta v)_{L^2(G)}. \quad (3.9)$$

Thus,

$$\|v\|_1^2 = \sum_{n=1}^{\infty} |(\lambda_n v_n, v)_{L^2}|^2 = \sum_{n=1}^{\infty} |(v_n, \Delta v)_{L^2}|^2 = \|\Delta v\|_{L^2}^2 \leq \|v\|_{H^2(G)}^2,$$

and with (2.7):

$$\|v\|_1^2 = \|\Delta \tilde{v}\|_{L^2(\mathbb{R}^d)}^2 = \sum_{|\alpha|=2} \|D^\alpha \tilde{v}\|_{L^2(\mathbb{R}^d)}^2 \geq C \|v\|_{H^2(G)}^2.$$

This is (3.6) for $k = 2$.

The general case $k \in \mathbb{N}$ of eq. (3.5) is now easily proved by induction: Let $v \in D_{k/2+1}$, $k \in \mathbb{N}$, i.e. $\mathcal{A}v \in D_{k/2} = H^k(G)$. With (3.9) one obtains

$$\mathcal{A}v = \sum_{n=1}^{\infty} (\lambda_n v_n, v)_{L^2} v_n = - \sum_{n=1}^{\infty} (v_n, \Delta v)_{L^2} v_n = -\Delta v;$$

thus, $\Delta v \in H^k(G)$ and standard regularity arguments imply $v \in H^{k+2}(G)$. Similarly, computing $\|v\|_{k/2+1}$ for $v \in H^{k+2}(G)$ yields

$$\|v\|_{k/2+1}^2 = \sum_{n=1}^{\infty} \lambda_n^k |(\lambda_n v_n, v)_{L^2}|^2 = \sum_{n=1}^{\infty} \lambda_n^k |(v_n, \Delta v)_{L^2}|^2 < \infty,$$

since $\Delta v \in H^k(G) = D_{k/2}$.

Finally, the equivalence relation (3.6) follows with $\|\cdot\|_{k/2+1} = \|\Delta \cdot\|_{k/2}$ and $\|\cdot\|_{H^{k+2}} \sim \|\Delta \cdot\|_{H^k}$.

□

4 The evolution problem

We solve in this section the evolution problem (1.1) by means of the spaces D_α provided in the last section. According to eq. (3.7), $v \in D_{1/2}$ has a harmonic extension \tilde{v} on \mathbb{R}^d ; so, when working with these spaces it is sufficient to consider problem (1.1) on the simpler domain $G \times \mathbb{R}_+$. However, a nontrivial asymptotic function u_∞ does not fit into this framework. Therefore, in a first step, u_∞ is eliminated by the time-dependent shift $u - u_\infty := u_s$. In terms of u_s problem (1.1) reads

$$\partial_t u_s - a \Delta u_s = b \cdot \nabla u_s + c u_s + f \quad \text{in } G \times \mathbb{R}_+, \quad (4.1a)$$

$$\Delta u_s = 0 \quad \text{in } \widehat{G} \times \mathbb{R}_+, \quad (4.1b)$$

$$u_s \text{ and } \nabla u_s \text{ continuous} \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+, \quad (4.1c)$$

$$u_s(x, t) \rightarrow 0 \quad \text{for } |x| \rightarrow \infty, t \in \mathbb{R}_+, \quad (4.1d)$$

$$u_s(\cdot, 0) = u_{s0} \quad \text{on } G \times \{t = 0\} \quad (4.1e)$$

with $f := c u_\infty - \frac{d}{dt} u_\infty$.

The “simplified” problem takes then the form

$$\dot{v} = -a \mathcal{A} v + \mathcal{B} v + f, \quad (4.2a)$$

$$v(0) = v_0, \quad (4.2b)$$

with the operator \mathcal{A} as defined in Definition 3.2 and the lower-order operator \mathcal{B} defined by $\mathcal{B} v := b \cdot \nabla v + c v$. Here v is a mapping from $[0, T)$, $T > 0$ into some function space over G . As explained above a reasonable such space is $D_{k/2} = H^k(G)$ with at least $k = 1$ (cf. Theorem 3.3). Moreover, for $T > 0$, when starting with $v_0 \in D_{1/2}$ and taking into account parabolic smoothing we expect $v \in L^2((0, T), D_1)$ which means in view of (4.2a) $\dot{v} \in L^2((0, T), L^2(G))$. This motivates the

Definition 4.1 *Let $T > 0$ and $v_0 \in D_{1/2}$. A function $v \in L^2((0, T), D_1)$ with weak time derivative $\dot{v} \in L^2((0, T), L^2(G))$ satisfying (4.2a) as equality in $L^2((0, T), L^2(G))$ and (4.2b) as equality in $D_{1/2}$ is called weak solution of problem (4.2).*

Condition (4.2b) makes sense for weak solutions due to the following interpolation result:

Lemma 4.2 *Let G be a bounded domain with smooth boundary, $T > 0$, and $k \in \mathbb{N}_0$. Let, furthermore, $v \in L^2((0, T), H^{k+1}(G))$ and $\dot{v} \in L^2((0, T), H^{k-1}(G))$. Then*

$$v \in C([0, T], H^k(G));$$

moreover, the mapping $t \rightarrow \|v(t)\|_{L^2(G)}^2$ is absolutely continuous with derivative

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2(G)}^2 = \langle \dot{v}(t), v(t) \rangle \quad (4.3)$$

for a.e. $t \in [0, T]$.

For a proof we refer to [7, p287f.]. We note only in the case $k = 0$ that $D_{-1/2}$ is the dual space of $D_{1/2} = H^1(G)$, thus $D_{-1/2} \subset H^{-1}(G)$. $\langle \cdot, \cdot \rangle$ denotes the dual pairing as explained after (3.4).

Theorem 4.3 (Weak solution of the evolution problem) *Let $T > 0$ and $a, b, c \in C(\overline{G} \times [0, T])$, $a \geq a_0 > 0$. Let, furthermore, $v_0 \in D_{1/2}$ and $f \in C([0, T], L^2(G))$. Then problem (4.2) has a unique weak solution v .*

PROOF: We start with the construction of Galerkin approximations using the complete system $\{v_\nu : \nu \in \mathbb{N}\}$ from Theorem 3.1. Let P_ν be the orthogonal projection in $L^2(G)$ onto $\text{span}\{v_\nu\}$, $P^{(n)} := \bigoplus_{\nu=1}^n P_\nu$, and let $v^{(n)}(t) \in P^{(n)}L^2(G)$ be the unique solution of the following finite-dimensional initial-value problem

$$\frac{d}{dt}v^{(n)} = P^{(n)}(-a\mathcal{A}v^{(n)} + \mathcal{B}v^{(n)} + f), \quad (4.4a)$$

$$v^{(n)}(0) = P^{(n)}v_0. \quad (4.4b)$$

Note that $P^{(n)}$ commutes with \mathcal{A} but not with a or \mathcal{B} . From standard results about ordinary differential equations follows $v^{(n)} \in C^1([0, T], D_1)$ for any $n \in \mathbb{N}$.

Next we derive some a-priori estimates for $v^{(n)}$, uniform in n , which allow to extract a weakly convergent subsequence of the sequence $(v^{(n)})$ of Galerkin approximations. We first show that

$$\max_{[0, T]} \|v^{(n)}\|_{1/2}^2 \leq C = C[v_0, f; T]. \quad (4.5)$$

Taking the scalar product of (4.4a) with $\mathcal{A}v^{(n)}$ we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v^{(n)}\|_{1/2}^2 &= \frac{1}{2} \frac{d}{dt} (\mathcal{A}^{1/2}v^{(n)}, \mathcal{A}^{1/2}v^{(n)})_{L^2} = (\mathcal{A}v^{(n)}, \frac{d}{dt}v^{(n)})_{L^2} \\ &= (\mathcal{A}v^{(n)}, -a\mathcal{A}v^{(n)} + \mathcal{B}v^{(n)} + f)_{L^2} \\ &\leq -a_0 \|v^{(n)}\|_1^2 + (\mathcal{A}v^{(n)}, \mathcal{B}v^{(n)})_{L^2} + (\mathcal{A}v^{(n)}, f)_{L^2}. \end{aligned} \quad (4.6)$$

Observing that \mathcal{B} is a bounded operator from $C([0, T], D_{1/2})$ into $C([0, T], L^2(G))$ there is a constant C_1 such that

$$(\mathcal{A}v^{(n)}, \mathcal{B}v^{(n)})_{L^2} \leq (C_1 a_0)^{1/2} \|v^{(n)}\|_1 \|v^{(n)}\|_{1/2} \leq \frac{a_0}{2} \|v^{(n)}\|_1^2 + \frac{C_1}{2} \|v^{(n)}\|_{1/2}^2.$$

Setting $\max_{[0, T]} \|f\|_{L^2}^2 =: C_2 a_0$ we thus obtain

$$\frac{d}{dt} \|v^{(n)}\|_{1/2}^2 \leq C_1 \|v^{(n)}\|_{1/2}^2 + C_2, \quad (4.7)$$

and Gronwall's inequality yields

$$\|v^{(n)}\|_{1/2}^2 \leq e^{C_1 t} \|v^{(n)}(0)\|_{1/2}^2 + \frac{C_2}{C_1} (e^{C_1 t} - 1) \leq e^{C_1 T} \|v_0\|_{1/2}^2 + \frac{C_2}{C_1} (e^{C_1 T} - 1)$$

on $[0, T]$, and hence (4.5).

To obtain a bound on $v^{(n)}$ in $L^2((0, T), D_1)$ we estimate similarly to (4.6):

$$\frac{1}{2} \frac{d}{dt} \|v^{(n)}\|_{1/2}^2 \leq -a_0 \|v^{(n)}\|_1^2 + \frac{a_0}{4} \|v^{(n)}\|_1^2 + C_1 \|v^{(n)}\|_{1/2}^2 + \frac{a_0}{4} \|v^{(n)}\|_1^2 + C_2. \quad (4.8)$$

Using (4.5) we rewrite (4.8) in the form

$$a_0 \|v^{(n)}\|_1^2 \leq -\frac{d}{dt} \|v^{(n)}\|_{1/2}^2 + 2(C_1 C + C_2);$$

thus, integrating over $[0, T]$ and observing once more (4.5) yields the bound

$$\int_0^T \|v^{(n)}\|_1^2 dt \leq \widehat{C}. \quad (4.9)$$

With (4.9) the right-hand side in (4.4a) is obviously bounded in $L^2((0, T), L^2(G))$, i.e. there is \check{C} such that

$$\int_0^T \left\| \frac{d}{dt} v^{(n)} \right\|_{L^2}^2 dt \leq \check{C} \quad (4.10)$$

for the sequence of (classical) derivatives $\left(\frac{d}{dt} v^{(n)}\right)$.

The bounds (4.9) and (4.10) imply that there is a subsequence (n_l) and functions $v \in L^2((0, T), D_1)$, $\dot{v} \in L^2((0, T), L^2(G))$ such that v is the weak limit of $(v^{(n_l)})$ in $L^2((0, T), D_1)$ and \dot{v} that of $\left(\frac{d}{dt} v^{(n_l)}\right)$ in $L^2((0, T), L^2(G))$, respectively, and moreover \dot{v} is the (weak) derivative of v .

Testing (4.4a) with functions w of the form $w(t) = \sum_{\nu=1}^m d_\nu(t) v_\nu \in C^1([0, T], L^2(G))$, where $m \leq n$ and $d_\nu : [0, T] \rightarrow \mathbb{R}$ are smooth functions, and integrating over $[0, T]$ yields

$$\begin{aligned} \int_0^T \left(\frac{d}{dt} v^{(n)}, w \right)_{L^2} dt &= \int_0^T (P^{(n)}(-a \mathcal{A} v^{(n)} + \mathcal{B} v^{(n)} + f), w)_{L^2} dt \\ &= \int_0^T (-a \mathcal{A} v^{(n)} + \mathcal{B} v^{(n)} + f, w)_{L^2} dt. \end{aligned} \quad (4.11)$$

Setting $n = n_l$ we find in the limit $l \rightarrow \infty$:

$$\int_0^T (\dot{v}, w)_{L^2} dt = \int_0^T (-a \mathcal{A} v + \mathcal{B} v + f, w)_{L^2} dt. \quad (4.12)$$

Since test functions of this type are dense in $L^2((0, T), L^2(G))$, eq. (4.12) holds for any $w \in L^2((0, T), L^2(G))$. This proves (4.2a) to be an equality in $L^2((0, T), L^2(G))$.

Inserting $w \in C^1([0, T], L^2(G))$ with $w(T) = 0$ in (4.12) we find after integration by parts on the left-hand side $-\int_0^T (v, \frac{d}{dt} w)_{L^2} dt + (v(0), w(0))_{L^2}$. Doing the same in (4.11) yields in the limit $n_l \rightarrow \infty$ on the left-hand side $-\int_0^T (v, \frac{d}{dt} w)_{L^2} dt + (v_0, w(0))_{L^2}$. Since $w(0) \in L^2(G)$ is arbitrary we have $v(0) = v_0$ in $L^2(G)$. These results prove v to be a weak solution of problem (4.2).

Finally, to prove uniqueness of the weak solution consider $v_1 - v_2 =: v_0$ satisfying

$$\dot{v}_0 = -a \mathcal{A} v_0 + \mathcal{B} v_0, \quad v_0(0) = 0.$$

Setting in eq. (4.12) $v := v_0 \in L^2((0, T), D(\mathcal{A}))$, $w := \mathcal{A} v_0$, and $f = 0$ we obtain for a.e. $t \in [0, T]$:

$$(\dot{v}_0, \mathcal{A} v_0)_{L^2} = -(a \mathcal{A} v_0, \mathcal{A} v_0)_{L^2} + (\mathcal{B} v_0, \mathcal{A} v_0)_{L^2}. \quad (4.13)$$

The left-hand side of (4.13) takes with (4.3) the form

$$(\dot{v}_0, \mathcal{A} v_0)_{L^2} = \langle \mathcal{A}^{1/2} \dot{v}_0, \mathcal{A}^{1/2} v_0 \rangle = \frac{1}{2} \frac{d}{dt} \|\mathcal{A}^{1/2} v_0\|_{L^2}^2 = \frac{1}{2} \frac{d}{dt} \|v_0\|_{1/2}^2,$$

whereas estimates analogous to that leading to (4.7) show that the right-hand side of (4.13) can be bounded by $\frac{1}{2} C_1 \|v_0\|_{1/2}^2$. Since $t \mapsto \|v_0(t)\|_{1/2}$ is absolutely continuous by Lemma 4.2, applying Gronwall on the inequality

$$\frac{d}{dt} \|v_0\|_{1/2}^2 \leq C_1 \|v_0\|_{1/2}^2 \quad \text{for a.e. } t \in [0, T]$$

with $\|v_0(0)\|_{1/2} = 0$ yields the desired result $v_0 \equiv 0$. \square

Remark 4.4 *As to the original (shifted) problem (4.1) Theorems 3.3 and 4.3 imply that $u_s(x, t) := [\tilde{v}(t)](x)$, where $\tilde{v}(t)$ is the harmonic extension of $v(t)$, satisfies (4.1a,b) for a.e. $(x, t) \in \mathbb{R}^d \times [0, T]$.*

Higher regularity of the weak solution depends on the smoothness of the coefficients and the initial-value. There holds

Theorem 4.5 (Higher regularity) *Let $T > 0$, $k \in \mathbb{N} \setminus \{1\}$, and $a, b, c \in C_1^k(\overline{G} \times [0, T])$.⁴ Let, furthermore, $v_0 \in D_{(k+1)/2}$, $f \in C^1([0, T], D_{k/2})$, and v being the weak solution of problem (4.2). Then*

$$v \in L^2((0, T), D_{k/2+1}), \quad \dot{v} \in L^2((0, T), D_{k/2}), \quad \ddot{v} \in L^2((0, T), D_{k/2-1}).$$

PROOF: Higher spatial regularity is easily obtained via the operator \mathcal{A} : Applying $\mathcal{A}^{k/2}$ on (4.2) and setting $\mathcal{A}^{k/2}v =: w$, $\mathcal{A}^{k/2}f =: f^{(k)}$, $\mathcal{A}^{k/2}\mathcal{B}\mathcal{A}^{-k/2} =: \mathcal{B}^{(k)}$, $(a - \mathcal{A}^{k/2}a\mathcal{A}^{-k/2})\mathcal{A} =: \mathcal{C}^{(k)}$, and $\mathcal{A}^{k/2}v_0 =: w_0$ we obtain

$$\dot{w} = -a\mathcal{A}w + \mathcal{B}^{(k)}w + \mathcal{C}^{(k)}w + f^{(k)}, \quad (4.14a)$$

$$w(0) = w_0. \quad (4.14b)$$

$f^{(k)}$ and w_0 fulfill the prerequisites of Theorem 4.3, $\mathcal{B}^{(k)} : C([0, T], D_{1/2}) \rightarrow C([0, T], L^2(G))$ is again a bounded operator, and $\mathcal{C}^{(k)}$ is of the same type as $\mathcal{B}^{(k)}$. Thus Theorem 4.3 applies to (4.14) with the result

$$w \in L^2((0, T), D_1), \quad \dot{w} \in L^2((0, T), L^2(G)),$$

i.e.

$$v \in L^2((0, T), D_{k/2+1}), \quad \dot{v} \in L^2((0, T), D_{k/2}).$$

To obtain higher temporal regularity we need some more a-priori estimates for the Galerkin approximations $v^{(n)}$. Note that a, b, c are at least $\in C_1^2(\overline{G} \times [0, T])$, $v_0 \in D_{3/2}$, and $f \in C^1([0, T], D(\mathcal{A}))$. So, inserting $w^{(n)} := \mathcal{A}v^{(n)}$ into (4.14a) and taking the scalar product with $\mathcal{A}w^{(n)}$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|w^{(n)}\|_{1/2}^2 = (\mathcal{A}w^{(n)}, -a\mathcal{A}w^{(n)} + \mathcal{B}^{(2)}w^{(n)} + \mathcal{C}^{(2)}w^{(n)} + f^{(2)})_{L^2}.$$

This is analogous to (4.6) and the subsequent estimates leading to (4.5) yield now

$$\max_{[0, T]} \|w^{(n)}\|_{1/2}^2 = \max_{[0, T]} \|v^{(n)}\|_{3/2}^2 \leq C_3, \quad (4.15)$$

and, via (4.4a),

$$\max_{[0, T]} \left\| \frac{d}{dt} v^{(n)} \right\|_{1/2}^2 \leq C_4. \quad (4.16)$$

On the other hand, differentiating (4.4a) with respect to t , setting $\frac{d}{dt}v^{(n)} =: \dot{v}^{(n)}$, $\frac{d}{dt}f =: \dot{f}$, and $(\partial_t b \cdot \nabla v^{(n)} + \partial_t c v^{(n)}) =: \dot{\mathcal{B}}v^{(n)}$ we obtain

$$\frac{d}{dt} \dot{v}^{(n)} = P^{(n)}(-a\mathcal{A}\dot{v}^{(n)} + \mathcal{B}\dot{v}^{(n)} + \dot{f} - \partial_t a\mathcal{A}v^{(n)} + \dot{\mathcal{B}}v^{(n)}). \quad (4.17)$$

We now modify for (4.17) the argument which leads from (4.5) to (4.9). Taking the scalar product of (4.17) with $\mathcal{A}\dot{v}^{(n)}$ and observing that $\dot{\mathcal{B}}$ is of the same type as \mathcal{B} , thus using

⁴In this notation the upper index at ‘‘C’’ refers to the order of spatial derivatives and the lower one (omitted if zero) to temporal derivatives; so, $a \in C_1^k(\overline{G} \times [0, T])$ means a , $\partial_t a$, and $D_x^\alpha a$, $|\alpha| \leq k$ are all continuous functions on $\overline{G} \times [0, T]$.

the bounds $\|\mathcal{B}\dot{v}^{(n)}\|_{L^2} \leq (C_1 a_0)^{1/2} \|\dot{v}^{(n)}\|_{1/2}$, $\|\dot{\mathcal{B}}v^{(n)}\|_{L^2} \leq (C_1 a_0)^{1/2} \|v^{(n)}\|_{1/2}$, as well as $\max_{[0,T]} \|\dot{v}^{(n)}\|_1^2 \leq C_5$, (4.5), and $a \geq a_0$, $|\partial_t a| \leq A$, $\max_{[0,T]} \|\dot{f}\|_{L^2}^2 \leq C_2 a_0$ we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\dot{v}^{(n)}\|_{1/2}^2 &\leq -a_0 \|\dot{v}^{(n)}\|_1^2 + \frac{a_0}{8} \|\dot{v}^{(n)}\|_1^2 + 2C_1 \|\dot{v}^{(n)}\|_{1/2}^2 + \frac{a_0}{8} \|\dot{v}^{(n)}\|_1^2 + 2C_2 \\ &\quad + \frac{a_0}{8} \|\dot{v}^{(n)}\|_1^2 + 2\frac{A}{a_0} C_5 + \frac{a_0}{8} \|\dot{v}^{(n)}\|_1^2 + 2C_1 C. \end{aligned}$$

This estimate is analogous to (4.8), so we have

$$\int_0^T \|\dot{v}^{(n)}\|_1^2 dt \leq C_6$$

and, using once more (4.17),

$$\int_0^T \left\| \frac{d}{dt} \dot{v}^{(n)} \right\|_{L^2}^2 dt \leq C_7.$$

Recalling the reasoning after (4.10) there is thus a subsequence $(n_{l_m}) =: (m)$ of (n) and a function $\ddot{v} \in L^2((0, T), L^2(G))$ such that \ddot{v} is the weak limit of $(\frac{d}{dt} \dot{v}^{(m)})$ and the weak time derivative of \dot{v} . Finally, inspecting again (4.17) we find that there is enough regularity of the data a, b, c, f , and v_0 left to improve the spatial regularity of \ddot{v} by the order $k - 2$, i.e. we have $\ddot{v} \in L^2((0, T), D_{k/2-1})$. \square

In view of Lemma 4.2 and Sobolev's embedding theorems Theorem 4.5 implies the existence of smooth solutions. Concerning the original evolution problem (1.1) we have

Corollary 4.6 (Classical solution of the evolution problem) *Let $G \subset \mathbb{R}^d$, $d \geq 3$ be a bounded domain with $C^{k+3/2}$ -boundary. Let, furthermore, $a, b, c \in C_1^k(\overline{G} \times [0, T])$, $u_\infty \in C^2([0, T])$, and $u_0 \in C^{k+1}(\overline{G})$ for some $k > 1 + d/2$ and any $T > 0$. Then problem (1.1) has a unique classical solution u , i.e. $u \in C_1^2(G \times \mathbb{R}_+) \cap C^2(\widehat{G} \times \mathbb{R}_+)$ satisfies pointwise eqs. (1.1).*

PROOF: Fixing some $T > 0$ we find with Theorem 3.3, Lemma 4.2, and Sobolev's embedding theorems for v from Theorem 4.5:

$$\begin{aligned} v &\in C([0, T], H^{k+1}(G)) \subset C([0, T], C^2(\overline{G})), \\ \dot{v} &\in C([0, T], H^{k-1}(G)) \subset C([0, T], C(\overline{G})). \end{aligned}$$

Thus, setting $u_s(x, t) := [\tilde{v}(t)](x)$ with $\tilde{v}(t)$ being the harmonic extension of $v(t) \in D_{1/2}$, we have $u_s \in C_1^2(\overline{G} \times [0, T])$ satisfying (4.1a) and (4.1e). Since $\tilde{v}(t) \in \mathcal{H}_0$ is harmonic in \widehat{G} conditions (4.1b) and (4.1d) hold for u_s as well. To prove (4.1c) note that this condition holds for \tilde{v}_n . Recalling the maximum principle for harmonic functions this implies

$$\max_{x \in \mathbb{R}^d} |D_x^\alpha \tilde{v}_n(x)| = \max_{x \in \overline{G}} |D_x^\alpha \tilde{v}_n(x)|$$

for any multiindex α with $|\alpha| \leq 1$. Setting $S_{mn}(x, t) := \sum_{\nu=m}^n c_\nu(t) \tilde{v}_\nu(x)$ we have then with Sobolev and relation (3.5)

$$\begin{aligned} \max_{t \in [0, T]} \sum_{|\alpha| \leq 1} \max_{x \in \mathbb{R}^d} |D_x^\alpha S_{mn}(x, t)| &= \max_{t \in [0, T]} \sum_{|\alpha| \leq 1} \max_{x \in \overline{G}} |D_x^\alpha S_{mn}(x, t)| \\ &\leq C \max_{t \in [0, T]} \|S_{mn}(\cdot, t)\|_{H^k(G)} \leq \tilde{C} \|S_{mn}\|_{C([0, T], D_{k/2})}. \end{aligned} \tag{4.18}$$

So, convergence of $v(t) = \sum_{n=1}^\infty c_n(t) v_n$ in $C([0, T], D_{k/2})$ implies convergence of $\tilde{v}(t) = \sum_{n=1}^\infty c_n(t) \tilde{v}_n$ in $C([0, T], C^1(\mathbb{R}^d))$, i.e. $u_s \in C^1(\mathbb{R}^d \times [0, T])$.

Similarly, fixing any $K \Subset \widehat{G}$ and using the interior derivative estimate (cf. [9, p23])

$$\max_{x \in K} |D_x^\alpha \widetilde{v}_n(x)| \leq \widehat{C} \max_{x \in \widehat{G}} |\widetilde{v}_n(x)| \leq \check{C} \max_{x \in \overline{G}} |\widetilde{v}_n(x)|$$

with $|\alpha| = 2$ we find $u_s \in C^2(K \times [0, T])$ and hence $u_s \in C^2(\widehat{G} \times [0, T])$.

Finally, since $T > 0$ is arbitrary, $u := u_s + u_\infty$ is a classical solution of problem (1.1). \square

Remark 4.7 *If the principal coefficient a in eq. (1.1a) does not depend on t , smooth solutions can be established in a much simpler way than in the time-dependent case (see Appendix E).*

Appendices

A The non-radial-flow problem

In the framework of magnetohydrodynamics the kinematic dynamo problem reads [4]:

$$\partial_t B + \nabla \times (\eta \nabla \times B) = \nabla \times (v \times B), \quad \nabla \cdot B = 0 \quad \text{in } G \times \mathbb{R}_+, \quad (\text{A.1a})$$

$$\nabla \times B = 0, \quad \nabla \cdot B = 0 \quad \text{in } \widehat{G} \times \mathbb{R}_+, \quad (\text{A.1b})$$

$$B \text{ continuous} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \quad (\text{A.1c})$$

$$B(x, t) = O(|x|^{-3}) \quad \text{for } |x| \rightarrow \infty, t \in \mathbb{R}_+, \quad (\text{A.1d})$$

$$B(\cdot, 0) = B_0 \quad \text{on } G \times \{t = 0\}. \quad (\text{A.1e})$$

Here, the induction equation (A.1a) describes the generation of the magnetic field B by the motion (with prescribed flow field v) of a conducting fluid (with conductivity $\eta > 0$) in a bounded region $G \subset \mathbb{R}^3$. Outside the fluid region there are no further sources of magnetic field. Thus, B continues in $\widehat{G} = \mathbb{R}^3 \setminus \overline{G}$ as a vacuum field and vanishes at spatial infinity.

If G is a ball B_R (or a spherical shell) the so-called poloidal-toroidal decomposition of solenoidal fields is especially useful [4, 14]:

$$B = B_P + B_T = -\nabla \times \Lambda S - \Lambda T, \quad \langle S \rangle = \langle T \rangle = 0.$$

Λ denotes here the non-radial derivative operator $\Lambda := x \times \nabla$, $\Lambda \cdot \Lambda =: \mathcal{L}$ is the Laplace-Beltrami-operator on the unit sphere S_1 , and $\langle \cdot \rangle$ denotes the spherical mean. The poloidal and toroidal scalars S and T , resp., are uniquely determined (e.g. in $L^2(B_R)$) by B :

$$x \cdot B = -\mathcal{L} S, \quad x \cdot \nabla \times B = -\mathcal{L} T.$$

In the following we refer to $P := x \cdot B$ instead of S as poloidal scalar.

In the case of a non-radial flow field, i.e. $v \cdot x \equiv 0$, and spherically symmetric conductivity problem (A.1) implies the scalar sub-problem (1.3) for P : (1.3a) is just the radial component of the first part of (A.1a) and (1.3b) is obtained by applying Λ on the first part of (A.1b). Condition (1.3c) is in fact enough to ensure a continuous magnetic field B , i.e. continuous second-order derivatives of $S = -\mathcal{L}^{-1}P$, since B involves not more than one radial derivative of S . The equivalence of (A.1d) with (1.3d) is clear for $B \cdot (x/|x|)$ and follows for the non-radial components with the divergence-constraint.

Remark A.1 In the mathematical treatment of problem (1.1) it turns out to be useful to consider functions v satisfying the integral condition $\int_{\mathbb{R}^d} |\nabla v|^2 dx < \infty$. In the context of

problem (1.3) this condition can be interpreted as one guaranteeing finite total magnetic energy. In fact, the total energy of the poloidal magnetic field reads

$$E[B_P] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times \Lambda S|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \mathcal{L}^{1/2} S|^2 dx,$$

and with the variational estimate

$$\inf_{f \neq 0, \langle f \rangle = 0} \frac{\|\mathcal{L} f\|_{L^2(S_1)}}{\|f\|_{L^2(S_1)}} = 2$$

one obtains the bound on $E[B_P]$:

$$E[B_P] \leq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \mathcal{L} S|^2 dx = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla P|^2 dx.$$

B The axisymmetric problem

The central assumption is here an axisymmetric magnetic field with representation

$$B = \nabla P \times \nabla \phi + A \nabla \phi = -\frac{1}{\rho} \partial_z P e_\rho + \frac{1}{\rho} A e_\phi + \frac{1}{\rho} \partial_\rho P e_z \quad (\text{B.1})$$

by two scalar quantities, the poloidal one P and the toroidal or azimuthal one A , depending (besides on t) on ρ and z with (ρ, ϕ, z) being cylindrical coordinates in $\mathbb{R}^3 \setminus \{\rho = 0\}$. Inserting (B.1) into the dynamo equation (A.1) the following sub-problem for the poloidal scalar P arises [3, 12, 10]:

$$\partial_t P - \eta \Delta_* P = -v \cdot \nabla P \quad \text{in } G_2 \times \mathbb{R}_+, \quad (\text{B.2a})$$

$$\Delta_* P = 0 \quad \text{in } \widehat{G}_2 \times \mathbb{R}_+, \quad (\text{B.2b})$$

$$P \text{ and } \nabla P \text{ continuous} \quad \text{in } H \times \mathbb{R}_+, \quad (\text{B.2c})$$

$$P(\rho, z, t) \rightarrow 0 \quad \text{for } \rho \rightarrow 0, (z, t) \in \mathbb{R} \times \mathbb{R}_+, \quad (\text{B.2d})$$

$$|\nabla P(\rho, z, t)| = O(\rho) \quad \text{for } \rho \rightarrow 0, (z, t) \in \mathbb{R} \times \mathbb{R}_+, \quad (\text{B.2e})$$

$$\left| \frac{1}{\rho} \nabla P(\rho, z, t) \right| = O((\rho^2 + z^2)^{-3/2}) \quad \text{for } \rho^2 + z^2 \rightarrow \infty, t \in \mathbb{R}_+, \quad (\text{B.2f})$$

$$P(\cdot, \cdot, 0) = P_0, \quad P_0 \text{ satisfying (B.2d,e)} \quad \text{on } G_2 \times \{t = 0\}. \quad (\text{B.2g})$$

Δ_* is the elliptic operator $\Delta_* := \partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \partial_z^2$ on the half-plane $H := \mathbb{R}_+ \times \mathbb{R}$. $G_2 \subset H$ is the ‘‘cross-section’’ of some bounded region $G_3 \subset \mathbb{R}^3$; more precisely, $\overline{G_3} \setminus \partial \overline{G_3}$ with $G_3 := G_2 \times \{0 \leq \phi < 2\pi\}$ is a bounded domain in \mathbb{R}^3 with smooth boundary. Note that the axisymmetric flow field need not be solenoidal, the azimuthal component, however, w.l.o.g. can assumed to be zero. Condition (B.2e) ensures a finite magnetic field on the symmetry axis $\{\rho = 0\}$. It implies the limit $\lim_{\rho \rightarrow 0} P(\rho, \cdot, t) = P_s(t)$, where P_s depends only on t . As P_s does not affect the magnetic field it is set to zero for simplicity (condition (B.2d)). Note that in [12, 10] conditions (B.2d,e) are replaced by

$$P(\rho, z, t) = O(\rho^2) \quad \text{for } \rho \rightarrow 0, (z, t) \in \mathbb{R} \times \mathbb{R}_+. \quad (\text{B.3})$$

The cautious Backus [3] requires (B.2e) *and* (B.3). In fact, conditions (B.2d,e) imply (B.3), but not vice versa. In the view of the original problem (A.1), condition (B.2e) seems to be the natural one. Similarly, in these references the ‘‘natural’’ condition (B.2f) is replaced by

$$P(\rho, z, t) = O((\rho^2 + z^2)^{-1/2}) \quad \text{for } \rho^2 + z^2 \rightarrow \infty, t \in \mathbb{R}_+. \quad (\text{B.4})$$

These conditions are in fact equivalent for solutions of (B.2b) as becomes clear in the subsequent formulation of problem (B.2).

There is an elegant way to eliminate the “coordinate–singularity” at $\rho = 0$ in problem (B.2), namely by embedding (B.2) in \mathbb{R}^5 . P is then considered as an axisymmetric function in \mathbb{R}^5 with symmetry axis in x_5 -direction. Identifying ρ^2 with $\sum_{i=1}^4 x_i^2$ and z with x_5 , $x \in \mathbb{R}^5$, and introducing $Q(x, t) := \tilde{Q}(\rho, z, t) := P(\rho, z, t)/\rho^2$ the crucial observation is [5]

$$\Delta_* P = (\partial_\rho^2 - \frac{1}{\rho} \partial_\rho + \partial_z^2) P = \rho^2 (\partial_\rho^2 + \frac{3}{\rho} \partial_\rho + \partial_z^2) \tilde{Q} = \rho^2 \Delta_5 Q$$

with Δ_5 being the Laplacian in \mathbb{R}^5 . With the further definitions

$$\begin{aligned} b_i(x, t) &:= -v_\rho(\rho, z, t) \frac{x_i}{\rho}, \quad i = 1, \dots, 4, & b_5(x, t) &:= -v_z(\rho, z, t). \\ c(x, t) &:= -2v_\rho(\rho, z, t)/\rho, & a(x, t) &:= \eta(\rho, z, t) \end{aligned}$$

problem (B.2) takes in \mathbb{R}^5 the form

$$\partial_t Q - a \Delta_5 Q = b \cdot \nabla Q + c Q \quad \text{in } G_5 \times \mathbb{R}_+, \quad (\text{B.5a})$$

$$\Delta_5 Q = 0 \quad \text{in } \widehat{G}_5 \times \mathbb{R}_+, \quad (\text{B.5b})$$

$$Q \text{ and } \nabla Q \text{ continuous} \quad \text{in } \mathbb{R}^5 \times \mathbb{R}_+, \quad (\text{B.5c})$$

$$Q(x, t) = O(|x|^{-3}) \quad \text{for } |x| \rightarrow \infty, t \in \mathbb{R}_+, \quad (\text{B.5d})$$

$$Q(\cdot, 0) = Q_0, \quad Q_0 \text{ axisym.} \quad \text{on } G_5 \times \{t = 0\}. \quad (\text{B.5e})$$

G_5 is now an axisymmetric bounded region in \mathbb{R}^5 . An axisymmetric initial field Q_0 implies axisymmetry of $Q(\cdot, t)$ for all $t > 0$. A condition on the symmetry axis is no longer necessary; conditions (B.2d,e) (as well as (B.3)) are automatically satisfied by $P := \rho^2 \tilde{Q}$. As to the behaviour for large x , Q is in \widehat{G} an exterior harmonic function with representation (C.1); thus, (B.5d) implies $|\nabla Q(x, \cdot)| = O(|x|^{-4})$ for $|x| \rightarrow \infty$ and hence (B.2f) as well as (B.4). On the other side, in the view of (C.4) either of the conditions (B.2f) and (B.4) implies (B.5d).

Remark B.1 Stredulinsky et al. [16] doubt the correctness of the boundary condition (B.4) in [10] and cite their own results (Theorem 2) about solutions with nonvanishing asymptotic value $P_\infty(t)$ at spatial infinity. They mention the possibility of $\lim_{t \rightarrow \infty} P_\infty(t) \neq 0$, even when $\lim_{t \rightarrow 0} P_\infty(t) = 0$. In fact, condition (B.4) is correct as demonstrated above, which means $P_\infty \equiv 0$ in the axisymmetric problem. The discrepancy arises because in [16] the authors are especially interested in the two-dimensional case where their problem (1) makes physical sense (“dynamo problem with plane symmetry”). In $d = 2$, in fact, $P_\infty \neq 0$ cannot be avoided in general. In $d > 2$, however, problem (1) is underdetermined and the condition $P_\infty \equiv 0$ can be added. Observe in this context that in [16] the authors do not claim uniqueness for solutions of problem (1), only for weak solutions which are “minimizers” (this is more than “harmonic”) in \widehat{G} (Theorem 1).

C Exterior harmonic functions

An exterior harmonic function u is called “harmonic at infinity” iff $u(x) \rightarrow 0$ for $|x| \rightarrow \infty$. Those functions have in $d \geq 3$ dimensions the series representation

$$u(x) = \sum_{n=0}^{\infty} |x|^{2-n-d} Y_n(x/|x|), \quad Y_n \in H_n, \quad (\text{C.1})$$

absolutely and uniformly converging in the exterior of any ball $B_r \subset \mathbb{R}^d$ with $r > 1$ (cf. [8, p115]. H_n is the space of all harmonic homogeneous polynomials of degree n in \mathbb{R}^d restricted to the unit sphere S^{d-1} with dimension $D_n := \dim H_n = (2n + d - 2)(n + d - 3)[n!(d - 2)!]^{-1}$ (cf. [8, p98f]. In particular, $\dim H_0 = 1$ and $Y_0 = \text{const}$; any other Y_n has vanishing spherical mean, $\langle Y_n \rangle = 0$, $n \in \mathbb{N}$. The total of spaces H_n spans $L^2(S^{d-1})$: $L^2(S^{d-1}) = \bigoplus_0^\infty H_n$. So, choosing orthonormal bases $\{Y_{nm} \mid 1 \leq m \leq D_n\}$ in H_n , any $f \in L^2(S^{d-1})$ allows a unique representation

$$f = \sum_{n=0}^{\infty} \sum_{m=1}^{D_n} c_{nm} Y_{nm} \quad (\text{C.2})$$

with coefficients $c_{nm} := (f, Y_{nm})_{L^2} \in \mathbb{R}$. Obviously, (C.2) is the higher-dimensional analogue of the well-known representation

$$f = \sum_{n=0}^{\infty} \sum_{m=-n}^n c_{nm} Y_{nm} \quad (\text{C.3})$$

by spherical harmonics $\{Y_{nm} \mid n \in \mathbb{N}_0, |m| \leq n\}$ in $d = 3$. Note that the Y_{nm} are in our setting real quantities; by taking suitable linear combinations this is true for spherical harmonics as well.

Exterior harmonic functions u satisfying the condition $\int_{\widehat{B}_R} |\nabla u|^2 dx < \infty$ for some $R > 0$ are harmonic at infinity up to a constant c_0 (cf. [15, p41]. With (C.1) this implies the representation

$$u(x) = c_0 + \sum_{n=0}^{\infty} |x|^{2-n-d} Y_n(x/|x|), \quad c_0 \in \mathbb{R}, \quad Y_n \in H_n. \quad (\text{C.4})$$

(C.4) holds also in the case of exterior harmonic functions with asymptotic conditions $u(x) = O(1)$ for $|x| \rightarrow \infty$ (cf. [2, p64]) $|\nabla u| = O(|x|^{1-d})$ for $|x| \rightarrow \infty$. The latter statement follows from the former and the estimate

$$|u(x)| \leq |u(Rx/|x|)| + \int_R^{|x|} |\nabla u(r x/|x|)| dr, \quad |x| \geq R \geq 0.$$

Let us, finally, consider the exterior boundary-value problem

$$\Delta u = 0 \quad \text{in } \widehat{G}, \quad (\text{C.5a})$$

$$u = \phi \quad \text{on } \partial \widehat{G}, \quad (\text{C.5b})$$

$$u(x) \rightarrow c \quad \text{for } |x| \rightarrow \infty. \quad (\text{C.5c})$$

Here, $\widehat{G} \subset \mathbb{R}^d$, $d \geq 3$ is an exterior region, i.e. $\widehat{G} = \mathbb{R}^d \setminus \overline{G}$ for some bounded domain $G \subset \mathbb{R}^d$, with C^1 -boundary $\partial \widehat{G}$. For $\phi \in C(\partial \widehat{G})$ and $c \in \mathbb{R}$ we call $u \in C(\widehat{G} \cup \partial \widehat{G}) \cap C^2(\widehat{G})$ satisfying (C.5) a classical solution. A weak formulation is based on the spaces $\widehat{\mathcal{H}} := \{v \in H_{loc}^1(\widehat{G}) \mid \|v\|_{\widehat{\mathcal{H}}} < \infty\}$ and $\widehat{\mathcal{H}}_0 := \text{clos}\{C_0^\infty(\widehat{G}), \|\cdot\|_{\widehat{\mathcal{H}}}\}$ with $\|v\|_{\widehat{\mathcal{H}}}^2 := \int_{\widehat{G}} |\nabla v|^2 dx$. Describing the boundary and asymptotic conditions by a function $w \in \widehat{\mathcal{H}}$, a weak version of (C.5) reads:

$$\int_{\widehat{G}} \nabla u \cdot \nabla v dx = 0 \quad \text{for any } v \in \widehat{\mathcal{H}}_0, \quad (\text{C.6a})$$

$$u - w \in \widehat{\mathcal{H}}_0. \quad (\text{C.6b})$$

For problem (C.6) holds:

Theorem C.1 (Solution of the exterior Dirichlet problem) *The exterior boundary-value problem (C.6) with given $w \in \widehat{\mathcal{H}}$ has a unique solution $u \in \widehat{\mathcal{H}}$. Moreover, $u \in C^\infty(\widehat{G})$ and $\Delta u = 0$ in \widehat{G} . If $w|_{\partial\widehat{G}} = \phi \in C(\partial\widehat{G})$ and $w \rightarrow c$ for $|x| \rightarrow \infty$, then u is a classical solution of (C.5). Furthermore, u is the unique minimizer of the functional $\|\cdot\|_{\widehat{\mathcal{H}}}$ on the set $\{v + w \mid v \in \widehat{\mathcal{H}}_0\}$.*

For a proof we refer to [6, p543]. We note only that $\|\cdot\|_{\widehat{\mathcal{H}}}$ -convergence already implies $\|\cdot\|_{L^2_{loc}(\widehat{G})}$ -convergence, thus $\widehat{\mathcal{H}}_0 \subset \widehat{\mathcal{H}}$. In fact, the Gagliardo-Nirenberg-Sobolev-inequality (cf. [7, p263]) implies the estimate

$$\|v\|_{L^p(\widehat{G})} \leq C\|v\|_{\widehat{\mathcal{H}}}, \quad p = \frac{2d}{d-2}$$

for any $v \in \widehat{\mathcal{H}}_0$ and a constant C depending only on d . So, fixing some $K \Subset \widehat{G}$ we obtain:

$$\|v\|_{L^2(K)} \leq C(K)\|v\|_{L^p(K)} \leq C(K)\|v\|_{L^p(\widehat{G})} \leq C(K)C\|v\|_{\widehat{\mathcal{H}}}.$$

In particular, our $\widehat{\mathcal{H}}_0$ coincides with the corresponding space $B_0^1(\widehat{G})$ in [6].

D Poloidal free decay modes

The poloidal free decay modes are a countable set of explicit solutions of the eigenvalue problem (3.1) if G is a ball B_R in \mathbb{R}^3 . In terms of spherical Bessel functions j_l and spherical harmonics (cf. Appendix C) they take the form

$$\tilde{p}_{lnm}(x) := \sqrt{\frac{2}{R^3}} \begin{cases} \frac{j_n(i_l^{n-1}|x|/R)}{j_n(i_l^{n-1})} Y_{nm}(x/|x|) & \text{in } B_R \\ (|x|/R)^{-n-1} Y_{nm}(x/|x|) & \text{in } \widehat{B}_R \end{cases}, \quad l \in \mathbb{N}, n \in \mathbb{N}_0, |m| \leq n$$

with eigenvalues $\lambda_{lnm} := \lambda_{ln} := (i_l^{n-1}/R)^2$; i_l^n is the l -th positive zero of j_n . For their restrictions $p_{lnm} := \tilde{p}_{lnm}|_{B_R}$ holds

Theorem D.1 *The set of functions $\{p_{lnm} : B_R \rightarrow \mathbb{R} \mid l \in \mathbb{N}, n \in \mathbb{N}_0, |m| \leq n\}$ constitutes a complete orthonormal system in $L^2(B_R)$.*

PROOF: The orthonormality of the p_{lnm} can be checked by explicit calculation using the orthonormality of the Y_{nm} and the corresponding relation for the j_n (cf. [1, p485, eq. 11.4.5]). To prove the completeness we show any solution of problem (3.1) being a linear combination of the \tilde{p}_{lnm} . Theorem 3.1 yields then the completeness of the p_{lnm} . So, let u be a solution of (3.1) with eigenvalue $\lambda > 0$. The representation (C.3) for L^2 -functions on the unit sphere implies the representation

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^n u_{nm}(|x|) Y_{nm}(x/|x|) \tag{D.1}$$

for u with coefficients u_{nm} depending only on $|x|$. For $|x| > R$ follows with (C.1)

$$u_{nm}(|x|) = c_{nm}|x|^{-n-1}$$

with $c_{nm} \in \mathbb{R}$, whereas inserting (D.1) into (3.1a) yields the differential equation for u_{nm} on $(0, R)$:

$$-\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} u_{nm} \right) + \frac{1}{r^2} n(n+1) u_{nm} = \lambda u_{nm} \tag{D.2}$$

with $r := |x|$. After rescaling $s := \sqrt{\lambda}r$, (D.2) takes the spherical form of Bessel's differential equation for $v_{nm}(s) := u_{nm}(s/\sqrt{\lambda})$:

$$\left(s^2 \frac{d^2}{ds^2} + 2s \frac{d}{ds} + s^2 - n(n+1)\right)v_{nm} = 0$$

with (in $s = 0$) regular solutions j_n , $n \in \mathbb{N}_0$. Matching inner and outer solutions according to (3.1c) yields the condition

$$\left(s \frac{d}{ds} j_n(s) + (n+1)j_n(s)\right)\Big|_{s=\sqrt{\lambda}R} = (s j_{n-1}(s))\Big|_{s=\sqrt{\lambda}R} = 0 \quad (\text{D.3})$$

fixing the eigenvalue at

$$\lambda = \lambda_{ln} = (i_l^{n-1}/R)^2. \quad (\text{D.4})$$

Note that (D.3) holds also in the case $n = 0$ with $j_{-1} = \cos s/s$.

According to Theorem 3.1, the eigenvalue λ is of finite multiplicity. Thus, only finitely many pairs (l, n) satisfy (D.4), and (D.1) is in fact a (finite) linear combination of the \tilde{p}_{lnm} . \square

Remark D.2 The condition of vanishing spherical mean eliminates all $n = 0$ modes; thus, $\{p_{lnm} \mid l \in \mathbb{N}, n \in \mathbb{N}, |m| \leq n\}$ is a complete orthonormal system in $\{v \in L^2(B_R) \mid \langle v \rangle = 0\}$.

E The case of time-independent principal coefficient

We outline here a simpler way (than for the general case in section 4) to obtain smooth solutions of the evolution problem. The basic idea is to absorb the principal coefficient a into the definition of the operator \mathcal{A} . For this purpose we solve the Poisson problem (2.1) with (2.1a) replaced by

$$-a \Delta u = f \quad \text{in } G,$$

where $a \in C(\overline{G})$, $a \geq a_0 > 0$. This can easily be achieved by replacing in the weak formulation (2.8) the L^2 -scalar product on the right-hand side by a weighted version:

$$(u, v)_{\mathcal{H}} = \int_G f v \frac{1}{a} dx =: (f, v)_{L_a^2(G)}.$$

The corresponding Green's operator $\mathcal{G}_a : L_a^2(G) \rightarrow L_a^2(G)$ yields then a complete orthonormal system $\{w_n : n \in \mathbb{N}\}$ with eigenvalues μ_n^{-1} in $L_a^2(G)$.⁵ The associated harmonic extension \tilde{w}_n satisfies instead of (3.3):

$$(\tilde{w}_n, v)_{\mathcal{H}} = \mu_n (w_n, v)_{L_a^2} \quad \text{for any } v \in \mathcal{H}_0,$$

and solves problem (3.1) with (3.1a) replaced by

$$-a \Delta w_n = \mu_n w_n \quad \text{in } G.$$

Note, however, that the regularity of \tilde{w}_n depends now on that of a ; for instance, $a \in C^k(\overline{G})$, $k > (d+1)/2$ implies $\tilde{w} \in C^2(G \cup \hat{G}) \cap C^1(\mathbb{R}^d)$ (cf. Theorem 2.3). The system $\{w_n : n \in \mathbb{N}\}$ allows the representation

$$w = \sum_{n=1}^{\infty} (w_n, w)_{L_a^2} w_n$$

⁵ $L_a^2(G)$ means just $L^2(G)$ equipped with $(\cdot, \cdot)_{L_a^2(G)}$.

for any $w \in L^2(G)$ and to define an operator \mathcal{A}_a analogous to \mathcal{A} . Theorem 3.3 holds then for \mathcal{A}_a as well, provided a is sufficiently smooth (for instance, $a \in C^{k-2}(\overline{G})$ implies $w_n \in H^k(G)$, $k \geq 2$).

We are now going to solve the initial-value problem

$$\dot{w} + \mathcal{A}_a w = \mathcal{B}w + f, \quad (\text{E.1a})$$

$$w(0) = w_0 \quad (\text{E.1b})$$

with the lower-order operator $\mathcal{B}w := b \cdot \nabla w + cw$ on the right-hand side. A suitable notion of weak solution reads now:

Definition E.1 *Let $T > 0$ and $w_0 \in D_{1/2}$. A function $w \in C([0, T], D_{1/2}) \cap C^1([0, T], D_{-1/2})$ satisfying (E.1a) as equality in $C([0, T], D_{-1/2})$ and (E.1b) as equality in $D_{1/2}$ is called weak solution of problem (E.1).*

Theorem E.2 *Let $T > 0$ and $a \in C(\overline{G})$, $a \geq a_0 > 0$, and $b, c \in C(\overline{G} \times [0, T])$. Let, furthermore, $w_0 \in D_{1/2}$ and $f \in C([0, T], L^2(G))$. Then problem (E.1) has a unique weak solution w ; moreover, $w \in L^2((0, T), D_1)$.*

Furthermore, if $a \in C^{k-2}(\overline{G})$,⁶ $b, c \in C^k(\overline{G} \times [0, T])$, $w_0 \in D_{(k+1)/2}$, and $f \in C([0, T], D_{k/2})$ for some $k \in \mathbb{N}$, then

$$w \in C([0, T]D_{(k+1)/2}) \cap C^1([0, T], D_{(k-1)/2}).$$

PROOF: The proof proceeds for the present similar to that of Theorem 4.2, i.e. we construct Galerkin approximations $w^{(n)}$ based, this time however, on the complete system $\{w_\nu : \nu \in \mathbb{N}\}$ introduced above. $w^{(n)} \in C^1([0, T], D(\mathcal{A}_a))$ is now the unique solution of the initial-value problem

$$\frac{d}{dt} w^{(n)} = -\mathcal{A}_a w^{(n)} + Q^{(n)}(\mathcal{B}w^{(n)} + f), \quad (\text{E.2a})$$

$$w^{(n)}(0) = Q^{(n)}w_0. \quad (\text{E.2b})$$

with $Q^{(n)} := \bigoplus_{\nu=1}^n Q_\nu$ and Q_ν denoting the orthogonal projection in $L_a^2(G)$ onto $\text{span}\{w_\nu\}$.

The a-priori estimate

$$\max_{[0, T]} \|w^{(n)}\|_{1/2}^2 \leq C \quad (\text{E.3})$$

follows completely analogously to (4.5) in the proof of Theorem 4.3. Differently from the general case, $(w^{(n)})$ can here be shown to be a Cauchy sequence in $C([0, T], D_{1/2})$:

Let $w_{nm} := w^{(n)} - w^{(m)}$ with $n \geq m$. Observing that $\mathcal{B} : C([0, T], D_{1/2}) \rightarrow C([0, T], L^2(G))$ is a bounded operator we obtain

$$\begin{aligned} (\mathcal{A}_a w_{nm}, Q^{(n)}\mathcal{B}w^{(n)} - Q^{(m)}\mathcal{B}w^{(m)})_{L_a^2} &= (\mathcal{A}_a w_{nm}, Q^{(n)}\mathcal{B}w_{nm} + (Q^{(n)} - Q^{(m)})\mathcal{B}w^{(m)})_{L_a^2} \\ &\leq C_1^{1/2} \|w_{nm}\|_1 \|w_{nm}\|_{1/2} + C_1^{1/2} \|w_{nm}\|_1 \|Q^{(n)} - Q^{(m)}\|_{L_a^2 \rightarrow L_a^2} \|w^{(m)}\|_{1/2} \\ &\leq \frac{1}{2} \|w_{nm}\|_1^2 + C_1 \|w_{nm}\|_{1/2}^2 + C_1 C \varepsilon_{nm}. \end{aligned}$$

In the last line we used (E.3) and introduced the abbreviation $\varepsilon_{nm} := \|Q^{(n)} - Q^{(m)}\|_{L_a^2 \rightarrow L_a^2}^2$. Scalar multiplication of (E.2a) with $\mathcal{A}_a w_{nm}$ yields now

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w_{nm}\|_{1/2}^2 &= (\mathcal{A}_a w_{nm}, -\mathcal{A}_a w_{nm} + Q^{(n)}(\mathcal{B}w^{(n)} + f) - Q^{(m)}(\mathcal{B}w^{(m)} + f))_{L_a^2} \\ &\leq C_1 \|w_{nm}\|_{1/2}^2 + C_1 C \varepsilon_{nm} + \frac{1}{2} C_2 \varepsilon_{nm}, \end{aligned}$$

⁶In the case $k = 1$ " $C^{-1}(\overline{G})$ " means $C(\overline{G})$.

where $C_2 := \max_{[0,T]} \|f\|_{L^2_a}^2$, and with Gronwall we arrive at

$$\max_{[0,T]} \|w_{nm}\|_{1/2}^2 \leq e^{2C_1 T} \|w_{nm}(0)\|_{1/2}^2 + (2C_1 C + C_2) \varepsilon_{nm} \frac{e^{2C_1 T} - 1}{2C_1}.$$

Observing $\|w_{nm}(0)\|_{1/2}^2 \leq \varepsilon_{nm} \|w_0\|_{1/2}^2$ and $\lim_{m \rightarrow \infty} \varepsilon_{nm} = 0$ (uniformly in $n \geq m$) we find $(w^{(n)})$ being a Cauchy sequence in $C([0, T], D_{1/2})$ with limit function w .

To see that w is a weak solution in the sense of Definition E.1 consider the sequence of (classical) derivatives $(\frac{d}{dt} w^{(n)})$. Going back to eq. (E.2a) we find the right-hand side converging in $C([0, T], D_{-1/2})$ with limit $-\mathcal{A}_a w + \mathcal{B} w + f$. As a consequence $\frac{d}{dt} w$ exists (classically) and equals $-\mathcal{A}_a w + \mathcal{B} w + f$ in $C([0, T], D_{-1/2})$; this means, in particular, $w \in C^1([0, T], D_{-1/2})$.

Uniqueness of the weak solution and higher spatial regularity follow just as in the proofs of Theorems 4.3 and 4.5, respectively. □

Concerning the original evolution problem (1.1) let us remark that Theorem E.2 provides the prerequisites to apply Corollary 4.6 even under slightly weaker assumptions on the data b , c , and f .

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