Bernoulli D 1738 Specimen theoriae novae de mensura sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitannae*, Vol. 5, pp. 175–92. English translation: Bernoulli D (1954). Exposition of a new theory on the measurement of risk. *Econometrica* **22**: 23–36
Birnbaum M H 1997 Violations of monotonicity in judgment and decision-making. In: Marley A A J (ed.) *Choice, Decision, and Measurement: Essays in honor of R. Duncan Luce*. Erlbaum, Mahwah, NJ, pp. 73–100
Birnbaum M H 1999 Testing critical properties of decision making on the Internet. *Psychological Science* **10**: 399–407
Birnbaum M H, McIntosh W R 1996 Violations of branch independence in choices between gambles. *Organizational Behavior and Human Decision Processes* **67**: 91–110
Birnbaum M H, Navarrete J B 1998 Testing descriptive utility theories: Violations of stochastic dominance and cumulative independence. *Journal of Risk and Uncertainty* **17**: 49–78
Edwards W 1954 The theory of decision making. *Psychological Bulletin* **51**: 380–417
Ellsberg D 1961 Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics* **75**: 643–9
Kahneman D, Tversky A 1979 Prospect theory: An analysis of decision under risk. *Econometrica* **47**: 263–91
Luce R D 2000 *Utility of Gains and Losses: Measurement-theoretical and Experimental Approaches*. Erlbaum, Mahwah, NJ
Quiggin J 1993 *Generalized Expected Utility Theory: The Rank-dependent Model*. Kluwer, Boston, MA
Savage L J 1954 *The Foundations of Statistics*. Wiley, New York
Tversky A, Kahneman D 1992 Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty* **5**: 297–323
von Neumann J, Morgenstern O 1947 *Theory of Games and Economic Behavior*, 2nd edn. Princeton University Press, Princeton, NJ
Wakker P, Tversky A 1993 An axiomatization of cumulative prospect theory. *Journal of Risk and Uncertainty* **7**: 147–76
Wu G 1994 An empirical test of ordinal independence. *Journal of Risk and Uncertainty* **9**: 39–60
Wu G, Gonzalez R 1998 Common consequence conditions in decision making under risk. *Journal of Risk and Uncertainty* **16**: 115–39

M. H. Birnbaum

Decision and Choice: Random Utility Models of Choice and Response Time

Introduction

The task of choosing a single 'best' option from some available, potentially infinite set of options R has received considerable study in psychology and economics. Random utility models of choice account for the stochastic variability underlying these choices (i.e., the same choice is not necessarily made on repeated presentations of the same set of options) by assuming that there exist a random variable $U(x)$ for each option x and a joint probability measure for these random variables such that the probability of choosing a particular option x from the set of available options is equal to the probability that $U(x)$ takes on a value greater or equal to the values of all other random variables (e.g., see *Luce's Choice Axiom*; *Utility and Subjective Probability: Contemporary Theories*). The basic choice paradigm is extended here by considering, in addition to the option chosen, the point in time that a choice is made from the set of available options. The random utility model can be tailored to cover this situation by replacing the utility variable $U(x)$ by $V(x) = \phi[U(x)]$ (ϕ some monotonically decreasing transformation), where $V(x)$ can be interpreted as decision time for choosing option x, and by replacing the maximum utility rule by a minimum decision time rule: the option chosen is the one which happens to be associated with the minimum choice (or decision) time with respect to all options in the available set. This model will be referred to as 'horse race' random utility model.

A number of problems arise in the study of 'horse race' random utility models for choice and response time that have found only partial solutions: (a) What conditions on the observable choice probabilities and decision times are necessary and sufficient for a random utility representation? (b) What are the consequences of assuming stochastic independence between the time of choice and the identity of the option chosen, and which assumptions on the joint distribution function do imply this independence? (c) What are possible generalizations to other choice paradigms? The presentation here is partly based on Marley and Colonius 1992 and Marley 1992, 1989. Some related results (not explicitly referred to in the following) can be found in Bundesen 1993, Robertson and Strauss 1981, and Vorberg 1991.

1. 'Horse Race' Random Utility Models

Let $R = \{x, y, \ldots\}$ be a finite set of potential choice options (for an extension of the theory to infinite choice sets, see Resnick and Roy 1992); the subset X of R containing at least two elements is the currently available choice set. $T(X)$ is a random variable denoting the time at which a choice is made; and $C(X)$ is a random variable denoting the element chosen from X. For $t \geqslant 0$

$$P_X(x;\, t) = Pr[C(X) = x \cap T(X) > t]$$

is the probability that options x is chosen from X after time t, and $\{P_X(x;\, t)\colon x \in X \subseteq R\}$, or (R, P) for short, is called joint structure of choice probabilities and response times. (R, P) is called complete if it is defined for all subsets of X of $R$with $|X| \geqslant 2$.

A complete joint structure of choice probabilities and response times (R, P) is said to satisfy a '*horse race*' random utility model if for any $X \subseteq R$ with

$|X| \geqslant 2$ there exists a probability measure $Pr_X[\ ]$ such that for the collection of nonnegative random variables $\{V(x)\colon x \in X\}$

$$P_X(x; t) = Pr_X[t < V(x) < V(y)\colon y \in X \backslash \{x\}] \quad (1)$$

For simplicity, all distributions are assumed to be absolutely continuous, i.e., they must possess a density function. However, as observed in Resnick and Roy 1992, absolute continuity is not necessary for the results cited here. Note that with

$$\begin{aligned} P_X(x) &= \lim_{t \downarrow 0} Pr_X[t < V(x) < V(y)\colon y \in X \backslash \{x\}] \\ &= Pr_X[V(x) < V(y)\colon y \in X \backslash \{x\}] \end{aligned} \quad (2)$$

the collection of choice probabilities $\{P_X(x)\colon x \in X \subseteq R\}$ constitutes a system of choice probabilities with

$$P_X(x) = Pr_X[V(x) = \min\{V(y)\colon y \in X\}] \quad (3)$$

satisfying a random utility representation with min replacing the usual max.

In this article the problem of finding conditions on the—at least theoretically—observable choice probabilities and response time distributions that are necessary and sufficient for the existence of a particular representation in terms of a collection of underlying random variables is referred to as a characterization problem.

Given a (complete) joint structure of choice probabilities and response times, what conditions on the (survival) functions $P_X(x; t)$ are necessary and sufficient for the existence of a (possibly dependent) 'horse race' random utility representation for that joint structure?

A complete answer to this problem is still open. However, a number of results in special cases have been found.

1.1 Representation by Independent Random Variables

A partial answer is given by the following
Theorem 1 (Marley and Colonius 1992).

(a) For a collection of choice probabilities and response times $\{P_X(x; t)\colon x \in X\}$ on a fixed set X with $|X| \geqslant 2$, there exist independent random variables $V(x)$, $x \in X$, with unique distributions, such that Eqn. (1) holds provided $P_X(x; t)$ is absolutely continuous and positive for all $t \geqslant 0$.

(b) A complete joint structure of choice probabilities and response times (R, P) can be represented by an independent 'horse race' random utility model, with unique distributions, if the conditions of (a) hold and

$$\left[\sum_{y \in X} P_X(y; t) \right]^{-1} (\mathrm{d}/\mathrm{d}t) P_X(x, t) \quad (4)$$

is independent of $X \subseteq R$ for all $t \geqslant 0$.

Thus, given the above regularity conditions on the response time distributions, any set of choice probabilities and response times on a fixed finite set X can be represented by independent random variables such that the 'horse race' Eqn. (1) holds. Following Berman 1963 the proof of (a) is based on setting up a differential equation with the following unique solution in terms of the (survival) distributions of independent random variables

$$\begin{aligned} G_x^X(t) &\equiv Pr[V(x) > t] \\ &= \exp\left\{ \int_0^t Pr[T(X) > s]^{-1}\, \mathrm{d}P_X(x; s) \right\} \\ &= \exp\left\{ \int_0^t \left[\sum_{y \in X} P_X(y; s) \right]^{-1} \mathrm{d}P_X(x; s) \right\} \\ &= \exp\left\{ \int_0^t \left[\prod_{y \in X} G_y^X(s) \right]^{-1} \mathrm{d}P_X(x; s) \right\} \end{aligned} \quad (5)$$

A related result, in the context of parallel-serial processing analysis, appears in Townsend 1976. Some of the above distributions, but not all, may be improper, i.e., they may have positive measure at infinity implying an infinite decision time for that alternative, even if the distribution of response time $T(X)$ is finite almost surely (see Dzhafarov 1993 for a related interpretation and further results under weaker conditions in a different context). Note that Theorem 1 (a) does not say that there always exists a unique set of independent random variables representing a complete joint structure of choice probabilities and response times irrespective of the subset X. The latter postulate is in fact paraphrased in Theorem 1 (b). Thus, the general characterization problem is still unsolved leaving open the possibility that for some joint structures no independent representation over all subsets X exists.

1.2 Hazard Rate Reformulation

In the theory of competing risks (e.g., David and Moeschburger 1978) the 'event' associated with each $x \in X$ is reinterpreted as a cause (e.g., of failure or death), and thus the selection $x \in X$ then corresponds to the cause associated with x being the first to occur when X contains the possible causes. The instantaneous rate of failure from cause x when all causes in X act simultaneously, called cause-specific (or crude hazard rate), is defined as

$$h_x^X(t) = \lim_{\delta\downarrow 0} Pr[t < T(X) \leqslant t+\delta \cap C(x) = x | T(X) > t] \quad (6)$$

In the choice context this can be interpreted as the instantaneous rate of choosing option x from X at time t given that no choice was made before t. Provided that $Pr[T(X) > t] \neq 0$, going over to the unconditioned probability in Eqn. (6) gives

$$h_x^X(t) = [(-\mathrm{d}/\mathrm{d}t)P_X(x;t)] \bigg/ \left(\sum_{y\in X} P_X(y;t) \right) \quad (7)$$

Note that the right hand side of Eqn. (7) agrees (except in sign) with that given in the statement of the Marley-Colonius Theorem (b). Thus, for a fixed set X, the crude hazard rate leads directly to the desired random variable representation by setting

$$Pr[V(x) > t] = \exp\left[-\int_0^t h_x^X(s)\,\mathrm{d}s \right] = G_x^X(t) \quad (8)$$

and for a common representation on all subsets X of R, h_a^X must not depend on X (as in part (b) of Theorem 1).

$P[T(X) > t]$ is commonly referred to as the overall survival distribution (for set X) with a corresponding overall hazard rate $h^X(t)$; i.e., by definition

$$h^X(t) = -\mathrm{d}/\mathrm{d}t\, Pr[T(X) > t]/Pr[T(X) > t] \quad (9)$$

and so from

$$Pr[T(X) > t] = \sum_{y\in X} P_X(y;t)$$

and Eqn. (7), it follows that

$$h^X(t) = \sum_{x\in X} h_x^X(t) \quad (10)$$

Thus, under the stated regularity conditions, the independent random variables $V(x), x\in X$, constructed above can be used to generate the survival distributions $P_X(x;t)$. In competing risks theory this is a well-known result, often referred to as the 'non-identifiability problem.' It means that any set of competing risk data can be explained by some independent risk model which, in most empirical situations, appears to be unrealistic. One way to establish identifiability of a dependent model is to assume that the $V(x)$ follow some (flexible) parametric family, to estimate the parameters of the multivariate distribution, and to test for independence (Tsiatis 1975).

1.3 Independence of Option Chosen from Time of Choice

The following condition is often made in the competing risks theory and it turns out to be equivalent to a condition specifying a large family of random utility models. A joint structure of choice probabilities and response times (S, P) satisfies the proportional hazard rate (PHR) condition if and only if for any subset X of R there are constants $C_X(x)$ such that

$$h_x^X(t) = C_X(x)h^X(t) \quad (11)$$

where $h^X(t)$ is the overall hazard rate for $T(X)$ and $h_x^X(t)$ are the cause specific hazard rates. After some algebra,

$$P_X(x;t) = C_X(x)Pr[T(X) > t]$$

In particular, because $P_X(x) = \lim_{t\downarrow 0} P_X(x;t)$,

$$P_X(x;t) = Pr[C(X) = x]Pr[T(X) > t] \quad (12)$$

Reversing the steps leads to the following result (see Kochar and Proschan 1991).

Theorem 2. The proportional hazard rate condition (PHR) is equivalent to assuming stochastic independence between $T(X)$ and $C(X)$ for any $X \subseteq R$.

Intuitively, proportionality of the hazard rates means that the time of choice gives no information as to the identity of the element chosen, and vice versa. This seems like a rather strong assumption in an empirical context, and the following result concerning Luce's choice model (see *Luce's Choice Axiom*) supports this view. Luce's choice model holds for a system of choice probabilities $\{P_X(x): x\in X \subseteq R\}$ provided there is a ratio scale v on R such that for $P_X(x) \neq 0, 1$

$$P_X(x) = \frac{v(x)}{\sum_{y\in X} v(y)} \quad (13)$$

For simplicity in the following, we assume that all choice probabilities are nonzero. The result can be generalized when this is not the case by adding a connectivity and a transitivity condition (Luce 1959, Theorem 4, p. 25).

Theorem 3 (Marley and Colonius 1992). Consider a (complete) independent 'horse race' random utility model (R, P) where for each $x \in X \subseteq R$ with $|X| \geqslant 2$, $P_X(x; t)$ is positive and absolutely continuous for all $t > 0$. If $T(X)$ is stochastically independent of $C(X)$, then the choice probabilities satisfy Luce's choice model.

Given the known limited empirical validity of Luce's choice model in empirical preference situations, this result implies that one must study dependent 'horse race' random utility models as long as one is not willing to drop the assumption of independence between $T(X)$ and $C(X)$.

1.4 Generalized Stable Survival Functions

A large class of (possibly) dependent 'horse race' random utility models based on extreme value distributions can be generated from the concept of a generalized stable survival function.

For simplicity, in this section assume $R = \{x_1, \ldots, x_n\}$. We call

$$P_R(t_1, \ldots, t_n) = Pr[V(x_1) > t_1, \ldots, V(x_n) > t_n)$$

a (multivariate) survival function. P_R is called generalized stable if there is a strictly monotone decreasing function η and a constant $\mu > 0$ such that for all $\alpha > 0$, $t_i \geqslant 0$, $i = 1, \ldots, n$,

$$(\eta \circ P_R)(\alpha t_1, \ldots, \alpha t_n) = \alpha^{\mu}(\eta \circ P_R)(t_1, \ldots, t_n) \quad (14)$$

where $\circ$ denotes concatenation of functions. Letting $G_R = (\eta \circ P_R)$ this amounts to

$$G_R(\alpha t_1, \ldots, \alpha t_n) = \alpha^{\mu} G_R(t_1, \ldots, t_n)$$

i.e., G_R is homogeneous of degree μ. Furthermore, a survival function P_R is called a strictly monotone transform of a generalized stable survival function Q_R if there is a strictly monotone increasing function γ (of the random variables associated with Q_R) with $\gamma(0) = 0$, $\gamma(\infty) = \infty$ such that for $t_i \geqslant 0$, $i = 1, \ldots, n$,

$$P_R(t_1, \ldots, t_n) = Q_R(\gamma(t_1), \ldots, \gamma(t_n))$$

Theorem 4 (Marley 1989). Any 'horse race' random utility model that is generated by a strictly monotone transform of a generalized stable survival function is such that $T(X)$ is stochastically independent of $C(X)$ for any $X \subseteq R$.

As shown by way of an example in Resnick and Roy 1992 (Sect. 5) the converse of this result does not hold in general, i.e., independence between $T(X)$ and $C(X)$ does not lead back to the class of generalized stable survival functions. Robertson and Strauss 1981 show that the converse of Theorem 4 is true if the survival function P_R (or some strictly monotone transformation of it) belongs to the generalized Thurstone class: for $t_i \geqslant 0$, $i = 1, \ldots, n$,

$$P_R(t_1, \ldots, t_n) = P(u(x_1)t_1, \ldots, u(x_n)t_n)$$

where $u(x_i) \geqslant 0$ and P is a survival function that is independent of R.

2. General Feature Model

Marley 1989 presents a general class of models which lends itself easily to a process interpretation for the choice paradigm. With $R = \{x_1, \ldots, x_n\}$ suppose each option x_i has m (random) components. Thus, there is a matrix of random variables $(Z_{ik})_{i=1,\ldots,n}^{k=1,\ldots,m}$, and it is assumed that the previous random variables $V(x_i)$ are given by

$$V(x_i) = \min_{k=1,\ldots,m} Z_{ik}$$

In the process interpretation Z_{ik} corresponds to the 'time of occurrence' of feature (or dimension) $k$for option i, and that option is selected which has the earliest such event over all features (or dimensions). Thus, the survival function is

$$\begin{aligned} P(t_1, \ldots, t_n) &= Pr[V(x_1) > t_1, \ldots, V(x_n) > t_n] \\ &= Pr\left[\bigcap_{i=1}^{n} \{V(x_i) > t_i\}\right] \\ &= Pr\left[\bigcap_{i=1}^{n} \left\{\min_{k=1,\ldots,m} Z_{ik} > t_i\right\}\right] \\ &= Pr\left[\bigcap_{i=1}^{n}\bigcap_{k=1}^{m} \{Z_{ik} > t_i\}\right] \end{aligned} \quad (15)$$

For the survival function P to be generalized stable, restrictions must be placed on the distribution of the random variables Z_{ik}. However, no independence of $(Z_{i1}, \ldots, Z_{im})$ for distinct i, $1 \leqslant i \leqslant n$ is assumed as one might be able to give a 'process' interpretation of such dependencies.

Example: Multivariate exponential distribution (Marley 1989, Pickands 1981).

Let $A = (a_{ik})$ be a $n \times m$ matrix with non-negative and finite entries such that there is at least one positive value in each row, and no two values are identical in any column. Also, $Z_1, \ldots, Z_m$ are mutually independent random variables with standard exponential distribution, i.e., $Pr[Z_k > t] = \exp(-t)$, and $V(x_1), \ldots, V(x_n)$ have components

$$V(x_i) = \min_{k=1,\ldots,m} \frac{Z_k}{a_{ik}} \tag{16}$$

Inserting this into Eqn. (15) yields the survival function

$$P(t_1, \ldots, t_n) = \exp - \left[\sum_{k=1}^{m} \max_{i=1,\ldots,n} a_{ik} t_i \right]$$

or,

$$-\ln P(t_1, \ldots, t_n) = \sum_{k=1}^{m} \max_{i=1,\ldots,n} a_{ik} t_i \tag{17}$$

With $G_R \equiv (-\text{In} \circ P)$ this clearly satisfies the condition of Theorem 4. Marley 1989 demonstrates how this model easily explains the standard 'bicycle/pony' or 'red bus/blue bus' examples that Luce's choice model cannot adequately represent.

3. *Subset Choices*

The above development can be extended to cases where one is allowed to choose a subset of 'acceptable' options from the set of available options rather than a single element (e.g., approval voting). Extending the previous notation, let $Q_X(Y; t)$, $t \geqslant 0$, be the probability that the nonempty subset $Y \in X$ is selected after time t, and assume that

$$Q_X(Y; t) = Pr[C(X) = Y \cap T(X) > t] \tag{18}$$

where (as before) $T(X)$ is a random variable denoting the time at which the subset choice is made, and $C(X)$ is a random variable denoting the subset chosen. Such a collection $\{Q_X(Y; t)\colon \emptyset \in Y \subseteq X \subseteq R\}$, or $\{R, Q\}$ for short, will be called a joint structure of transition probabilities and response times, and (R, Q) will be called complete if it is defined over all subsets of R, as before. A (complete) joint structure of transition probabilities and response times (R, Q) satisfies a random (subset) advantage model provided there exists for any $X \subseteq R$ a probability measure $Pr_X[\ ]$ such that for the collection of nonnegative random variables $\{V(Y)\colon \emptyset \subset Y \subseteq X \subseteq R\}$

$$Q_X(Y; t) = Pr_X[t < V(Y) < V(Z)\colon \text{ for all } Z \text{ with } \emptyset \subset Z \subseteq X, Z \neq Y] \tag{19}$$

If the $V(Y)$, for a fixed X, are independent, then the random (subset) advantage model is independent. The idea here is that some 'event' is associated with each non empty subset of the available choice set, and that the time of choice and the subset selected are determined by the occurrence of the first such event. As before, assuming absolute continuity,

$$\begin{aligned} Q_X(Y) &= Pr_X[V(Y) < V(Z)\colon \text{ for all } Z \text{ with } \emptyset \subset Z \subseteq X, Z \neq Y] \\ &= \lim_{t \downarrow 0} Q_X(Y; t) \end{aligned} \tag{20}$$

is the probability of selecting the subset Y of X. The following is an immediate result paralleling Theorem 1 (a).

Theorem 5 (Marley and Colonius 1992). A collection of transition probabilities and response times $\{Q_X(Y; t)\colon \emptyset \subset Y \subseteq X\}$ can be uniquely represented by an independent random (subset) advantage model provided $Q_X(Y; t)$, $\emptyset \subset Y \subseteq X$, are absolutely continuous and positive for all $t \geqslant 0$.

4. *Structures of Transition Probabilities*

Subset selection as defined in the preceding section might only be an intermediate stage in the determination of a single 'best' element by repeatedly selecting subsets from the subset determined in the previous subset selection. The class of Markovian transition structures has the recursive property

$$P_X(x) = \sum_{Y \subseteq X} Q_X(Y) P_Y(x) \tag{21}$$

However, Eqn. (21) is compatible with any set of choice probabilities, e.g., define

$$Q_X(Y) = \begin{cases} P_X(x) & \text{if } Y = \{x\} \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

Thus, some restrictions must be put on the transition probabilities. The following *proportionality* condition leads to Tversky's 1972a, b elimination-by-aspects model (EBA):

$$\frac{Q_X(Y)}{Q_X(Z)} = \frac{\sum_{J \cap X = Y, \emptyset \subset J \subset R} Q_R(J)}{\sum_{J \cap X = Z, \emptyset \subset J \subset R} Q_R(J)} \tag{23}$$

for $\emptyset \subset Y, Z \subseteq R$, provided the denominators are both positive, and if one denominator vanishes, so does the other.

To formulate a generalization of Theorem 3 in the context of subset choices, the following plausible min-consistency constraint on the response time distribution is needed which, however, does not constrain the distribution of the $V(Y)$ to have any specific form:

$$Pr_X[V(Y) \leqslant t] = Pr_R[\min\{V(J) | J \cap X = Y, \emptyset \subset J \subseteq R\} \leqslant t] \quad (24)$$

Theorem 6. (Marley and Colonius 1992). Consider an independent random (subset) advantage model (R, Q) where for each $\emptyset \subset Y \subseteq X \subseteq R$, $Q_X(Y; t)$ is absolutely continuous and positive for all $t \geqslant 0$ and $Q_X(Y)$ is nonzero. If the model satisfies min-consistency, and if the subset chosen, $C(Y)$, is independent of the time of choice, $T(Y)$, then the transition probabilities satisfy proportionality.

Thus, independence between $C(Y)$ and $T(Y)$ in this subset choice context leads to the transition probabilities of the EBA model paralleling the result (Theorem 3) for the single-element choice context with the special case of the EBA model, i.e., Luce's choice model. Finally, it has been shown (Marley 1989) that the general feature model can easily be adjusted such that it contains the EBA model as a special case with a generalized stable survival function where

$$-\ln P(t_1, \ldots, t_n) = \sum_{J \subseteq R} V_R(J) \max_{i \in J} t_i \quad (25)$$

where $V_R(J) \geqslant 0$ for all $J \subseteq R$.

5. *Conclusions*

The introduction of the notion of response (or decision) time into the random utility approach to modeling choice behavior has been very fruitful. It has brought about new insights into existing stochastic choice models and their characterizations, and it has generated new questions, many of which have only found partial solutions up to now.

See also: Bayesian Theory: History of Applications; Dynamic Decision Making; Luce's Choice Axiom; Stochastic Dynamic Models (Choice, Response, and Time); Utility and Subjective Probability: Contemporary Theories; Utility and Subjective Probability: Empirical Studies

Bibliography

Berman S M 1963 Note on extreme values, competing risks and semi-Markov processes. *Annals of Mathematical Statistics* **34**: 1104–6

Bundesen C 1993 The relationship between independent race models and Luce's Choice Axiom. *Journal of Mathematical Psychology* **37**: 446–71

David H A, Moeschburger M L 1978 *The Theory of Competing Risks*. Griffin, London

Dzhafarov E N 1993 Grice-representability of response time distribution families. *Psychometrika* **58**: 281–314

Kochar S C, Proschan F 1991 Independence of time and cause of failure in the multiple dependent competing risks model. *Statistica Sinica* **1**: 295–9

Luce R D 1959 *Individual Choice Behavior*. Wiley, New York

Marley A A J 1989 A random utility family that includes many of the 'classical' models and has closed form choice probabilities and reaction times. *British Journal of Mathematical Statistical Psychology* **42**: 13–36

Marley A A J 1992 A selective review of recent characterizations of stochastic choice models using distribution and functional equation techniques. *Mathematical Social Sciences* **23**: 5–29

Marley A A J, Colonius H 1992 The 'horse race' random utility model for choice probabilities and reaction times, and its competing risks interpretation. *Journal of Mathematical Psychology* **36**: 1–20

Pickands J III 1981 *Multivariate Negative Exponential and Extreme Value Distributions*. Manuscript, Department of Statistics, University of Pennsylvania, Philadelphia, PA

Resnick S I, Roy R 1992 On min-stable horse races with infinitely many horses. *Mathematical Social Sciences* **23**: 119–45

Robertson C A, Strauss D J 1981 A characterization theorem for random utility variables. *Journal of Mathematical Psychology* **23**: 184–9

Townsend J T 1976 Serial and within-stage independent parallel model equivalence on the minimum completion time. *Journal of Mathematical Psychology* **14**: 219–38

Tsiatis A 1975 A non-identifiability aspect of the problem of competing risks. *Proceedings of the National Academy of Science, USA* **72**: 20–2

Tversky A 1972a Choice by elimination. *Journal of Mathematical Psychology* **9**: 341–67

Tversky A 1972b Elimination by aspects: A theory of choice. *Psychology Review* **79**: 281–99

Vorberg D 1991 *Characterizations of Independent Random Utility Models Which Imply the Choice Axiom*. Technical Report, Technische Universität Braunschweig, Braunschweig

H. Colonius

Decision Biases, Cognitive Psychology of

Normative analyses of rational decision making dictate *a priori* how decisions ought to be made. Descriptive analyses of decision making are based on experimental studies and emphasize the role of information processing in people's decisions. The empirical evidence indicates that people's decisions are often at odds with the assumptions of the rational theory. The following describes some of the cognitive mechanisms that underlie decision behavior and cause it to depart form the normative benchmark, yielding systematic decision biases.

1. *Normative and Descriptive Analyses*

The study of decision making is an interdisciplinary enterprise involving economics, political science, and psychology, as well as statistics and philosophy. One

International Encyclopedia of the Social & Behavioral Sciences ISBN: 0-08-043076-7