

FROM FECHNER TO FECHNER, AND BEYOND

Ehtibar N. Dzhafarov

Purdue University (ehtibar@purdue.edu)

Hans Colonius

Oldenburg University (hans.colonius@uni-oldenburg.de)

Abstract

Dzhafarov and Colonius' theory of Fechnerian Scaling deals with arbitrary spaces of stimuli endowed with probability-of-different functions $\psi(\mathbf{x}, \mathbf{y})$. The cornerstone of this theory is the Law of Regular Minimality, according to which $h(\mathbf{x}) = \arg \min_{\mathbf{y}} \psi(\mathbf{x}, \mathbf{y})$ and $g(\mathbf{y}) = \arg \min_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{y})$ are well-defined functions, and $g \equiv h^{-1}$. Due to this law the psychometric increments $\Psi^{(1)}(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x}, \mathbf{y}) - \psi(\mathbf{x}, h(\mathbf{x}))$ and $\Psi^{(2)}(\mathbf{y}, \mathbf{x}) = \psi(\mathbf{x}, \mathbf{y}) - \psi(g(\mathbf{y}), \mathbf{y})$ can be cumulated along "allowable paths" within stimulus spaces yielding the paths' lengths. The Fechnerian distance $G(\mathbf{a}, \mathbf{b})$ between \mathbf{a} and \mathbf{b} is defined as the infimum of the lengths of all allowable closed loops from \mathbf{a} to \mathbf{b} and back. The notion of an allowable loop, however, has been defined differently for continuous spaces (piecewise smooth arcs) and discrete spaces (finite chains). In this paper we introduce a new mathematical notion, Dissimilarity Function, and assume that $\Psi^{(1)}$ and $\Psi^{(2)}$ are dissimilarity functions. Allowable loops in all stimulus spaces in this new version of the theory can be defined as finite cycles of stimuli, $G(\mathbf{a}, \mathbf{b})$ being the infimum of the sums of the chained psychometric increments along all such cycles containing \mathbf{a} and \mathbf{b} . This construction includes previous developments of Dzhafarov-Colonius' theory as special cases, but it rather radically departs from the spirit of Fechner's original theory.

In Fechner's original theory every point on a unidimensional stimulus continuum is assigned a measure of its dissimilarity from its "immediate neighbors," and this dissimilarity measure is integrated over the interval between two stimuli to yield the subjective distance between them. Since Fechner assumed that the unidimensional continuum of stimuli monotonically corresponds to a unidimensional continuum of "sensations," the local dissimilarity measure in this theory can be derived from greater-less judgments, e.g., by computing the slopes of the probability-of-greater functions at their medians or, as a crude approximation, the reciprocals of JNDs. In Dzhafarov (2001) and Dzhafarov & Colonius (1999a) we argue that the integration of a local dissimilarity measure constitutes the essence of Fechner's theory. Departing from this idea (the "*from Fechner*" of the title) we developed a theory we called Multidimensional Fechnerian Scaling, MDfS (Dzhafarov, 2002a-d, 2004; Dzhafarov & Colonius, 1999a, b, 2001): unidimensional stimulus continua are replaced in MDfS with arbitrary open regions of n -dimensional real-valued vectors, the greater-less judgments are replaced with same-different ones, and the dissimilarity measure (*psychometric increments*) is computed from the probability-of-different functions. When this dissimilarity measure is integrated along a piecewise smooth (i.e., continuously differentiable) arc connecting two stimuli, one gets this arc's length (we call it *psychometric length*); and the infimum of all such lengths across all piecewise smooth closed loops containing given two stimuli yields the subjective (Fechnerian) distance between these stimuli.

In Dzhafarov & Colonius (2005a) this theory was generalized to arbitrary spaces in which one can define piecewise smooth arcs. Unlike in MDfS, in this generalized theory all topological, geometrical, analytical, and metric properties of stimulus spaces are derived solely from

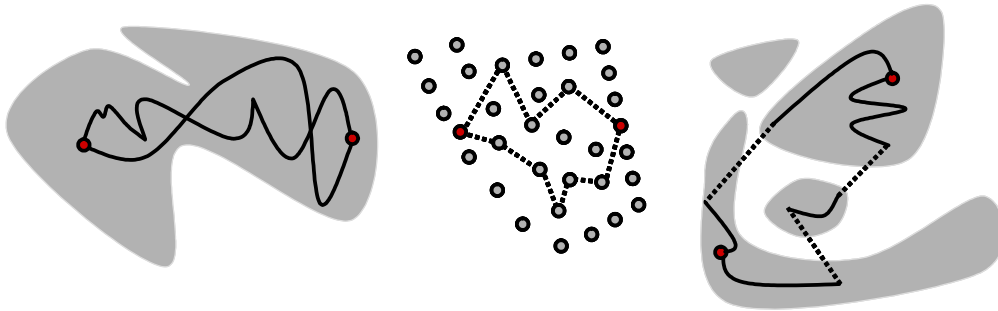


Figure 1: Allowable closed loops in (from left to right) continuous, discrete, and hybrid (discrete-continuous) stimulus spaces.

the properties of discrimination probabilities, with no reference to physical properties of stimuli. Therefore we call this theory *purely psychological* (“psychophysics without physics”). As a special case, one can return to Fechner’s original theory (“*to Fechner*”) and reformulate it with no reference to the representability of stimuli by real numbers, their Euclidean topology, order, additivity, or any other physical property explicitly or implicitly used, e.g., in the derivation of the celebrated logarithmic law. Physical measurements are only used as stimulus labels, each label exhaustively characterized by its discrimination probabilities from other labels (with regards to its *observation area*, as explained below); the Fechnerian distance between stimuli labeled **a** and **b** will remain precisely the same if the unidimensional continuum of stimuli is bijectively mapped, say, in a unit square.

[To prevent confusion: the “purely psychological” does not mean that one cannot or should not use physical measures, it only means that the Fechnerian distances among stimuli are logically and computationally independent of the choice of physical measures. Nothing prevents one, once Fechnerian distances have been computed, to ask whether the Fechnerian distance between stimulus and absolute threshold is, say, a power function of a specific physical measure of stimulus.]

In parallel to the Generalized Fechnerian Scaling for continuous spaces we developed another purely psychological theory, Fechnerian Scaling of Discrete Object Sets, FSDOS (Dzhafarov & Colonius, 2005b, in press a, c). Discrete object (or stimulus) sets are those in which the dissimilarity of any stimulus from all other stimuli cannot fall below some positive quantity. No two distinct stimuli in a discrete space can be connected by an arc, and the notion of a piecewise smooth closed loop here is replaced with that of a closed finite chain of stimuli (leading from **a** to **b** and back). Each link of such a chain is characterized by the dissimilarity of its endpoints, and the psychometric length is simply the sum of the dissimilarity values of its links. Although we do call these computations Fechnerian and have established useful relations between them and the continuous Fechnerian computations (Dzhafarov & Colonius, in press a, c), there is a conceptual gap between them, especially apparent when we consider hybrid spaces consisting of isolated continuous components (Dzhafarov & Colonius, 2005b): an allowable path here consists of piecewise smooth arcs within components and discrete jumps between them (Fig. 1). This situation seems undesirable as we do not have a clear operational boundary between continuous and discrete spaces. One might even argue that every discrete space (e.g., that of the written letters of an alphabet) is only an approximation to, or a categorization of, a continuous one (graphical modifications of written letters).

The revision we propose is seemingly simple: while the notion of an arc is not defined for discrete spaces, the notion of a finite chain is defined for all stimulus spaces without exception; the proposal therefore is to define the subjective distance between **a** and **b** as the

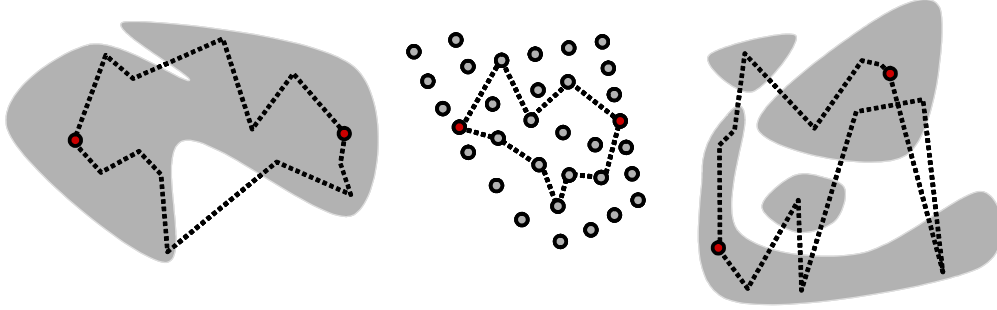


Figure 2: In Universal Fechnerian Scaling the allowable closed loops in all possible stimulus spaces are finite chains of stimuli.

infimum of psychometric length for all closed finite chains containing \mathbf{a} and \mathbf{b} . In other words, the computations in all possible stimulus spaces are the same as in FSDOS (see Fig. 2). The implementation of this simple idea, however, requires thorough rethinking of the basic concepts upon which we base our theory. In the following we present outlines of this new development (necessarily schematic due to the lack of space).

In essence, what we propose amounts to elevating FSDOS from somewhat of an afterthought it has been to becoming the very core of Fechnerian Scaling. One might challenge the continued use of the adjective “Fechnerian” for this theory: Fechner’s idea of the integration of infinitesimally small dissimilarities is being replaced here, on the most basic level, by summation of finite dissimilarities along finite stimulus chains (although in continuous spaces, as explained below, this may lead to arcs and integration “in the limit,” if the number of links in the chains whose lengths approach the infimum increases beyond bounds). In other words, with reference to the title, it may very well be the case that the present development leads us “*beyond Fechner.*” At this stage, however, we prefer to retain the “Fechnerian” terminology for continuity with our previous work: the new version of the theory can be called *Universal Fechnerian Scaling*, to emphasize that one and the same construction is applied to all possible stimulus spaces.

Dissimilarity Function: A New Mathematical Notion

Let \mathfrak{S} be a set with elements (stimuli) denoted by boldface lowercase symbols ($\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \dots$). A string of k such symbols, $\mathbf{x}_1 \dots \mathbf{x}_k$, represents a stimulus chain of cardinality k . We use the notation of the form $[prefix]\mathbf{x}_1 \dots \mathbf{x}_k$ to denote various numerical characteristics of chain $\mathbf{x}_1 \dots \mathbf{x}_k$ (e.g., $D\mathbf{a}\mathbf{b}$, $D\mathbf{x}_1 \dots \mathbf{x}_k$, $\psi\mathbf{x}\mathbf{y}$, ...).

Consider a function $D : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, 1]$ (in the last section we will show how to generalize this to mapping into \mathbb{R}^+). We call D a *deviation function* if

$$D\mathbf{a}\mathbf{b} = 0 \iff \mathbf{a} = \mathbf{b}. \quad (1)$$

A deviation function D is called a *dissimilarity function* if it has the following two properties:

$$(D\mathbf{a}'_n \mathbf{a}_n \rightarrow 0) \wedge (D\mathbf{b}'_n \mathbf{b}_n \rightarrow 0) \implies D\mathbf{a}'_n \mathbf{b}'_n - D\mathbf{a}_n \mathbf{b}_n \rightarrow 0, \quad (2)$$

and, for any sequence of chains $\{\mathbf{a}\mathbf{x}_1^n \dots \mathbf{x}_{k_n}^n\}_{n \in \mathbb{N}}$,

$$D\mathbf{a}\mathbf{x}_1^n \dots \mathbf{x}_{k_n}^n \rightarrow 0 \implies D\mathbf{a}\mathbf{x}_{k_n}^n \rightarrow 0, \quad (3)$$

where $D\mathbf{x}_1 \dots \mathbf{x}_k$ is defined as $\sum_{i=1}^{k-1} D\mathbf{x}_i \mathbf{x}_{i+1}$. Note that if D is a metric, then (1), (2), and (3) are satisfied trivially.

A dissimilarity function D induces a *metric* $G : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}^+$ by means of the following construction:

$$G\mathbf{a}\mathbf{b} = \inf_{\{\mathbf{a},\mathbf{b}\} \subseteq \{\mathbf{x}_1, \dots, \mathbf{x}_k\}, k \in \mathbb{N}} D\mathbf{x}_1 \dots \mathbf{x}_k \mathbf{x}_1 = \inf_{\mathbf{y}_1 \dots \mathbf{y}_{k'}, k' \in \mathbb{N}} Day_1 \dots \mathbf{y}_{k'} \mathbf{b} + \inf_{\mathbf{z}_1 \dots \mathbf{z}_{k''}, k'' \in \mathbb{N}} D\mathbf{b}\mathbf{z}_1 \dots \mathbf{z}_{k''} \mathbf{a}, \quad (4)$$

where the infimum on the left is taken over all closed loops $\mathbf{x}_1 \dots \mathbf{x}_k \mathbf{x}_1$ in \mathfrak{S} containing \mathbf{a} and \mathbf{b} . The topologies based on open D -balls and open G -balls coincide. Generally, $G\mathbf{a}\mathbf{b} \leq D\mathbf{a}\mathbf{b} + D\mathbf{a}\mathbf{b}$, but if D is an oriented metric (a metric which is not necessarily symmetrical), $G\mathbf{a}\mathbf{b} = D\mathbf{a}\mathbf{b} + D\mathbf{a}\mathbf{b}$. Denoting by $\min k_{\text{inf}}(\mathbf{a}, \mathbf{b})$ the smallest cardinality k for which

$$\inf_{\mathbf{y}_1 \dots \mathbf{y}_i, i \leq k} Day_1 \dots \mathbf{y}_i \mathbf{b} = \inf_{\mathbf{y}_1 \dots \mathbf{y}_i, i \in \mathbb{N}} Day_1 \dots \mathbf{y}_i \mathbf{b},$$

we can have three cases: $\min k_{\text{inf}}(\mathbf{a}, \mathbf{b}) = 0$ (this will hold for all \mathbf{a}, \mathbf{b} if D is an oriented metric), $0 < \min k_{\text{inf}}(\mathbf{a}, \mathbf{b}) < \infty$, and $\min k_{\text{inf}}(\mathbf{a}, \mathbf{b}) = \infty$. In the latter case, with some additional assumptions about \mathfrak{S} , $G\mathbf{a}\mathbf{b}$ may coincide with the infimum of lengths of all closed arcs containing \mathbf{a}, \mathbf{b} .

Psychometric Increments as Dissimilarity Functions

We begin by briefly recapitulating the basic conceptual setup of our theory (Dzhafarov & Colonius, 2005a). A *discrimination system* is $(\mathfrak{S}_1^*, \mathfrak{S}_2^*, \psi^*)$ where $\mathfrak{S}_1^*, \mathfrak{S}_2^*$ are stimuli in, respectively, first and second *observation areas* (e.g., stimuli presented first and stimuli presented second); $\psi^* : \mathfrak{S}_1^* \times \mathfrak{S}_2^* \rightarrow [0, 1]$ is a probability-of-different function (see Dzhafarov & Colonius, 2005a, b and in press a,b, for a variety of meanings of the notions “observation area” and “same-different”). A *reduced discrimination system* is $(\mathfrak{S}_1, \mathfrak{S}_2, \tilde{\psi})$, obtained from $(\mathfrak{S}_1^*, \mathfrak{S}_2^*, \psi^*)$ by lumping together *psychologically equal* stimuli in each of the observation areas: \mathbf{a} and \mathbf{b} in \mathfrak{S}_1^* (or in \mathfrak{S}_2^*) are equal if $\psi^* \mathbf{a}\mathbf{y} = \psi^* \mathbf{b}\mathbf{y}$ (resp., $\psi^* \mathbf{x}\mathbf{a} = \psi^* \mathbf{x}\mathbf{b}$) for all $\mathbf{y} \in \mathfrak{S}_2^*$ (resp., $\mathbf{x} \in \mathfrak{S}_1^*$).

The cornerstone of our theory is the *Law of Regular Minimality*: for any (reduced) discrimination system $(\mathfrak{S}_1, \mathfrak{S}_2, \tilde{\psi})$, and for any $\mathbf{a} \in \mathfrak{S}_1$ and $\mathbf{b} \in \mathfrak{S}_2$, $\arg \min_{\mathbf{y} \in \mathfrak{S}_2} \tilde{\psi} \mathbf{a}\mathbf{y}$ and $\arg \min_{\mathbf{x} \in \mathfrak{S}_1} \tilde{\psi} \mathbf{x}\mathbf{b}$ are well-defined functions, and

$$\arg \min_{\mathbf{y} \in \mathfrak{S}_2} \tilde{\psi} \mathbf{a}\mathbf{y} = \mathbf{b} \iff \arg \min_{\mathbf{x} \in \mathfrak{S}_1} \tilde{\psi} \mathbf{x}\mathbf{b} = \mathbf{a}. \quad (5)$$

[$\arg \min_x f(\dots, x, \dots)$ is the value of x at which f reaches its global minimum.] Stimuli \mathbf{a}, \mathbf{b} related to each other by (5) are called each other’s *Points of Subjective Equality*, PSE. This law was first formulated in Dzhafarov (2002d) and was later shown to have unexpectedly strong consequences for a variety of issues, including the (im)possibility of modeling same-different discriminations by random images selectively attributable to stimuli being compared (Dzhafarov, 2003a-c). We view the Law of Regular Minimality as the most fundamental law of sensory discrimination (for empirical evidence and detailed discussion, see Dzhafarov 2002d; Dzhafarov & Colonius, 2005a, in press a).

Due to Regular Minimality the stimuli in $\mathfrak{S}_1, \mathfrak{S}_2$ can always be relabeled so that any two mutual PSEs are assigned identical labels (for details, see Dzhafarov & Colonius, 2005a). Denoting the set of these common labels by \mathfrak{S} , we transform $(\mathfrak{S}_1, \mathfrak{S}_2, \tilde{\psi})$ into a *canonical discrimination system* (\mathfrak{S}, ψ) , in which Regular Minimality holds in its simplest, canonical form:

$$\arg \min_{\mathbf{y} \in \mathfrak{S}} \psi \mathbf{a}\mathbf{y} = \mathbf{a} \text{ and } \arg \min_{\mathbf{x} \in \mathfrak{S}} \psi \mathbf{x}\mathbf{b} = \mathbf{b}. \quad (6)$$

Dealing with a canonical system (\mathfrak{S}, ψ) one should keep in mind that the physical identity of a stimulus is encoded by its label and its observation area (ordinal position within a pair); in particular, the two \mathbf{a} 's in (\mathbf{a}, \mathbf{a}) may be physically different.

Due to (6) the *canonical psychometric increments* $\Psi^{(1)}\mathbf{ab} = \psi\mathbf{ab} - \psi\mathbf{aa}$ and $\Psi^{(2)}\mathbf{ab} = \psi\mathbf{ba} - \psi\mathbf{aa}$ satisfy (1), i.e., they are deviation functions. We obtain the present (“universal”) version of Fechnerian Scaling by assuming, in addition, that $\Psi^{(1)}$ and $\Psi^{(2)}$ are dissimilarity functions, and defining the Fechnerian metric in \mathfrak{S} as the metric G induced by either of them in accordance with (4). It is irrelevant which of the two, $\Psi^{(1)}$ or $\Psi^{(2)}$, is used, they induce one and same metric G (in the previous versions of our theory this proposition was called the Second Main Theorem of Fechnerian Scaling). In terms of the (reduced) discrimination system $(\mathfrak{S}_1, \mathfrak{S}_2, \tilde{\psi})$ this means that if $\mathbf{a}', \mathbf{b}' \in \mathfrak{S}_2$ are the PSEs for, respectively, $\mathbf{a}, \mathbf{b} \in \mathfrak{S}_1$, then the subjective distance between \mathbf{a} and \mathbf{b} in \mathfrak{S}_1 is the same as the subjective distance between \mathbf{a}' and \mathbf{b}' in \mathfrak{S}_2 . The topologies based on open $\Psi^{(1)}$ -balls and open $\Psi^{(2)}$ -balls coincide, and both ψ and G are (uniformly) continuous in this topology.

Fechnerian Scaling with Transformed Probabilities

One of the open problems in our theory (or at least in the interpretation of Fechnerian distances as subjective distances) is that all Fechnerian computations applicable to $(\mathfrak{S}_1^*, \mathfrak{S}_2^*, \psi^*)$ are equally applicable to $(\mathfrak{S}_1^*, \mathfrak{S}_2^*, T\psi^*)$, where $T\psi^*$ is any monotone continuous transformation of ψ^* . For a discussion of how this problem can be tackled by, e.g., relating it to a response bias model, see Dzhafarov & Colonius, 2005a, in press b). Here we only mention this issue to point out that the psychometric increments derived from $T\psi^*$ may very well fall outside $[0, 1]$ or another finite interval of reals, because of which the notion of a dissimilarity function has to be generalized from $D : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, 1]$ to $D : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}^+$. The generalization is achieved by supplementing property (3) with

$$D\mathbf{a}x_{k_n}^n \rightarrow \infty \implies D\mathbf{a}x_1^n \dots x_{k_n}^n \rightarrow \infty \quad (7)$$

and predicating property (2) on the condition “if $\{\mathbf{a}_n\}_{n \in \mathbb{N}}$ and $\{\mathbf{b}_n\}_{n \in \mathbb{N}}$ are bounded sequences.” A sequence $\{\mathbf{a}_n\}_{n \in \mathbb{N}}$ is called bounded (above) if, for some \mathbf{a} and M , $D\mathbf{a}\mathbf{a}_n \leq M$. The theory essentially remains the same, except that the convergence and uniform continuity statements now generally have to be related to appropriate boundedness constraints.

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