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The "Horse Race" Random Utility Model for Choice Probabilities and Reaction Times, and Its Competing Risks Interpretation

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Random utility models have traditionally been applied to probabilistic choice data, with little attention to reaction times. We describe the class of "*horse race*" *random utility models* that can be applied to both choice probabilities and reaction times. We show that any (well behaved) set of choice probabilities and reaction times on a fixed set can be represented by an independent "horse race" random utility model, and relate this result to work in the theory of competing risks. We use the latter theory to motivate the condition that the option chosen and the time of choice be independent, a condition that is satisfied by a large class of (extreme value) "horse race" random utility models. Combining the latter condition with the assumption of an independent "horse race" random utility model yields a new characterization of Luce's choice model, and a generalization of these conditions to subset choices (as opposed to choosing a single "best" element) yields the transition probabilities of Tversky's elimination-by-aspects model.

1. INTRODUCTION

Random utility models hold an important position in theoretical, experimental, and applied work in the areas of preference, ranking and choice (Colonius, 1984, provides a recent extensive summary of much of the theoretical work, and Ben-Akiva and Lerman, 1985, and Train, 1986, give detailed discussions of statistical properties and applications). These models are normally applied to situations where a person has to select a single "best" option from an available set, but (see later)

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they can also be generalized to include cases where a person may select a subset (possibly including two or more options) for further consideration or final choice. A challenging theoretical question in this domain has been: under what conditions does a set of choice probabilities satisfy a random utility model? Falmagne (1978) gave the solution for probabilities of choice (of single "best" elements) defined on *all* subsets of a finite set, and Cohen (1980) extended this result to infinite sets; Marley (1990) summarizes work on the case where only *binary* choice probabilities are available—no satisfactory solution has yet been developed in that case. Also, Barbera and Pattanaik (1986) use techniques paralleling those of Falmagne (1978) to characterize random utility models where a person can select a subset (possibly including two or more options) for further consideration or final choice. Such (*transition*) *random utility models* (Corbin & Marley, 1974) are related to, but not identical to, the *random* (*subset*) *advantage models* that we discuss later in this paper.

An interesting extension of random utility models and of the associated theoretical questions arises when we also consider *reaction times*. In the preference area, there are few theoretical or experimental studies of the extended problem, exceptions being Busemeyer, Forsyth, and Nozawa (1988) and Marley (1989) for theoretical developments, and Petrusic and Jamieson (1978) for experimental work. The present paper summarizes some elementary theoretical results regarding the joint representation of choice probabilities and reaction times by (appropriately defined) random utility models and poses the general representational problem. In particular, we show that (under quite general conditions), *any* set of choice probabilities and reaction times on a *fixed* choice set can be represented by an (appropriately defined) independent random utility model. However, it is not possible to reduce a general (dependent) random utility model over *all* subsets of some master set to an independent random utility model, and the problem of characterizing such dependent random utility models (for choice probabilities *and* reaction times) is still open. We relate our results to work in *theory of competing risks*, and use techniques from that theory to study models where the time of choice and the option chosen are independent.

This independence condition is satisfied by Marley's (1989) class of random utility models, which includes all models based on multivariate (dependent) Weibull distributions—in particular, Luce's (1959) choice model, the transition probabilities in Tversky's (1972a, b) elimination-by-aspects (EBA) modsel, and McFadden's (1978) generalized extreme value class (GEV) all have such a representation. An open question is whether there are (random utility) models not in Marley's class that satisfy independence of the time of choice and option chosen.

We also characterize Luce's (1959) choice model as the only model (for choice of the single "best" option from the available set) that satisfies both sets of independence conditions mentioned above. Finally we generalize these models and conditions to allow choices of subsets, rather than single elements, and show that the transition probabilities of Tversky's (1972a, b) elimination-by-aspects (EBA) model are uniquely characterized by those generalized independence conditions.

For simplicity, the major notation and results of the paper are stated in terms of selection of a single "best" element. Section 5 then extends the results to subset choice.

2. "Horse Race" Random Utility Models

Let $S = \{x_1, ..., x_n\}$ be a finite set of potential choice options and let the subset X of S, with $|X| \geqslant 2$, be the currently available choice set.[1] $\mathbf{T}(X)$ is a random variable denoting the time at which a choice is made, $\mathbf{C}(X)$ is a random variable denoting the (single "best") element chosen, and $P_X(x; t)$, $t \geqslant 0$, is the probability that option x is chosen from X *after* time t. Thus

$$P_X(x; t) = \Pr[\mathbf{C}(X) = x \cap \mathbf{T}(X) > t] \tag{1}$$

may be thought of as a *survival function* (*for option* x). A collection of survival functions $\{P_X(x; t) : x \in X \subseteq S\}$ for a fixed $X \subseteq S$ will be called a *joint structure of choice probabilities and reaction times.* If the collection of survival functions ranges over all subsets of S, the joint structure is called *complete*, and denoted by (S, P). If nothing else is specified, we will tacitly assume below that the joint structures considered are complete.

We now investigate a simple model generating such a joint structure, namely the "horse race" random utility model (cf. Marley, 1989). Assume that associated with each option $x \in X$ there is a nonnegative random variable $\mathbf{t}(x)$ which denotes the time of occurrence of some "event" associated with that option; the element is chosen whose "event" occurs first. In terms of the above notation, for each $x \in X$,

$$P_X(x; t) = \Pr[t < \mathbf{t}(x) < \mathbf{t}(y) : \text{for all } y \in X - \{x\}]. \tag{2}$$

We assume that all distributions are absolutely continuous (Feller, 1966, pp. 136–140), i.e., they have densities and no singular components. Our main result (Theorem 1) need not hold if simultaneous occurrences of "events" can happen with positive probability (cf., e.g., Miller, 1977); absolute continuity ensures that a single "best" element is always defined by the above "events." (In fact, by associating "events" with subsets rather than with single elements, we can even study models of subset choice under this condition of absolute continuity—see Section 5).

Definition 1. A joint structure of choice probabilities and reaction times (S, P) satisfies a *"horse race" random utility model* if there is a nonnegative random vector $\mathbf{t} = (\mathbf{t}(x_1), ..., \mathbf{t}(x_n))$ on some probability space such that Eq. (2) holds for all subsets

[1] Note that the cardinality of the subset X is assumed to be $\geqslant 2$. For single-element subsets, it is not obvious why the time for a choice should vary at all. If it is taken to reflect the time to choose between that option and no choice at all, the *no-choice option* (Corbin and Marley, 1974) could be included in S which again restricts the subsets X to cardinality $\geqslant 2$.

X, with $|X| \geqslant 2$, of S. The random vector $\mathbf{t}$ is called a *random representation of the joint structure*. If the components of $\mathbf{t}$ are jointly independent, then the random representation (and the corresponding "horse race" random utility model) is called *independent*.

The term "random utility" in this definition derives from the following observation (see Marley, 1989). Setting

$$\begin{aligned} P_X(x) &= \lim_{t \to 0+} P_X(x; t) \\ &= \Pr[\mathbf{t}(x) < \mathbf{t}(y) : \text{for all } y \in X - \{x\}] \\ &= \Pr[\mathbf{t}(x) = \min\{\mathbf{t}(y) : y \in X\}], \end{aligned}$$

then $P_X(x)$ denotes the probability of choosing option x from X. Obviously, the collection of choice probabilities $\{P_X(x) : x \in X \subseteq S\}$ constitutes a *structure of choice probabilities* (Corbin and Marley, 1974), and the representing vector $\mathbf{t}$ also provides a (*choice*) *random utility representation* (Corbin and Marley, 1974) for these choice probabilities. Note that the usual representation is in terms of random variables $\mathbf{U}$ such that

$$P_X(x) = \Pr[\mathbf{U}(x) = \max\{\mathbf{U}(y) : y \in X\}].$$

Clearly, these two representations (in terms of a min or a max) are equivalent in that one can be derived from the other by applying an appropriately chosen strictly monotonic (decreasing) transformation on the relevant random variables, leaving the choice probabilities invariant.

The representation problem for a joint structure of choice probabilities and reaction times is then:

> Given a joint structure of choice probabilities and reaction times $\{P_X(x, t);\ x \in X \subseteq S\}$, what conditions on the survival functions $P_X(x; t)$ are necessary and sufficient for the existence of a "horse race" random utility representation of that joint structure? (RP)

One aim of this paper is to point out a close connection between a (partial) solution to the above problem and a basic result in the theory of competing risks (see, e.g., David and Moeschberger, 1978). The latter theory refers to situations in which an object is exposed to two or more causes of failures, or risks, simultaneously. There are postulated to be a fixed number of different mutually exclusive and collectively exhaustive causes which compete to bring about the eventual failure of the object under study. Obvious examples of such a situation arise in actuarial science and industrial reliability applications. A basic condition underlying most competing risks situations is that only the time of failure and the identity of its cause are observable (the *latent-failure time model*, cf. Gail, 1982) as opposed to the full failure distribution for all the causes.

If $Q_S(x; t)$ denotes the probability that a failure due to cause x is observed after time t when S is the set of possible causes (or risks), then in the theory of competing risks $Q_S(x; t)$ is commonly called the *crude survival function for cause x*. Returning to the "horse race" random utility model, $Q_S(x; t)$ obviously corresponds to the observable survival function $P_S(x; t)$ for choices from S. Moreover, it should be clear that going over to a subset X of feasible choices corresponds to a competing risks situation where all risks in S not contained in X are a priori eliminated.

We now present a simple proof, based on Berman (1963), that, under some regularity conditions, *any* set of choice probabilities and reaction times $\{P_X(x; t): x \in X\}$ on a *fixed* (finite) set X can be represented by *independent* random variables in such a way that Eq. (2) holds (Townsend, 1976, presents a related result). It then follows immediately from that result what the conditions are for a (complete) joint structure of choice probabilities and reaction times $\{P_X(x; t): x \in X \subseteq S\}$ to be representable by an independent horse race random utility model. We then restate these results in terms of *hazard functions*, relate them to work on *competing risks theory*, and study the special case where the option chosen and the time of choice are independent. Berman's (1963) proof is short, so we include a variant of it for completeness. Berman wrote his proof in terms of distribution functions corresponding to $1 - P_X(x; t)$, whereas we write ours in terms of the survival functions $P_X(x; t)$. Also, the distributions in the solution can be either *proper* (with zero measure at infinity) or *improper* (with positive measure at infinity)—we briefly discuss this fact after presenting the theorem and proof. (The interpretation of the solution given in Part(b) of Theorem 1 is given in Section 3).

For simplicity, we assume $P_X(x; t) > 0$ for all $t \geqslant 0$. The result is easily generalized to various weaker conditions (see Marley, 1992).

THEOREM 1. (a). *Consider a joint structure of choice probabilities and reaction times $\{P_X(x; t): x \in X\}$ on a fixed finite set X. If each $P_X(x; t)$, $x \in X$, is absolutely continuous and positive for all $t \geqslant 0$, then there exist unique independent random variables $\mathbf{t}^X(x)$, $x \in X$, such that Eq.* (2) *holds.*

(b) *A* (*complete*) *joint structure of choice probabilities and reaction times, i.e. $\{P_X(x; t),\ x \in X \subseteq S\}$, can be uniquely represented by an independent horse race random utility model if the conditions of* (a) *hold and*

$$[(d/dt)\, P_X(x; t)] \Big/ \Big(\sum_{y \in X} P_X(y, t) \Big)$$

is independent of $X \subseteq S$ for all $t \geqslant 0$.

Proof (also see Berman, 1963). Part (a). We wish to find, for the given $X \subseteq S$, (nonnegative) random variables $\mathbf{t}^X(x)$, $x \in X$, such that for $x \in X$ and for $t \geqslant 0$,

$$P_X(x; t) = \Pr[t < \mathbf{t}^X(x) < \mathbf{t}^X(z): \text{for all } z \in X - \{x\}]. \tag{3}$$

We now *assume* that $\mathbf{t}^X$ exists satisfying Eq. (3) and show how that assumption allows us to actually construct the solution.

If $\mathbf{t}^X$ exists, i.e., Eq. (3) holds, then with $\mathbf{T}(x) = \min_{y \in X} \mathbf{t}^X(y)$ we have

$$\begin{aligned}\sum_{y \in X} P_X(y; t) &= \sum_{y \in X} \Pr[t < \mathbf{t}^X(y) < \mathbf{t}^X(z) : \text{for all } z \in X - \{y\}] \\ &= \Pr[t < \min_{y \in X} \mathbf{t}^X(y)] \\ &= \Pr[t < \mathbf{T}(X)],\end{aligned}$$

and also letting $G_x^X(t) = \Pr[\mathbf{t}^X(x) > t]$, we have (using independence of the $\mathbf{t}^X(y)$, $y \in X$)

$$\prod_{y \in X} G_y^X(t) = \Pr[\mathbf{T}(X) > t].$$

Combining these two results, we have

$$\Pr[\mathbf{T}(X) > t] = \sum_{y \in X} P_X(y; t) = \prod_{y \in X} G_y^X(t), \tag{4}$$

each term being the probability that $\mathbf{T}(X)$ exceeds t. Inserting the right hand side of Eq. (4) in the right hand side of Eq. (3), where the right hand side of Eq. (3) equals

$$\int_t^\infty \prod_{y \in X - \{x\}} G_y^X(s)\, d[1 - G_x^X(s)],$$

we obtain

$$\begin{aligned}P_X(x; t) &= -\int_t^\infty \prod_{y \in X} G_y^X(s) \cdot [G_x^X(s)]^{-1}\, dG_x^X(s) \\ &= -\int_t^\infty \left[\sum_{y \in X} P_X(y; s)\right] d\{\log G_x^X(s)\}.\end{aligned}$$

Conversion to a differential equation yields

$$dP_X(x; s) = \left[\sum_{y \in X} P_X(y; s)\right] d\{\log G_x^X(s)\}, \tag{5}$$

with (unique, proper or improper) solution

$$G_x^X(t) = \exp\left\{\int_0^t \left[\sum_{y \in X} P_X(y; s)\right]^{-1} dP_X(x; s)\right\}. \tag{6}$$

(Remember, we are assuming that $P_X(x; t)$, $x \in X$, are positive and absolutely continuous for all $t \geqslant 0$). For suppose $\{H_x^X, x \in X\}$ is a second solution. Then it also satisfies Eq. (5) for all s, and therefore

$$d\{\log G_x^X(s)\} = d\{\log H_x^X(s)\},$$

i.e.,

$$\log G_x^X(s) = \log H_x^X(s) + C,$$

so

$$G_x^X(s) = AH_x^X(s).$$

But we require that for $s=0$, $G_x^X(s) = H_x^X(s) = 1$, so $A = 1$.

Thus the random vector $\mathbf{t}^X$ composed of the (proper or improper) random variables $\mathbf{t}^X(x)$, $x \in X$, with survival functions $G_x^X(t)$, $t \geqslant 0$, is the desired "horse race" random utility representation of $\{P_X(x; t), x \in X\}$.

Part (b). Clearly, to have a common random utility representation $\mathbf{t}$ of $\{P_X(x; t), x \in X \subseteq S\}$, we require that $\mathbf{t}^X(x)$ be independent of X for all $X \subseteq S$, i.e., that the right hand expression of Eq. (6) be independent of X, so in particular we require that

$$[(d/dt)\, P_X(x; t)] \Big/ \left(\sum_{y \in X} P_X(y; t) \right)$$

be independent of X for all $t \geqslant 0$. Q.E.D.

Some, but not all, of the distributions (and associated random variables) in the above solution may be *improper*—i.e., have positive measure at infinity. This occurs if the integral in Eq. (6) does not diverge (to minus infinity) as t goes to infinity. However, provided the reaction time distribution $\Pr[\mathbf{T}(x) > t]$ is *proper*, i.e., with zero measure at infinity (which it customarily would be), then at least one of the G_x^X, $x \in X$, must be proper because by Eq. (4) their product is then proper. Miller (1977) presents a quite elementary example with $\mathbf{T}$ proper, but with one of the distributions in the solution improper, and he also presents an alternative condition (to absolute continuity) under which solutions (proper or improper) exist.

In summary, given a set of reaction time distributions $\{P_X(x; t) : x \in X \subseteq S\}$, we can use the expression given in the statement of the theorem and its dependence or otherwise on $X \subseteq S$ to check whether these reaction time distributions are compatible with an independent "horse race" random utility model, and if they are, to obtain the (unique) solution.

We now relate this result to other work from the theory of competing risks. In particular, we show how the condition in Theorem 1(b) can be interpreted as saying that the *cause specific* (*or crude*) *hazard rate* associated with outcome (read: cause) x, $x \in X$, is independent of X.

3. Hazard Rate Reformulation

For an arbitrary subset X of S, define a *hazard function* (e.g., Luce, 1986, p. 13) with respect to the crude survival function for option x in context X as follows:

$$h_x^X(t) = \lim_{\delta \to 0+} (1/\delta) \Pr[t < \mathbf{T}(X) \leqslant t + \delta \cap \mathbf{C}(X) = x \mid \mathbf{T}(X) > t]. \tag{7}$$

Thus, $h_x^X(t)$ can be interpreted as the instantaneous rate of choosing option x from X at time t given that no choice was made before t. In terms of competing risks, $h_x^X(t)$ is the instantaneous rate of failure from cause x when all causes in X act simultaneously, commonly called the *cause-specific* (or *crude*) *hazard rate*. For $h_x^X(t)$ to exist, we have to assume absolute continuity of the crude survival function $P_X(x; t)$. Going over to the unconditioned probability in Eq. (7) and remembering the elementary fact that

$$\Pr[\mathbf{T}(X) > t] = \sum_{y \in X} P_X(y; t) \tag{8}$$

(see proof of Theorem 1a) gives

$$h_x^X(t) = [(-d/dt)\, P_X(x; t)] \bigg/ \bigg(\sum_{y \in X} P_X(y; t) \bigg), \tag{9}$$

provided $\Pr[\mathbf{T}(X) > t] \neq 0$.

Note that the right hand side expression in Eq. (9) agrees (except in sign) with that given in the statement of Theorem 1(b). Thus, the arguments below simply show how to construct the solution given in Theorem 1(a) from the hazard function perspective given by $h_x^X(t)$, $t > 0$. For a *fixed* set X, Eq. (9) leads directly to the desired random variable representation, and for a common representation on *all* subsets X of a given set S, h_x^X must not depend on X (as Theorem 1b).

$\Pr[\mathbf{T}(X) > t]$ is commonly referred to as the *overall survival distribution* (*for set* X) with a corresponding *overall hazard rate* $h^X(t)$; i.e., by definition

$$h^X(t) = \frac{-d/dt\, \Pr[\mathbf{T}(X) > t]}{\Pr[\mathbf{T}(X) > t]}, \tag{10}$$

and so from Eqs. (8), (9), (10)

$$h^X(t) = \sum_{x \in X} h_x^X(t).$$

The next step towards a random utility representation is to define a set of independent random variables $\mathbf{t}^X(x)$, $x \in X$, with survival distribution functions (SDFs) given by

$$\begin{aligned} \Pr[\mathbf{t}^X(x) > t] &= \exp\left[-\int_0^t h_x^X(s)\, ds \right] \qquad (11) \\ &= G_x^X(t). \end{aligned}$$

(This is the standard relationship between a hazard function $h_x^X(t)$ and its survival distribution $G_x^X(t)$—see, for instance, Luce, 1986, p. 15). But this representation of $\{G_x^X, x \in X\}$ is exactly the form given in the proof of Theorem 1 using Berman's

(1963) approach. Note that in order for $G_x^X(t)$, $t \geqslant 0$, to be a proper SDF, we must have that

$$\int_0^t h_x^X(s)\,ds$$

diverges to infinity as t tends to infinity.

Thus, under rather mild regularity conditions, the independent random variables $\mathbf{t}^X(x)$, $x \in X$, constructed above can be used to generate the survival functions $P_X(x; t)$. In competing risks theory, this is a well-known result, often referred to as the "nonidentifiability problem" (cf. Cox, 1959; Berman, 1963; Tsiatis, 1975; Elandt-Johnson & Johnson, 1980, p. 277 ff). It means that any set of competing risk data (in the absolutely continuous case) can be explained by some independent risk model. This is conceived as a problem since the independence model is often unrealistic (e.g., when systemic diseases are studied). The only way to establish identifiability of a dependent model in the standard competing risks situation is to assume that the distribution of the vector $\mathbf{t}$ belongs to some (flexible) parametric family. It is then often possible to estimate the parameters of the multivariate distribution and to test for independence (see, e.g., Basu & Gosh, 1978; Arnold & Brockett, 1983).

Remember that for a joint structure of choice probabilities and reaction times the random variables $\mathbf{t}^X(x)$ defined according to Eq. (11) do not completely solve the representation problem (RP) stated above. As already discussed, their SDF's, G_x^X, defined via h_x^X, in general depend on the specific subset X of feasible choice options, whereas (see Definition 1) we want the *same* random vector $(\mathbf{t}(x_1), ..., \mathbf{t}(x_n))$ to generate the survival functions $P_X(x; t)$ for arbitrary $x \in X \subseteq S$. However, as also mentioned in our discussion of Berman's result, it is trivial to state the additional conditions that are needed for the same independent random variables to represent $\{P_X(x, t) : x \in X\}$ for all $X \subseteq S$. Nonetheless, we see in the next section that it is *not* in general possible to represent a dependent "horse race" random utility model by some equivalent independent "horse race" random utility model—at least, not if the independent random variables are associated directly with the elements of S.

4. The Proportional Hazard Rate Condition

We now discuss a condition that has frequently been made in the competing causes literature (see, e.g., David and Moeschburger, 1978) and show that (somewhat surprisingly) it is equivalent to a condition that has arisen as a a result of the specification of a large family of random utility models. We then show that, within this family, the only model that can be represented by an independent "horse race" random utility model is Luce's choice model. Since the class without independence is significantly larger, including all of McFadden's (1978) generalized extreme value

(GEV) models and the transition probabilities of Tversky's (1972a, b) elimination-by-aspects (EBA) model, as well as the choice model, this result shows that for our purposes independence is usually too strong an assumption.

DEFINITION 2. A joint structure of choice probabilities and reaction times (S, P) satisfies the *proportional hazard rate* (*PHR*) *condition* iff for any subset X of S there are constants $C_X(x)$, $x \in X$, such that

$$h_x^X(t) = C_X(x)\, h^X(t), \tag{12}$$

where h_X is the overall hazard rate for $\mathbf{T}(X)$, and h_x^X are the cause specific hazard rates.

Although $C_X(x)$, $x \in X \subseteq S$, is not constrained in this definition, we now show that in fact we must have $C_X(x) = P_X(x)$ for all $x \in X \subseteq S$.

Combining Eqs. (8), (9) gives

$$h_x^X(t) = [(-d/dt)\, P_X(x; t)]/\Pr[\mathbf{T}(X) > t],$$

which with Eqs. (10), (12) implies that

$$\begin{aligned} &[(-d/dt)\, P_X(x; t)]/\Pr[\mathbf{T}(X) > t] \\ &\quad = h_x^X(t) \\ &\quad = C_X(x)\, h^X(t) \\ &\quad = C_X(x)[(-d/dt)\Pr[\mathbf{T}(X) > t]]/\Pr[\mathbf{T}(X) > t]. \end{aligned}$$

Cancelling the denominators and integrating with respect to t then easily gives

$$P_X(x; t) = C_X(x)\Pr[\mathbf{T}(X) > t].$$

In particular, because $P_X(x) = \lim_{t \to 0} P_X(x; t)$, this equation implies that $C_X(x) = \Pr[\mathbf{C}(X) = x]$, i.e., we have

$$P_X(x; t) = \Pr[\mathbf{C}(X) = x]\Pr[\mathbf{T}(X) > t], \tag{13}$$

and therefore the stochastic independence of the random variables $\mathbf{T}(X)$ and $\mathbf{C}(X)$ is established as a consequence of PHR with $C_X(x) = \Pr[\mathbf{C}(X) = x]$. Moreover, the converse is shown by reversing the above steps. Thus, the proportional hazard rate (PHR) condition (with $C_X(x) = \Pr[\mathbf{C}(X) = x]$) is equivalent to assuming stochastic independence between $\mathbf{T}(X)$ and $\mathbf{C}(X)$. Theorem 2 of Kochar and Proschan (1991) proves an equivalent result. Intuitively, the proportionality of the hazard rates means that the time of choice gives no information as to the identity of the choice option, and vice-versa.

Note that a special relationship holds between the SDFs G_x^X, $x \in X$, and the overall SDF $\Pr[\mathbf{T}(X)>t]$ when (PHR) holds with $C_X(x)=\Pr[\mathbf{C}(X)=x]=P_X(x)$ —namely, by Eq. (11),

$$\begin{aligned} G_x^X(t) &= \exp\left[-\int_0^t h_x^X(s)\,ds\right] \\ &= \exp\left[-\int_0^t C_X(x)\,h_X(s)\,ds\right] \\ &= \left[\exp-\int_0^t h_X(s)\,ds\right]^{C_X(x)} \\ &= \Pr[\mathbf{T}(X)>t]^{P_X(x)}. \end{aligned} \tag{14}$$

(The final step follows from the standard relationship between a hazard function and its survival distribution. See, for instance, Luce, 1986, p. 15.) Clearly (by rearranging the above steps), this power function relationship actually is equivalent to (PHR) with $C_X(x)=P_X(x)$, or, equivalently, to the stochastic independence of $\mathbf{T}(X)$ and $\mathbf{C}(X)$.

Equation (13), which states that the option chosen, x, is independent of the time of choice, t, is satisfied by (at least) the class of "horse race" random utility models generated by *generalized stable survival functions* (Marley, 1989)—a class that includes Luce's choice model, McFadden's generalized extreme value model, and (when absolute continuity is not required) the transition probabilities of Tversky's elimination-by-aspects model. This independence condition was first studied in detail in the psychological literature by Robertson and Strauss (1981), and has also been studied by Lee (1979), and Kochar and Proschan (1991). It is an open question whether this independence condition characterizes the above class of "horse race" random utility models.

We now use the two independence conditions mentioned in this paper to give an interesting characterization of Luce's (1959) choice model. Remember, Luce's choice model holds for a system of choice probabilities $\{P_X(x): x\in X\subseteq S\}$ provided there is a ratio scale v on S such that provided $P_X(x)\neq 0$, 1,

$$P_X(x)=\frac{v(x)}{\sum_{y\in X} v(y)}.$$

For simplicity in the following, we assume that all the choice probabilities are nonzero. The result can be generalized when this is not the case by adding a connectivity and a transitivity condition (Luce, 1959, Theorem 4, p. 25).

THEOREM 2. *Consider an independent horse race random utility model (S, P) where for each $x\in X\subseteq S$, $P_X(x;t)$ is absolutely continuous and positive for all $t\geqslant 0$ and $P_X(x)$ is nonzero. If the option chosen, $\mathbf{C}(X)$, is independent of the time of choice, $\mathbf{T}(X)$, then the choice probabilities satisfy Luce's choice model.*

Proof. Under the stated conditions, G_x^X, $x \in X$, of Eq. (14), must be independent of X, and so in particular for x, $y \in X \cap Y$, with X, $Y \subseteq S$

$$P_X(x) \log \Pr[\mathbf{T}(X) > t] = P_Y(x) \log \Pr[\mathbf{T}(Y) > t]$$

and

$$P_X(y) \log \Pr[\mathbf{T}(X) > t] = P_Y(y) \log \Pr[\mathbf{T}(Y) > t].$$

Dividing gives

$$\frac{P_X(x)}{P_X(y)} = \frac{P_Y(x)}{P_Y(y)}$$

which implies (Luce, 1959) that there is a ratio scale v on S such that for $x \in X \subseteq S$,

$$P_X(x) = \frac{v(x)}{\sum_{z \in X} v(z)},$$

i.e. the choice model representation. Q.E.D.

Continuing with the development in Theorem 2, in order to obtain the distributions $\mathbf{T}(X)$, $X \subseteq S$, define Ψ on the nonnegative reals by: for $t \geqslant 0$,

$$\Psi(t) = \frac{\log \Pr[\mathbf{T}(S) > t]}{\sum_{z \in S} v(z)}.$$

Substituting this representation and those of the choice probabilities in the first equation of the proof, and taking $Y = S$, we obtain

$$\log \Pr[\mathbf{T}(X) > t] = \sum_{z \in X} v(z) \cdot \Psi(t),$$

i.e.,

$$\Pr[\mathbf{T}(X) > t] = \exp \sum_{z \in X} v(z) \cdot \Psi(t),$$

where $\Psi(0) = 0$, $\Psi(\infty) = -\infty$.

Remember that $\Psi(t)$ is defined in terms of the survival function $\Pr[\mathbf{T}(S) > t]$, and is thus, as developed, not an arbitrary function. However, as we discuss later, if we only constrain the development to satisfy Luce's choice model (for the choice probabilities), then we do have considerable freedom in the choice of the survival function, and hence of Ψ.

Related (but not identical) characterizations of the choice model are given by Robertson and Strauss (1981, Theorem 2), who refer to Strauss (1979, Theorem 4) for proof, and by Bundesen (1991) and Vorberg (undated manuscript).

5. Choices on Subsets

The above development assumes that a *single* element must be chosen from the set of available options. However, one needs to also consider results for the case where a subset with more than one element may be selected. Such subset selection might only be an intermediate stage in the determination of a single "best" element, as in *(Markov) transition models* (Tversky, 1972b; Corbin and Marley, 1974); or the selected subset could be the final choice set. In the former case, the overall reaction time distribution is determined recursively via a generalization of Luce's (1960) techniques (Marley, 1975). Such generalizations are complex, and require further developments in the light of more recent results. Therefore, here we only discuss reaction time models for one stage (not the full recursion) of subset choice.

Extending and reinterpreting the notation of Section 2, let $Q_X(Y; t)$, $t \geqslant 0$, be the probability that the nonempty[2] subset $Y \subseteq X$ is "selected" (for further consideration) after time t when X is the available choice set, and assume that

$$Q_X(Y; t) = \Pr[\mathbf{C}(X) = Y \cap \mathbf{T}(X) > t],$$

where (as in Section 2) $\mathbf{T}(X)$ is a random variable denoting the time at which a choice (of a subset) is made, and $\mathbf{C}(X)$ is now a random variable denoting the *subset* chosen. Such a collection $\{Q_X(Y; t) : \phi \subset Y \subseteq X \subseteq S\}$, or (S, Q) for short, will be called a *joint structure of transition probabilities and reaction times*, and as previously we usually focus on *complete* structures, i.e., those defined over all subsets of S. The following definition generalizes the idea of a "horse race" random utility model to such subset selection (Marley, 1991b, gives a parallel definition when a single "best" element must be selected).

DEFINITION 3. A joint structure of transition probabilities and reaction times (S, Q) satisfies a *random (subset) advantage model* provided there are nonnegative random variables $\mathbf{t}^X(Y)$, $\phi \subset Y \subseteq X \subseteq S$, such that

$$Q_X(Y; t) = \Pr[t < \mathbf{t}^X(Y) < \mathbf{t}^X(Z) : \text{for all } Z \text{ with } \phi \subset Z \subseteq X,\ Z \neq Y]. \tag{15}$$

If the $\mathbf{t}^X(Y)$, $\phi \subset Y \subseteq X$, for a fixed X are independent, then the random (subset) advantage model is *independent*.

Clearly, the idea behind this definition is that some "event" is associated with each nonempty subset of the available choice set, and that the time of choice and the subset selected are determined by the occurrence of the first such event. Note that the random variables $\mathbf{t}^X(Y)$ for each subset X are allowed to depend on X. One would then suspect, given the result of Theorem 1a, that any joint structure of transition probabilities and reaction times (S, Q) will satisfy an independent random (subset) advantage model (under absolute continuity). This is in fact the case (Theorem 3 below), so we will later strengthen the requirements of the representation.

[2] We assume that the selected subset must be nonempty (but might be the whole available set) as this is the usual assumption for Markov transition models (Tversky, 1972a, b). However, see Corbin and Marley (1974) for a discussion of the *no-choice* option where the person can select the empty set, i.e., reject all the options of the currently available choice set.

As before, assuming absolute continuity,

$$Q_X(Y) = \Pr[\mathbf{t}^X(Y) < \mathbf{t}^X(Z) : \text{for all } Z \text{ with } \phi \subset Z \subseteq X,\ Z \neq Y]$$
$$= \lim_{t \to 0+} Q_X(Y; t)$$

is the probability of selecting the subset Y of X (for further consideration). $\{Q_X(Y) : \phi \subset Y \subseteq X \subseteq S\}$ then constitutes a *structure of transition probabilities* (Corbin and Marley, 1974).

We now pose (and partially solve) representation questions regarding joint structures of transition probabilities and reaction times parallel to those we studied above for joint structures of choice probabilities and reaction times.

For instance, the proof of the following result follows immediately from the proof of Theorem 1(a), with random variables and distributions defined for all Y, $\phi \subset Y \subseteq X \subseteq S$ replacing those for all $x \in X \subseteq S$.

Paralleling the conditions of Theorems 1 and 2, for simplicity we assume $Q_X(Y; t) > 0$ for all $t \geqslant 0$.

THEOREM 3. *A set of transition probabilities and reaction times* $\{Q_X(Y; t) : \phi \subset Y \subseteq X\}$ *can be uniquely represented by an independent random (subset) advantage model provided* $Q_X(Y; t)$, $\phi \subset Y \subseteq X$, *are absolutely continuous and positive for all* $t \geqslant 0$.

Proof. As in Theorem 1(a), where instead of constructing random variables $\mathbf{t}^X(x)$, $x \in X \subseteq S$, it is necessary (by exactly the same techniques) to construct random variables $\mathbf{t}^X(Y)$, $\phi \subset Y \subseteq X \subseteq S$. Q.E.D.

To formulate a version of Theorem 1(b) in the current context, we need to understand what it might mean for a family of random variables $\mathbf{t}^X(Y)$, $\phi \subset Y \subseteq X \subseteq S$, to be "consistent." A plausible constraint is *min-consistency*: Let $k = 2^{|S|} - 1$, and let $g_1, \ldots, g_k$ be the nonempty subsets of S. There is a nonnegative random vector $\mathbf{t} = (\mathbf{t}(g_1), \ldots, \mathbf{t}(g_k))$ such that for $\phi \subset Y \subseteq X \subseteq S$,

$$\mathbf{t}^X(Y) = \min\{\mathbf{t}(J) \mid J \cap X = Y,\ \phi \subset J \subseteq S\}. \tag{16}$$

This is interpreted as follows. First, consider choices on the master set S. Associated with each nonempty subset J of S is a nonnegative random variable $\mathbf{t}(J)$ which denotes the time of occurrence of some "event" associated with that subset, and the subset is chosen whose "event" occurs first. Now consider choices that are constrained to the subsets of some presented set X, $X \subset S$. In this case, it is as if the person first determines some subset J, $\phi \subset J \subseteq S$, by the above process on the master set (via the random variables $\mathbf{t}(J)$, $\phi \subset J \subseteq S$), and then makes as his/her subset selection the (maximal) subset Y of J for which $Y \subseteq X$. (This dependence on the master set might appear to be inappropriate; however, there is an equivalent hidden dependence on the master set in all random utility models, including the previously discussed "horse race" random utility model. Also, Marley (1981) shows how to motivate min-consistency directly from an assumption regarding the existence of a set of "aspects" that characterize the elements of S).

Clearly, one can apply Eq. (16) to the random (subset) advantage representation of Theorem 3 to yield conditions (in terms of the transition probabilities and reaction times) under which min-consistency holds for the representation. This result appears of limited interest, so we turn to a version of Theorem 2 that applies to structures of transition probabilities and reaction times. We are able to show that an appropriate version of Theorem 2 in the current context gives the transition probabilities of Tversky's EBA model as the unique solution[3]—remember, the EBA model is a generalization of Luce's choice model. Before developing that result, it should be noted that if the $\mathbf{t}(J)$, $\phi \subset J \subseteq S$, are independent, then so are the $\mathbf{t}^X(Y)$, $Y \subseteq X$, for a *fixed* set X. However, min-consistency in isolation does not constrain the distribution of the nonnegative random vector $\mathbf{t}$ (nor of $\mathbf{t}^X(Y)$, $Y \subseteq X$) to have any specific form (e.g., exponential). Thus, the constraints placed on these distributions by Theorem 4 (next) must be due to additionally assuming that the subset chosen is independent of the time of choice.

To state the theorem, we need to know a little more about the transition probabilities in Tversky's EBA model. Tversky (1972b) motivated his structure of transition probabilities (S, Q) via *proportionality*: for $\phi \subset Y$, $Z \subseteq X \subseteq S$,

$$\frac{Q_X(Y)}{Q_X(Z)} = \frac{\sum_{J \cap X = Y, \phi \subset J \subseteq S} Q_S(J)}{\sum_{J \cap X = Z, \phi \subset J \subseteq S} Q_S(J)}$$

provided the denominators are both positive, and if one denominator vanishes, so does the other.

Remembering that we have reinterpreted $\mathbf{T}(X)$ as the time of the choice of a *subset* of X, and $\mathbf{C}(X)$ as the *subset* chosen, we obtain:

THEOREM 4. *Consider an independent random (subset) advantage model (S, Q) where for each $\phi \subset Y \subseteq X \subseteq S$, $Q_X(Y; t)$ is absolutely continuous and positive for all $t \geqslant 0$ and $Q_X(Y)$ is nonzero. If the model satisfies min-consistency, and if the subset chosen, $\mathbf{C}(Y)$, is independent of the time of choice, $\mathbf{T}(Y)$, then the transition probabilities satisfy proportionality.*

Proof. Given the representing random variables $\mathbf{t}^X(Y)$, $\phi \subset Y \subseteq X \subseteq S$, let

$$G_Y^X(t) = \Pr[\mathbf{t}^X(Y) > t], \qquad t \geqslant 0,$$

[3] The following technical point is important for relating this result to other characterizations of the EBA as a random utility model, although an understanding of this point is not necessary to understand the present result in isolation. As mentioned in the text, Marley (1981) characterized the transition probabilities of the EBA model in terms of a stochastic process over "aspects." Aspects are selected randomly, and options are retained for further study if they contain the selected aspect. At the level of *aspects*, the relevant random utility model belongs to the generalized extreme value family (Marley, 1989), but does not at *that* level satisfy absolute continuity—since, via the aspects, more than one "event" (selection of an option) can occur at the same time. Nonetheless, when this aspect based model is reinterpreted at the level of selection of *subsets* of elements, it *does* satisfy absolute continuity—in particular, only a single "event" (associated with the selection of a particular subset) can occur at a particular time.

(with $\mathbf{t}^S = \mathbf{t}$) be the associated survival functions. Then using min-consistency, Eq. (16), and the independence of the $\mathbf{t}(J)$, $\phi \subset J \subseteq S$, we have

$$G_Y^X(t) = \prod_{\substack{J \cap X = Y \\ \phi \subset J \subseteq S}} G_J^S(t),$$

i.e.,

$$\log G_Y^X(t) = \sum_{\substack{J \cap X = Y \\ \phi \subset J \subseteq S}} \log G_J^S(t).$$

Also, the arguments leading to Eq. (14) go through in an exactly parallel fashion under the present conditions to give, for $\phi \subset B \subseteq A \subseteq S$,

$$G_B^A(t) = \Pr[\mathbf{T}(A) > t]^{Q_A(B)},$$

i.e.,

$$\log G_B^A(t) = Q_A(B) \log \Pr[\mathbf{T}(A) > t],$$

which substituted in the above sum of logs expression gives

$$Q_X(Y) \log \Pr[\mathbf{T}(X) > t] = \sum_{\substack{J \cap X = Y \\ \phi \subset J \subseteq S}} Q_S(J) \log \Pr[\mathbf{T}(S) > t]. \tag{17}$$

However, Y such that $\phi \subset Y \subseteq X$ is arbitrary, so we also have, for any $\phi \subset Z \subseteq X$,

$$Q_X(Z) \log \Pr[\mathbf{T}(X) > t] = \sum_{\substack{J \cap X = Z \\ \phi \subset J \subseteq S}} Q_S(J) \log \Pr[\mathbf{T}(S) > t]$$

and dividing by the conditions of the theorem $Q_X(Z) > 0$ gives for $\phi \subset Y$, $Z \subseteq X$,

$$\frac{Q_X(Y)}{Q_X(Z)} = \frac{\sum_{J \cap X = Y, \phi \subset J \subseteq S} Q_S(J)}{\sum_{J \cap X = Z, \phi \subset J \subseteq S} Q_S(J)},$$

i.e., proportionality. Q.E.D.

To construct the representation of $\mathbf{T}(X)$, $\phi \subset X \subseteq S$, implied by this result, we use the following representation of a structure (S, Q) that satisfies proportionality (Corbin and Marley, 1974; Marley, 1981): there is a set of nonnegative scales v_X, $\phi \subset X \subseteq S$, such that for $\phi \subset Y \subseteq X \subseteq S$,

$$Q_X(Y) = \frac{v_X(Y)}{\sum_{\phi \subset C \subseteq X} v_X(C)}$$

with

$$v_X(Y) = \sum_{\substack{J \cap X = Y \\ \phi \subset J \subseteq S}} v_S(J).$$

Thus Eq. (17) can be rewritten as

$$\frac{v_X(Y)}{\sum_{\phi \subset C \subseteq X} v_X(C)} \log \Pr[\mathbf{T}(X) > t] = \frac{v_X(Y)}{\sum_{\phi \subset D \subseteq S} v_S(D)} \log \Pr[\mathbf{T}(S) > t],$$

i.e., letting

$$\Psi(t) = \frac{\log \Pr[\mathbf{T}(S) > t]}{\sum_{\phi \subset D \subseteq S} v_S(D)},$$

we obtain

$$\Pr[\mathbf{T}(X) > t] = \exp\left(\sum_{\phi \subset C \subseteq X} v_X(C) \cdot \Psi(t)\right).$$

Note that $\Psi(t)$ is defined in terms of the survival function $\Pr[\mathbf{T}(S) > t]$, and is thus, as developed, not an arbitrary function. However, as with Luce's choice model (see Section 6, Conclusion), if we only constrain the development to satisfy proportionality, then we do have considerable freedom in the choice of the function Ψ.

6. Conclusion

We have characterized the class of *independent* "horse race" random utility models (for choice probabilities and reaction times). It remains to characterize the class of general (i.e., not necessarily independent) "horse race" random utility models, and sub-classes of the general class, such as those that satisfy independence of the option chosen and the time of choice. One might hope to generalize Falmagne's (1978) approach to solve the former problem, and Marley (1989) discusses possible approaches to the latter problem.

Although this latter class includes many common models such as Luce's choice model (see earlier), the independence of the option chosen and the time of choice is probably too strong for most data. For instance (Marley, 1989), consider the following (independent) "horse race" representation for the choice model: for $x \in X$,

$$\Pr[\mathbf{t}(x) > t] = \exp - v(x)\, \Psi(t)$$

for $t > 0$, with v positive valued and Ψ a strictly increasing function with $\Psi(0) = 0$, $\Psi(\infty) = \infty$. Then it is easily checked that

$$\begin{aligned} P_X(x; t) &= \Pr[t < \mathbf{t}(x) < \mathbf{t}(y) : y \in X - \{x\}] \\ &= \frac{v(x)}{\sum_{y \in X} v(y)} \cdot \exp - \sum_{y \in X} v(y) \cdot \Psi(t), \end{aligned}$$

which is of the form discussed early, but now Ψ is an arbitrary function (satisfying the given conditions).

Now consider the hazard function

$$\frac{(-d/dt)\, P_X(x;t)}{P_X(x;t)} = \sum_{y \in X} v(y)\, \Psi'(t),$$

which is *separable* (Luce, 1986, p. 156) in the set X and the time t. It is known that such hazard functions for real data frequently are first increasing in t, then decreasing in t until they reach a stable asymptotic value (e.g., Luce, 1986, Chap. 4). Clearly $\Psi'(t)$ in the above can be selected with such a shape and still satisfy the conditions placed on Ψ. However, Maloney and Wandell (1984), discussed in Luce, 1986, p. 156), have studied such separable hazard functions in the domain of visual detection, and found them not entirely satisfactory.

A further interesting theoretical task is to complete the study of reaction times in (Markov) transition models. We have only presented results for one stage of the choice process (Theorems 3, 4). Combining these results with Luce's (1960) work on transition models based on discarding a single element at a time (he presented a family of gamma distributions as a solution) and with Stern's (1987) work on gamma processes, will probably lead to more realistic reaction time models that nonetheless retain some of the tractability of the models presented here (also see the discussion in Marley, 1989).

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