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A remark about the justification of the nonlinear Schrödinger equation in quadratic spatially periodic media

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Abstract. We prove the validity of a technical assumption necessary in a proof of the validity of the nonlinear Schrödinger equation as envelope equation in quadratic spatially periodic media.

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The dynamics of the envelopes of spatially and temporarily oscillating wave packets advancing in dispersive spatially periodic media can be approximated by solutions of a Nonlinear Schrödinger equation. In [1] the semilinear wave equation

$$\partial_t^2 u(x,t) = \chi_1(x)\partial_x^2 u(x,t) - \chi_2(x)u(x,t) - \chi_3(x)u^{\vartheta}(x,t) \tag{1}$$

has been considered as a model problem for this, where $x \in \mathbb{R}, t \in \mathbb{R}, u(x,t) \in \mathbb{R}, \vartheta = 2$ or $\vartheta = 3$, and $\chi_j(x) = \chi_j(x+2\pi)$ for $j = 1,2,3$.

Under a number of technical assumptions in [1] a proof for the validity of the nonlinear Schrödinger equation as an amplitude equation has been given. In the present paper we explain that the technical assumption (7) in [1] for the much more advanced quadratic case $\vartheta = 2$ is always satisfied if $\chi_1 \in C^2_{\mathrm{per}}$.

The linearized problem

$$\partial_t^2 u(x,t) = \chi_1(x)\partial_x^2 u(x,t) - \chi_2(x)u(x,t) \tag{2}$$

is solved by the Bloch waves

$$u(x,t) = f_n(\ell,x)\mathrm{e}^{\mathrm{i}\ell x}\mathrm{e}^{\pm\mathrm{i}\omega_n(\ell)t}$$

where $n \in \mathbb{N}, \ell \in (-1/2,1/2]$, with $\omega_n(\ell) \in \mathbb{R}$ satisfying $\omega_{n+1}(\ell) \geq \omega_n(\ell)$, and $f_n(x,\ell)$ satisfying $f_n(\ell,x) = f_n(\ell,x+2\pi)$ and $f_n(\ell,x) = f_n(\ell+1,x)\mathrm{e}^{\mathrm{i}x}$.

Slow modulations in time and space of such a Bloch mode (indexed with n_0) are described by the ansatz

$$u(x,t) = \varepsilon A(\varepsilon(x+c_{\mathrm{g}}t),\varepsilon^2 t)f_{n_0}(\ell_0,x)\mathrm{e}^{\mathrm{i}\ell_0 x}\mathrm{e}^{\mathrm{i}\omega_{n_0}(\ell_0)t} + \mathrm{cc} + \mathrm{h.o.t.}, \tag{3}$$

where cc means complex conjugate, h.o.t. means terms of order ε^2 and higher, $0 < \varepsilon \ll 1$ is a small parameter, $c_{\mathrm{g}} = \partial_\ell \omega_{n_0}(\ell_0)$ is the negative group velocity, and where A is the slowly varying envelope. Plugging the ansatz into (1) one finds that A has to satisfy a NLS equation

$$\partial_T A = \mathrm{i}\nu_1 \partial_X^2 A + \mathrm{i}\nu_2 A|A|^2 \tag{4}$$

with coefficients $\nu_1 \in \mathbb{R}$ and $\nu_2 \in \mathbb{R}$. This describes via the complex valued amplitude $A(X,T) \in \mathbb{C}$ slow modulations in time $T = \varepsilon^2 t$, and space $X = \varepsilon(x + c_{\mathrm{g}} t)$, of the underlying wave $f_{n_0}(\ell_0, x)\mathrm{e}^{\mathrm{i}\ell_0 x}\mathrm{e}^{\mathrm{i}\omega_{n_0}(\ell_0)t}$.

Validity means that, given a solution A of (4) for $T \in [0, T_0]$, for all small $\varepsilon > 0$ the difference between the formal approximation and exact solutions of (1) stays small for all t in the long time interval $\left[0, T_0/\varepsilon^2\right]$. In [1], in order to prove this, besides a number of non-resonance conditions, in case $\vartheta = 2$ we also needed a technical assumption on the quadratic interaction of the Bloch modes $f_n(\ell)$, namely: there exists an $\alpha > 1/2$ and a $C > 0$ such that for all $j, j_1, j_2 \in \mathbb{N}$ and $\ell_1, \ell_2, \ell_3 \in (-1/2, 1/2]$ we have

$$\left| \frac{1}{2\pi} \int_0^{2\pi} f_j(\ell_1, x)\chi_3(x)\overline{f_{j_1}(\ell_2, x) f_{j_2}(\ell_3, x)} \frac{1}{\chi_1(x)} \,\mathrm{d}x \right| \leq \left(\frac{C}{1 + |j - j_1 - j_2|} \right)^{\alpha}. \tag{5}$$

In [1] assumption (5) has been verified with $\alpha = 2 - \delta$ for a $\delta > 0$ arbitrary in case that χ_1 is independent of x. In the present paper we prove (5) in case $\chi_1 \in C^2_{\mathrm{per}}$ not being a constant by applying a change of coordinates making χ_1 a constant.

The idea is based on [2] where the spectral problem has been discussed. Introducing

$$\tilde{u}(y,t) = \chi_1^{-1/4}(x) u(x,t) \quad \text{where} \quad y = \int_0^x \chi_1^{-1/2}(\xi)\,\mathrm{d}\xi, \tag{6}$$

equation (1) with $\vartheta = 2$ transforms into

$$\partial_t^2 \tilde{u}(y,t) = \partial_y^2 \tilde{u}(y,t) - \tilde{\chi}_2(y)\tilde{u}(y,t) - \tilde{\chi}_3(y)\tilde{u}^2(y,t), \tag{7}$$

where

$$\tilde{\chi}_2(y) = \chi_2(x) - \chi_1^{3/4}(x)(\chi_1^{1/4}(x))'', \qquad \tilde{\chi}_3(y) = \chi_3(x)\chi_1^{1/4}(x),$$

with $\tilde{\chi}_j(y) = \tilde{\chi}_j(y + \tilde{L})$, $\tilde{L} = \int_0^{2\pi} \chi_1^{-1/2}(\xi)\,\mathrm{d}\xi$, and consequently $\ell \in (-1/(2\tilde{L}), 1/(2\tilde{L}))$ in the Bloch representation.

Thus (1) can be transformed via (6) into (7) with constant coefficient in front of the second spatial derivative and (5) is also satisfied in case χ_1 not being a constant. Since $\tilde{\chi}_3 \in C_b^2$ is needed in [1, Lemma A.1] we require at least $\chi_1 \in C^2_{\mathrm{per}}$. Moreover, in the variation of constant formula used to obtain local existence and uniqueness [1, Sections 4.2 and 5.2], given $u(\cdot,t) \in H^s$ we need

$$\left(\partial_x^2 + \chi_2 - \chi_1^{3/4}(\cdot)(\chi_1^{1/4}(\cdot))''\right)\left(\chi_3(\cdot)\chi_1^{1/4}(\cdot)u^2(\cdot,t)\right) \overset{!}{\in} H^{s-2}.$$

Here and in the following, we use the abbreviation H^s for $H^s(\mathbb{R},\mathbb{R})$ or $H^s(\mathbb{R},\mathbb{C})$.

Therefore, if we want an approximation result in high order Sobolev spaces, then we need additional regularity of χ_1 and χ_3. In detail, for $s \in \mathbb{R}$ we define

$\lceil s \rceil$ as the smallest integer greater or equal to s. The improved result then is as follows:

Theorem 1. *Let $s \in (1/2, 5/2), s_A \geq 4$, and assume that $\chi_2 \in C_{\mathrm{per}}^{\max\{0,\lceil s-2 \rceil\}}$ and $\chi_{1,3} \in C_{\mathrm{per}}^{\max\{2,\lceil s \rceil\}}$ in (1) are chosen in such a way that the nonresonance conditions*

$$\begin{aligned} \inf_{n \in \mathbb{Z}\setminus\{0\}, |j| \leq 4, (n,j) \notin \{-(n_0,1),(n_0,1)\}} |\omega_n(j\ell_0) - j\omega_{n_0}(\ell_0)| &> 0, \\ \inf_{r,n \in \mathbb{Z}\setminus\{0\}, \ell, m \in \left(-\frac{1}{2\pi}, \frac{1}{2\pi}\right], |\ell - m - \ell_0| < \delta} |-\omega_r(\ell) - \omega_{n_0}(\ell - m) + \omega_n(m)| &> 0, \end{aligned}$$

hold. Then for all C_1 and $T_0 > 0$ there exist $\varepsilon_0 > 0$ and $C_2 > 0$ such that for all solutions $A \in C([0,T_0], H^{s_A})$ of (1) with

$$\sup_{T \in [0,T_0]} \|A(\cdot,T)\|_{H^{s_A}} \leq C_1$$

the following holds. For all $\varepsilon \in (0, \varepsilon_0)$ there are solutions $u \in C(\left[0, T_0/\varepsilon^2\right], H^s)$ of (1) with

$$\sup_{t \in [0,T_0/\varepsilon^2]} \left\| u(\cdot,t) - \left(\varepsilon A\left(\varepsilon(\cdot + c_{\mathrm{g}}t), \varepsilon^2 t\right) f_{n_0}(\cdot,\ell_0) \mathrm{e}^{\mathrm{i}\ell_0 \cdot} \mathrm{e}^{\mathrm{i}\tilde{\omega}_{n_0}(\ell_0)t} + \mathrm{cc} \right) \right\|_{H^s} \leq C_2 \varepsilon^{3/2}.$$

Proof. Theorem 1 has been established in [1] under the additional condition (5). Hence, it remains to prove the validity of (5).

Denote the inverse of $y = \int_0^x \chi_1^{-1/2}(\xi)\,\mathrm{d}\xi$ by $x = h(y)$. The solutions

$$u(x,t) = f_n(\ell,x)\mathrm{e}^{\mathrm{i}\ell x}\mathrm{e}^{\pm \mathrm{i}\omega_n(\ell)t}$$

of (2) transform under (6) into

$$\tilde{u}(y,t) = \chi_1^{-1/4}(h(y)) f_n(\ell, h(y)) \mathrm{e}^{\mathrm{i}\ell h(y)} \mathrm{e}^{\pm \mathrm{i}\omega_n(\ell)t}.$$

Since $\mathrm{e}^{\mathrm{i}\ell h(y)}$ can be split into $\mathrm{e}^{\mathrm{i}\ell 2\pi y/\tilde{L}} \mathrm{e}^{\mathrm{i}\ell(h(y) - 2\pi y/\tilde{L})}$, where the second factor is $\tilde{L}$-periodic w.r.t. y due to $h(\tilde{L}) = 2\pi$ this can be written as

$$\tilde{u}(y,t) = \tilde{f}_n(\tilde{\ell}, y) \mathrm{e}^{\mathrm{i}\tilde{\ell} y} \mathrm{e}^{\pm \mathrm{i}\omega_n(\tilde{\ell})t}$$

where

$$\tilde{f}_n(\tilde{\ell}, y) = \chi_1^{-1/4}(h(y)) f_n(\ell, h(y)) \mathrm{e}^{\mathrm{i}\ell(h(y) - 2\pi y/\tilde{L})}$$

and $\tilde{\ell} = 2\pi\ell/\tilde{L}$.

Since $\mathrm{d}y = \chi_1^{-1/2}(x)\,\mathrm{d}x$ the condition (5) transforms into

$$\left| \int_0^{\tilde{L}} \tilde{f}_j(\ell_1, y) \check{\chi}_3(y) \overline{\tilde{f}_{j_1}(\ell_2, y) \tilde{f}_{j_2}(\ell_3, y)}\, \mathrm{d}y \right| \leq C \left(\frac{C}{1 + |j - j_1 - j_2|} \right)^{\alpha} \tag{8}$$

where $\check{\chi}_3(y) = \tilde{\chi}_3(y) \mathrm{e}^{\mathrm{i}(\ell_1 - \ell_2 - \ell_3)(h(y) + 2\pi y/\tilde{L})}$. Since the $\tilde{f}_j(\tilde{\ell}, y)\mathrm{e}^{\mathrm{i}\tilde{\ell} y}$ are solutions of

$$\partial_y^2 u - \tilde{\chi}_2 \tilde{u} = -\omega^2 \tilde{u},$$

since $\tilde{\ell} = 2\pi\ell/\tilde{L}$ is linearly related with ℓ, and since $\check{\chi}_3$ satisfies the assumptions of [1, Lemma A.1], the validity of (8) has already been established in [1, Lemma A.1]. Therefore, we are done. $\square$

In most photonic crystals the χ_j only take two values, i.e., the χ_j are only in L^∞ and the validity of (5) is still an open question.

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