Soliton transport in tubular networks: transmission at vertices in the shrinking limit

Hannes Uecker\textsuperscript{a}, Daniel Grieser\textsuperscript{a}, Zarif Sobirov\textsuperscript{b}, Doniyor Babajanov\textsuperscript{c} and Davron Matrasulov\textsuperscript{c}

\textsuperscript{a} Institut für Mathematik, Universität Oldenburg, D26111 Oldenburg, Germany
\textsuperscript{b} Tashkent Financial Institute, 60A, Amir Temur Str., 100000, Tashkent, Uzbekistan
\textsuperscript{c} Turin Polytechnic University in Tashkent, 17 Niyazov Str., 100095, Tashkent, Uzbekistan

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Soliton transport in tube-like networks is studied by solving the nonlinear Schrödinger equation (NLSE) on finite thickness (“fat”) graphs. The dependence of the soliton solution and of the reflection at vertices on the graph thickness and on the angle between its bonds is studied and related to a special case considered [1], in the limit when the thickness of the graph goes to zero. It is found that both the wave function and reflection coefficient reproduce the regime of reflectionless vertex transmission studied in [1].

I. INTRODUCTION

Particle and wave transport in branched structures is of importance for different topics of contemporary physics such as optics, cold atom physics, fluid dynamics and acoustics. For instance, such problems as light propagation in optical fiber networks, BEC in network type traps and acoustic waves in discrete structures deal with wave transport in branched systems. In most of the practically important cases such transport is described by linear and nonlinear Schrödinger equations (NLSE) on graphs. The latter has become the topic of extensive study during past few years [1–10] and is still rapidly progressing. Such interest in the NLSE on networks is mainly caused by possible topology-dependent tuning of soliton transport in branched structures which is relevant to many technologically important problems such as BEC in network type traps [11–13], information and charge transport in DNA double helix [14–15], light propagation in waveguide networks [16] etc.

Soliton solutions of the NLSE on simplest graphs and connection formulae are derived in [1], showing that for certain relations between the nonlinearity coefficients of the bonds soliton transmission through the graph vertex can be reflectionless (ballistic). Dispersion relations for linear and nonlinear Schrödinger equations on networks are discussed in [2]. The problem of fast solitons on star graphs is treated in [3] where estimates for the transmission and reflection coefficients are obtained in the limit of high velocities. The problem of soliton transmission and reflection is studied in [2] by solving numerically the stationary NLSE on graphs. More recent progress in the study of the NLSE on graphs can be found in [17–19]. Scattering solutions of the stationary NLSE on graphs are obtained in [3], and analytical solutions of the stationary NLSE on simplest graphs are derived in [4].

In metric graphs the bonds and vertices are one and zero dimensional, respectively. However, in realistic systems such as electromagnetic waveguides and tube-like optical fibers, the wave (particle) motion may occur along both longitudinal and transverse directions [17–19]. Therefore it is important to study below which (critical) thickness the transverse motions become negligible and the wave(particle) motion can be treated as one-dimensional. In other words, studying the regime of motion when wave dynamics in such tube-like network can be considered the same as that in metric graph is of importance.

In this paper we study the NLSE on so-called fat graphs, i.e. on two-dimensional networks having finite thickness. The geometry will be explained in more detail below, but see Fig. 1 for a sketch. In particular, we consider the same relations between the bond nonlinearity coefficients as those in the paper [1] and study shrinking of the fat graph into the metric graph keeping such relations. Initial conditions for the NLSE on fat graph are taken as quasi 1D solitons. By solving the NLSE on fat graphs we find that in the shrinking limit such fat graphs reproduce the reflectionless regime of transport studied in [1], i.e., the vertex transmission become ballistic.

![FIG. 1: Sketch of a metric graph θ and a fat graph Ω = V_0 ∪ B_1 ∪ B_2 ∪ B_3, with bonds of width w_j, where w_j = O(ε). Ideally, the lengths l_1, l_2, l_3 of the bonds are infinite, but for numerical simulations of the NLSE we use finite lengths with Dirichlet boundary conditions (DBC) at the ends, and homogeneous Neumann boundary conditions (NBC) else.](image-url)
The main problem to be solved in the treatment of the Schrödinger equation on fat graphs is reproducing of vertex coupling rules in the shrinking limit, i.e., when the fat graph shrinks to the metric graph. In case of metric graphs, “gluing” conditions, or vertex coupling rules, are needed to ensure self-adjointness of the Schrödinger equation. The most important example of a vertex coupling is the Kirchhoff condition. For fat graphs there are no such coupling rules; they only appear in the shrinking limit, and their form depends on specifics of the fat graph, for example on the boundary conditions imposed at the lateral boundary. For Neumann boundary conditions, the resulting vertex coupling is the Kirchhoff condition, as was shown in [20, 21], who study convergence of the eigenvalue spectrum of the Schrödinger equation, and in a series of papers by Exner and Post [22–24], who study various aspects of the Schrödinger equation with Neumann boundary conditions (including transport, resonances and magnetic field effects). The vertex couplings obtained in the shrinking limit of the Schrödinger equation on the fat graph with Dirichlet and other boundary conditions were obtained in [31–35]. Recent studies of the linear Schrödinger equation on fat graphs focused on the inverse problem of finding a suitable fat graph problem which reproduces a given coupling rule in the shrinking limit [28]. Further references on linear Schrödinger equation on fat graphs are [26, 27, 32, 33, 37–42], and the boundary conditions imposed at the lateral boundary. For Neumann boundary conditions, the resulting vertex coupling is the Kirchhoff condition. For fat graphs there are no such coupling rules; they only appear in the shrinking limit, including the dependence of reflection coefficient on the angle between the graph bonds. The last section presents come concluding remarks.

II. THE NLSE ON METRIC AND FAT GRAPHS

Consider the nonlinear Schrödinger equation

$$\partial_t \psi_k = i(\psi_k'' + \beta_k |\psi_k|^2 \psi_k), \quad k = 1, 2, 3,$$  \hspace{1cm} (1)

on a metric star graph $\Gamma$ with 3 edges $\Gamma_k$, and nonlinearity coefficients $\beta_k > 0$. The graph is assumed to have semi-infinite bonds $\Gamma_1 = (-\infty, 0)$, $\Gamma_2, 3 = (0, \infty)$, but the main part of our analysis will be numerical, for which we assume finite lengths $l_k$ of bonds, with coordinates $\xi_k \in (-l_k, 0)$, $\xi_{2,3} \in (0, l_{2,3})$, and homogeneous Dirichlet boundary conditions at $\xi_1 = -l_1, \xi_{2,3} = l_{2,3}$. Furthermore, we assume that the solutions, $\psi_k = \psi_k(t, \xi_k) \in \mathbb{C}$ obey the vertex (at $\xi_k = 0$) conditions

$$\alpha_1 \psi_1 = \alpha_2 \psi_2 = \alpha_3 \psi_3,$$  \hspace{1cm} (2)

with parameters $\alpha_k$, where it is understood that $\psi_1' (\psi_2', \psi_3')$ denote the derivatives from the left (right). In the following we call Eqs. (1) and (2) problem (P0).

Soliton solutions of the problem (P0) that propagate without reflection (i.e., ballistically) were obtained analytically in [1] for the special case when the nonlinearity coefficients satisfy the relation

$$\frac{1}{\beta_1} = \frac{1}{\beta_2} + \frac{1}{\beta_3},$$  \hspace{1cm} (3)

These solutions have, after properly identifying $\xi$ with $\xi_k$ on $\Gamma_k$ the form

$$\psi_k(t, \xi) = \sqrt{\frac{2}{\beta_k}} \text{sech}(\eta(\xi - \xi_0 - ct)) e^{-i(2\xi - (c^2 - 4\eta^2)t)}/4,$$  \hspace{1cm} (4)

with free parameters amplitude $\eta > 0$, speed $c$ (wavenumber $c/2$), and reference position $\xi_0$. Fig. 2 presents amplitudes, $A_k = \max_{t \in \Gamma_k} |\psi_k(t, x)|$ for Kirchhoff boundary conditions ($\alpha_1 = \alpha_2 = \alpha_3 = 1$) and for the boundary conditions given by Eq. (2). The vertex boundary conditions given by [2] are one possibility to make the linear part of (1) skew-adjoint. The problem (P0) conserves the norm $N$ and the Hamiltonian $H$ given by

$$N = \sqrt{N_1^2 + N_2^2 + N_3^2}, \quad N_k^2(t) = \int_{\Gamma_k} |\psi_k(t, x)|^2 dx,$$  \hspace{1cm} (5)

$$H = H_1 + H_2 + H_3,$$  \hspace{1cm} (6)

$$H_k(t) = \int_{\Gamma_k} \left| \partial_\xi \psi_k(t, \xi) \right|^2 - \frac{\beta_k}{2} |\psi_k(t, x)|^4 dx.$$  \hspace{1cm} (7)
It is a question of normalization to set
\[ \alpha_1 = \beta_1 = 1, \quad (7) \]
which leaves 4 parameters for \((P_0)\), and, of course, the choice of the initial conditions.

Our goal is to compare exact and numerical solutions \((\psi_1, \psi_2, \psi_3)\) of \((P_0)\) with the numerical solutions \(\phi = \phi(t, x)\) of an associated NLSE on a fat graph presented in Fig. 1, i.e.,
\[ \partial_t \phi = i(\Delta \phi + \tilde{\beta}(x)|\phi|^2 \phi), \quad (8) \]
where \(\Delta = \partial^2_x + \partial^2_z, \quad x = (x_1, x_2) \in \Omega_\varepsilon, \) and \(\Omega_\varepsilon = V_\varepsilon \cup B_{1,\varepsilon} \cup B_{2,\varepsilon} \cup B_{3,\varepsilon}\) consists of a “vertex–region” \(V_\varepsilon\) of diameter \(O(\varepsilon)\), and \(O(\varepsilon)\)-tubes \(B_k\) around \(\Gamma_k\), see Fig. 1.

In the following Eq. (9) will be called the problem \((P_\varepsilon)\).

We also use the notation \(\phi_k\) for \(\phi|_{B_k}\).

It is clear that different versions of \(\Omega_\varepsilon\) are possible. Here we choose to give the following 5 parameters to \(\Omega_\varepsilon\) not a priori present in Eq. (1):

1. the angles \(\theta_2, \theta_3\) between the bonds \(B_2\) and \(B_3\) and the \(x_1\)-axis,
2. the widths \(w_1, w_2, w_3\) of the different bonds.

In the numerical calculations we impose homogeneous Dirichlet boundary conditions (DBC) for both, \((P_0)\) and \((P_\varepsilon)\), at the “ends” of bonds, and for \((P_\varepsilon)\) homogeneous Neumann boundary conditions (NBC) \(\partial_k \psi = 0\) everywhere else. As our simulations will run on time–scales where the solitons will be well separated from the ends of the bonds, we could as well pose NBC there. Also note that strictly speaking [1] is not a solution over the finite graph, but it is exponentially small at the ends of the bonds.

We take \(\tilde{\beta}(x)\) constant on bond \(k\) and with suitable jumps near 0. Furthermore, we set
\[ \varepsilon := w_1, \quad w_2 = \delta_2 \varepsilon \quad \text{and} \quad w_3 = \delta_3 \varepsilon \quad (9) \]
and write \(\Omega_\varepsilon\) for fixed \(\delta_k, \theta_k, \, k = 2, 3\). For definiteness we choose
\[ B_1 = \Omega_\varepsilon \cap \{x_1 < 0\}, \quad B_2 = \Omega_\varepsilon \cap \{x_2 > w_1/2\}, \quad B_3 = \Omega_\varepsilon \cap \{x_2 < -w_1/2\}, \quad (10) \]
and thus \(V_\varepsilon = \Omega_\varepsilon \setminus (B_1 \cup B_2 \cup B_3).\) Motivated by \(\frac{1}{\varepsilon} \int_{\Omega_\varepsilon} 1 dx \to \int_1 + \delta y_2 + \delta y_3\) as \(\varepsilon \to 0,\) corresponding to \(N\) on \(\Gamma\) we define the scaled norms
\[ N_\varepsilon(t) = \left( \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} |\phi(t, x)|^2 dx \right)^{1/2}, \quad (11) \]
\[ \text{and} \quad N_{k,\varepsilon}(t) := \left( \frac{1}{\varepsilon} \int_{B_k} |\phi(t, x)|^2 dx \right)^{1/2}. \quad (12) \]

Then \(N_\varepsilon\) is conserved for \((8)\), and the \(N_{k,\varepsilon}\) indicate how much “mass” is in the different bonds.

For the linear problem it is known, [25], that under the scaling
\[ \frac{w_1}{w_k} = \alpha_k^2, \quad \text{i.e.} \quad \delta_k = \frac{1}{\alpha_k^2}, \quad \text{and} \quad \psi_k = \frac{1}{\alpha_k} \phi_k|_{\Gamma_k}, \quad (13) \]
the vertex conditions [2] appear in the limit \(\varepsilon \to 0.\)

Then, at least formally, we can expect \((P_0)\) as a “limit” of \((P_\varepsilon)\) if
\[ \tilde{\beta}|_{B_k} = w_k \beta_k = \alpha_k^2 \beta_k. \quad (14) \]

If \(\alpha_2 \neq 1\) (or \(\alpha_3 \neq 1)\), then boundary condition presented in Eq. (2) gives jumps from \(\psi_1\) to \(\psi_2\) (resp. \(\psi_3\)) at the vertex. This, however, is merely a question of scaling.

For instance, setting \(\psi_k = \alpha_k \psi_k\) (cf. (13)), we obtain
\[ \partial_t \tilde{\psi}_k = i(\tilde{\psi}_k'' + \gamma_k |\tilde{\psi}_k|^2 \tilde{\psi}_k), \quad \tilde{\psi}_1 = \tilde{\psi}_2 = \tilde{\psi}_3, \quad (15) \]
\[ \tilde{\psi}_1' = \frac{1}{\alpha_2^2} \tilde{\psi}_2' + \frac{1}{\alpha_3^2} \tilde{\psi}_3', \quad \text{at} \ x = 0, \]
i.e., continuity at the vertex, where \(\gamma_k = \beta_k \alpha_k^2\), as in [14]. The scaling given by Eqs. 1, 2 is more customary than [15], and therefore we stick to 1, 2 as the “limit problem". Note that the angles \(\theta_{1,2}\) of the fat graph do not appear in \((P_0)\).

We expect that for \(\varepsilon \to 0\) solutions \(\phi_k\) of \((P_\varepsilon)\) behave like \(\frac{1}{\varepsilon} \alpha_k \psi_k\) with \(\psi_k\) being the solutions of \((P_0)\), i.e., are constant in transverse direction on each bond \(B_k\), with width \(w_k = \delta_k \varepsilon.\) Therefore, from Eqs. (12) and (13) we expect
\[ N_{k,\varepsilon}^2(t) = \frac{1}{\varepsilon} \int_{B_k} |\phi_k(t, x)|^2 dx \approx \delta_k \int_{\Gamma_k} |\phi_k|_{\Gamma_k}^2 d\xi_k \approx \delta_k \int_{\Gamma_k} |\alpha_k|^2 |\psi_k|^2 d\xi_k = N_k^2(t), \quad (16) \]

In the numerical calculations, in addition to \(N_{k,\varepsilon}\) we explore the following functions (dropping the dependence
on parameters $\varepsilon, \delta_{2,3}, \theta_{2,3}, c$ and $\eta$:

$$A_k(t) = \max_{x \in B_k} |\phi_k(t, x)| \quad \text{(scaled amplitude)}, \quad (17)$$

$$m_k(t) = \max_{x \in B_k} \|\tilde{\psi}_k(t, x)\| - \frac{1}{\alpha_k} |\phi_k(t, x)| \quad \text{(maximal amplitude distance between } (P_{\varepsilon}) \text{ and } (P_0)).$$

Here $\tilde{\psi}_k$ is the extension of $\psi_k$ to $B_k$, constant in transverse direction, and for $\psi_k$ we either use the explicit formula $[3]$ if $[3]$ holds, or numerics for $(P_0)$ if not. Note that $(18)$ ignores phase differences between $\tilde{\psi}_k$ and $\phi_k$, as these are less important from the viewpoint of applications.

### III. SOLITON TRANSPORT IN FAT GRAPHS

The main practically important problem in the context of wave propagation in branched systems is energy and information transport via solitary waves. Dependence of the soliton dynamics on the topology of a network makes such systems attractive from the viewpoint of tunable particle transport in low dimensional optical, thermal and electronic devices. Therefore treatment of the problems $(P_0)$ and $(P_{\varepsilon})$ from the viewpoint of vertex soliton transmission is of importance. Our main purpose is to compare propagation of solitons in $\Omega_{\varepsilon}$ with that in $\Gamma$, and in particular to “lift” the earlier results $[1]$ from $\Gamma$ to $\Omega_{\varepsilon}$. Transition from two- to one-dimensional wave motion in the shrinking limit is of special importance for this analysis.

In a typical simulation, for $(P_0)$ we use soliton-type initial condition given as

$$\psi_1(0, \xi_1) = \sqrt{2} \eta \text{sech}(\eta(x_1 - x_0)) e^{-i\xi_1/2}, \quad \psi_{2,3}(0, \cdot) \equiv 0 \quad (19)$$

where $x_0$ and $\eta$ are chosen in such a way that $\psi_1(0, 0)$ is very close to 0. Similarly, for $(P_{\varepsilon})$ we choose

$$\phi(0, x) = \begin{cases} \sqrt{2} \eta \text{sech}(\eta(x_1 - x_0)) e^{-i\xi x_1/2} & x_1 < 0, \\ 0 & \text{else} \end{cases} \quad (20)$$

i.e., we extend the initial condition $[19]$ trivially in transverse direction. We then run both, $(P_0)$ and $(P_{\varepsilon})$ until some final time $t_1$ such that the solitons launched by $[19]$ and $[20]$, respectively, have interacted with the vertex, and have been reflected or transmitted sufficiently far into the bonds. See the appendix for the numerical methods used. Our main solution diagnostics will be the time dependent norms $N_k(t)$, $N_k,\varepsilon(t)$, the amplitudes $A_k(t)$, $A_k,\varepsilon(t)$, the distances $m_k(t)$, and the reflection coefficients defined below.

For definiteness, we consider $\Gamma_1$ as the “incoming” bond and $\Gamma_{2,3}$ as “outgoing”. In Fig. 3 solutions of the problem $(P_{\varepsilon})$ for the Kirchhoff boundary conditions are presented for the case of a “relatively fat” graph ($\varepsilon = 0.5$), while Fig. 4 show the plots of the corresponding norms $N_k$ and amplitudes $A_k$ for the simulation for $(P_{\varepsilon})$ in Fig. 3 (Kirchhoff case), together with the respective quantities for $(P_0)$. At this relatively large $\varepsilon = 0.5$ there is a significant difference between $(P_{\varepsilon})$ and $(P_0)$.

In the following we focus on soliton reflection and
transmission in the shrinking limit $\varepsilon \to 0$, for the "ballistic" boundary conditions given by (3) on $(P_0)$. In Fig. 5 and 6 we plot the diagnostics defined above for different $\varepsilon$ on an otherwise fixed graph fulfilling the conditions of Eq. (3), i.e., for the ballistic case. As $\varepsilon \to 0$, the amplitudes and masses in the different bonds get close to the metric graph case, and also the (numerical) wave functions as a whole converge to the ones on the metric graph, with one small qualification: While the main mismatches of some parameters below. Thus, e.g., $r^A_{k,\varepsilon} = 0$ (and thus also $r^1_{1,\varepsilon} = 0$) means zero reflection of an incoming different scales in panels (a4), (b3) and (c3) strongly indicates the convergence of the $(P_0)$ wave function to the $(P_0)$ wave function in $L^\infty$ (modulo phases), uniformly on bounded time intervals.

From the viewpoint of practical applications, probably the most important question is how much of an incoming soliton is reflected resp. transmitted in the vertex region of a fat graph. To display this in a concise way, for $(P_0)$ we define the reflection and transmission coefficients

$$r^A_{k,\varepsilon} := A_{k,\varepsilon}(t_1)/A_{1,\varepsilon}(0) \quad \text{(amplitude reflection)},$$

$$r^N_{k,\varepsilon} := N_{k,\varepsilon}(t_1)/N_{1,\varepsilon}(0) \quad \text{(mass reflection)},$$

where again we dropped the dependence on parameters $w_{2,3}, \beta_{2,3}, c$ and $\eta$ here, but will plot $r_{k,\varepsilon}, r^N_{k,\varepsilon}$ as functions of some parameters below. Thus, e.g., $r^N_{1,\varepsilon} = 0$ (and
soliton at the vertex, while, e.g., \( r_{2,ε}^N = 1 \) means that all of the “mass” was transmitted to bond two. These extreme cases of course do not occur, but the goal is, e.g., to tune \( r_{k,ε}^{N,A} \). The corresponding quantities for \((P_0)\) are defined as

\[
\begin{align*}
    r_k^A &:= A_k(t_1)/A_1(0), & r_k^N &:= N_k(t_1)/N_1(0), \quad (23)
\end{align*}
\]

and the transmission formula \((3)\) means, e.g., that \( r_{1,ε}^{A,N} \rightarrow 0 \) in the limit of infinite bonds and of \( t_1 \rightarrow \infty \).

In Fig. 7(a) the vertex reflection coefficients (both for norm and amplitude) are plotted as functions of the graph thickness \( ε \) for the case described by \((3)\). The limit \( ε \rightarrow 0 \) again shows a rather smooth transition from the “scattering” regime to ballistic transmission. Figure 7(b) presents the dependence of \( r_{1,ε}^N \) on the graph thickness and the angles \( θ = θ_1 = θ_2 \). Even though the angles do not appear in the \( ε \rightarrow 0 \) limit \((P_0)\), at finite \( ε \) they of course play an important role. Ballistic transport through the vertex occurs in the shrinking limit as well as in the limit of small angles.

Besides the equal angle case \( θ_2 = θ_3 \) considered so far, we checked a variety of other configurations with \( θ_2 < θ_1 \) (angle of the thinner bond with the incoming bond smaller than the other angle), and vice versa, for various \( θ_{1,2} \) between \( π/20 \) and \( π/2 \). The results remain qualitatively similar to Figs. 5–7 i.e., in the limit \( ε \rightarrow 0 \) the reflection coefficients vanish, and as above the \((P_0)\) wave functions converge to the \( θ_{2,3} \) independent wave functions (\( ψ_1, α_2ψ_2, α_3ψ_3, α_3ψ_3 \)) of \((P_0)\). As the convergence for \( ε \rightarrow 0 \) is clearly linear, an interesting question is how to choose a first order in \( ε \) correction of the fat graph geometry or NLSE coefficients that minimizes \( r_{1,ε}^N \) also for finite \( ε > 0 \).

An important issue for particle and wave transport in fat graphs is the dependence of the scattering on initial soliton velocity and amplitude. In Fig. 8 reflection coefficients are plotted as functions of bond thickness \( ε \) for different initial velocities \((a)\) and amplitudes \((b)\). The dependence of reflection on initial data is significant for fat graphs, with, e.g., less reflection for slower and longer waves, as should be expected. However, in the shrinking limit the reflections again vanish in all cases considered.

Finally, although in Figs. 5–7 we focused on the ballistic case \( δ_2 + δ_3 = 1 \), for other values of \( δ_2, δ_3 \), as for instance \( δ_2 = δ_1 = 1 \) in Fig. 4 as \( ε \rightarrow 0 \) we have the same kind of convergence of \((N_k, ε, A_k, r_{k,ε}^{N,A}, m_{k,ε})\) to \((N_k, A_k, r_{k,ε}^{N,A}, 0)\) as above, and altogether of \( φ \) to \((ψ_1, α_2ψ_2, α_3ψ_3, \ldots, \) i.e., of \((P_ε)\) to \((P_0)\).

\[\text{IV. CONCLUSIONS}\]

We studied soliton transport in tube like networks modeled by the time-dependent NLSE on fat graphs, i.e. graphs with finite bond thickness. We numerically solved the NLSE on fat graphs for different values of thickness, and focussing on the ballistic case given by Eq. 6 studied the shrinking limit of the fat graph. It is found that in the shrinking limit solutions of the NLSE on fat graphs converge to those the associated metric graphs,
Moreover, we set\[\text{The vertex conditions then are}\]
\[\text{and hence that the conditions presented in Eq. (3) for reflectionless transport also work on fat graphs with small } \varepsilon. \text{ Dependence of the vertex reflection coefficient on the bond thickness and on the angle between the bonds of fat graph is also studied.}\]

At this point it is not clear in which norms we can expect or analytically show convergence of solutions of (P_\varepsilon) to solutions of (P_0), as \( \varepsilon \to 0 \). First, following [34] this will be discussed for the stationary case, including some potentials at the vertex in order to have nontrivial stationary solutions for the fat graph and the metric graph, cf. [5] [10]. An important point in the study of wave(particle) dynamics in fat graphs is the definition of the fat graph thickness at which one can neglect transverse motion and consider the system as one-dimensional. The above treatment allows us to define such a regime. However, the transition from two- to one dimensional motion is rather smooth and there is no critical value of the bond thickness at which a "jump" from fat to metric graph occurs. In any case, we believe that our numerical results should be considered as a first step in the way for the study of particle and wave transport described by nonlinear evolution equations on fat graphs, and can be useful for further analytical studies of NLSE on such graphs.

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Appendix: Details of numerical approach

We discretize (P_0) by second order spatial finite differences and denote \( w_j = u_j(t) = \psi_j(t, \xi_j), \xi_j = -l_j + j\delta, \) \( v_j = v_j(t) = \psi_j(t, \xi_{j,j}), w_j = v_j(t) = \psi_j(t, \xi_{j,j}), j = 1, \ldots, n-1 \), such that, e.g., \( u''_n = \frac{1}{\delta^2}(w_{n-1} - 2u_n + u_{n+1}) \).

Moreover, we set
\[ u_0 = \psi_1(-l_1) = 0, v_n = \psi_2(l_2) = 0, w_n = \psi_3(l_1) = 0, \]

and \( u_n = \psi_1(0), v_0 = \psi_2(0), \text{ and } w_0 = \psi_3(0). \)

The vertex conditions then are
\[ u_n = \alpha_2 v_0 = \alpha_3 w_0, \quad u'_n = \frac{1}{\alpha_2} v'_0 + \frac{1}{\alpha_3} w'_0. \]

Using one-sided FD for \( u'_n, v'_0 \) and \( w'_0 \) we have
\[ u'_n = \frac{1}{\delta}(u_n - u_{n-1}) = \frac{1}{\delta} \left( \frac{1}{\alpha_2} (v_1 - v_0) + \frac{1}{\alpha_3} (w_1 - w_0) \right), \]
\[ \Leftrightarrow u_n(1 + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}) = u_{n-1} + \frac{1}{\alpha_2} v_1 + \frac{1}{\alpha_3} w_1. \] (24)

which expresses \( u_n \) and hence \( v_0, w_0 \) in terms of \( u_{n-1}, v_1, w_1 \). The resulting system \( z^n := (u''_n, v'_0, w'_0)_{n=1, \ldots, n-1} \) can be best expressed by a matrix vector multiplication \( Mz \). The scheme differs from the one in [11], where the PDE is extended up to and including the vertex from the left, which works well to discretize the reflectionless solutions [4] in case of [3], but it introduces an asymmetry between the bonds not present in (P_0).

To integrate the resulting ODEs \( \partial_t z = i(Mz + \beta|z|^2z) \), where \( \beta = (\beta_u, \beta_v, \beta_w) \) with obvious meaning, we use an explicit scheme with stepsize \( h \) in \( t \), namely
\[ z^{n+1} = z^n + 2h(Mz^n + \beta|z^n|^2z^n), \] (25)

where \( \tilde{u}_i = \frac{1}{2}(u_{i-1} + u_{i+1}) \) and similar for \( \tilde{v}_i \) and \( \tilde{w}_i \). For \( h \leq \delta^2/4 \) this conserves \( N(t) \) with high accuracy, and also \( H(t) \).

To simulate (P_\varepsilon) we write it as a 2-component real system for \( z = (u, v) \) where \( \psi = u + iv \). We set up and discretize the domain \( \Omega \) using routines from pde2path [35] which are based on the FEM from the Matlab pdetoolbox. For efficiency it is quite useful to apply some local mesh refinement near the vertex. We typically work with meshes of 5000-20000 triangles. Eq. (8) then translates into the system of ODEs
\[ Mz = Kz + F(z) \] (26)

where \( M \) is the mass matrix, \( K = K_{\Delta} \) is the stiffness matrix, and \( F(z) \) is the FEM nonlinearity. For the time integration of (26) we use a semi-linear trapez rule, i.e., setting \( z_n = z(t_n), t_n = nh, \)
\[ [M + \frac{h}{2}K]z^{n+1} = [M + \frac{h}{2}K]z^n + hF(z^n). \] (27)

Over relevant time-scales (27) conserves (the discretized version of) \( N_\varepsilon \) from (12) reasonably well. We also tried the relaxation scheme from [40] which conserves \( N_\varepsilon \) slightly better, but becomes computationally much slower, mainly since one can no longer LU-pre-factorize \( M - \frac{\varepsilon}{4}K \). On the other hand, the stability requirements for explicit schemes like (26) become prohibitive for fine meshes near the vertex. For (27), typical calculation times for the propagation of a solitary wave through the network are on the order of 1 minute.