THE DISTRIBUTION OF PERIODIC AND APERIODIC PATTERN EVOLUTIONS IN RINGS OF DIFFUSIVELY COUPLED MAPS

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This paper reports histograms showing the detailed distribution of periodic and aperiodic motions in parameter-space of one-dimensional lattices of diffusively coupled quadratic maps subjected to periodic boundary conditions. Particular emphasis is given to the parameter domains where lattices support traveling patterns.

1. Introduction

This work reports a systematic investigation of the distribution of periodic and aperiodic pattern-evolution in rings of diffusively coupled quadratic maps. Our main motivation here is the wish to investigate whether lattices of coupled maps can be effectively used to simulate a plethora of phenomena associated with climatic variability and change. Presently, simulations of these phenomena reduce, essentially, to the solutions of large arrays of coupled differential equations, a quite computer-demanding task. As may be recognized from the contributions in books edited by Kaneko [1993b] and Schuster [1999], coupled map lattices (CMLs) are known to reproduce many interesting dynamical aspects associated with spatio-temporal complexity for large classes of phenomena. Some 30 years ago, meteorological studies awakened great interest in the investigation of using discrete maps as simple models of complex temporal dynamics. It is our hope that lattices of coupled maps might turn out to be an efficient framework for a variety of applications, from the interpolation of missing-field data, or the generation of synthetic series of climate elements, to the actual simulation of large-scale phenomena involving several time-scales, e.g. the El-Niño Southern Oscillation (ENSO) and the North Atlantic Oscillation.

What is presently known about the dynamics of CMLs that could be interesting for meteorological and climatic applications? A survey of the literature shows that most of the work so far concentrates on the so-called globally coupled maps, where the essential contribution driving locally each oscillator arises from the mean-field of all other oscillators, the environment. Such approach is strongly influenced by the methodology of statistical mechanics. For the application that we have in mind, however, diffusively coupled maps seem to be a proper

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starting point. Thus, we ask for what is known about the dynamics of diffusively coupled lattices.

The generic problem of understanding the dynamical evolution of CMLs is a difficult and intricate one and, despite all intensive work so far expended, much still remains to be done. By far, the most popular model today is based on coupling quadratic (logistic) maps (see e.g. [Franceschini & Vernia, 1998; He et al., 1997; Kaneko, 1989; Lemaitre et al., 1996; Willeboorse, 1993; Zanette, 1999] and several references therein). To represent local oscillators by quadratic maps seems a bit too simplistic for applications since a quadratic map is capable of supporting no more than just one single finite attractor for each value of the parameter. However, even though more

Fig. 1. Illustrative examples of the five classes of modular time-evolution found in CMLs: (a) Static evolution (class $S$); (b) Positive evolution (class $P$); (c) Negative evolution (class $N$); (d) Hesitating evolution (class $H$); (e) Chaotic evolution (class $C$).
realistic models call for the allowance of local multi-stability, in this paper we wish to reconsider a number of open questions concerning the dynamics of coupled quadratic maps, aiming at an understanding of a specific aspect of relevance for meteorological applications, namely wave-propagation phenomena in one-dimensional lattices of coupled maps subjected to periodic boundary conditions, i.e. in rings of maps.

Wave behavior is crucially important in geophysical fluid dynamics and, in particular, for weather and climate variability; planetary waves in the middle latitudes (e.g. Rossby waves), equatorial waves in the atmosphere and oceans (e.g. the Kelvin and mixed Rossby gravity waves) are supporting mechanisms of important phenomena like for instance ENSO [Indeje et al., 2000] and its connections to other tropical and middle latitude climate anomalies [James, 1994; Indeje et al., 2000].

Quasi-periodic oscillations (e.g. the intraseasonal Madden–Julian oscillation [Madden & Julian, 1971; Wang et al., 1996]) and weather regimes (e.g. blocking) are also manifestations of wave behavior that play an important role in climatic variability.

Traveling waves in CMLs were observed a few years ago by Oppo and Kapral [1986] and by Kaneko [1992, 1993a, 1993b] in lattices with diffusive coupling. Following suitable transients, the ring of coupled maps (oscillators) displays asymptotic configurations, to be referred here as “patterns”, which are seen to evolve either periodically or not. In a previous work [Lind et al., 2001] we argued that this dichotomic division might be conveniently subdivided into five primary classes of time-evolutions, three periodic and two aperiodic. The three periodic time-evolutions correspond to patterns that travel along a conventional “positive” direction in the ring (class P), along the contrary “negative” direction (class N) or remain static in space (class S), i.e. may alternate the configuration periodically in time but do not move spatially. When no discernable periodicity is observed as time evolves we call the time-evolution “chaotic” (class C). Among chaotic evolutions we distinguish a particularly interesting one, which we call “hesitation” (class H), in which the system evolves periodically, either as class N, S, or P, during a lapse of time, when it then changes more or less abruptly to a different sort of periodic evolution. In other words, the system has its periodic time-evolution intermittently changed from one periodic evolution to another, with the change of class happening either mediated by a chaotic evolution or just abruptly. Class H corresponds to the interesting atmospheric phenomenon known as vacillation [Lorenz, 1963; James, 1994].

Figure 1 shows illustrative examples of the five classes of time-evolution. As shown previously [Lind et al., 2001], a simple way to discriminate all classes of time-evolution is by looking at return maps constructed from the time series representing the time-evolution of any of the local oscillators.

A decisive advantage of introducing a broad classification of the time-evolutions is to allow the construction of “phase diagrams” indicating domains characterized by similar behaviors. This type of diagram is the main result reported here. As far as we know, this is the first classification of all possible behaviors over the entire range of parameter values. In addition, we also investigate the relative distributions of periodic patterns observed on the lattice, as a function of local parameters, coupling strength and lattice size.

2. Quantitative Characterization of Time-evolutions

The ring of diffusively coupled maps considered here is ruled by the following equation of motion

\[ x_{t+1}(i) = f(x_t(i)) + \varepsilon \left[ f(x_t(i)) - \frac{f(x_t(i+1)) + f(x_t(i-1))}{2} \right] \]

(1)

where, as usual, \( f(x) \) represents the local oscillator, \( x_t(i) \) is the value of a continuous variable \( x \) at discrete times \( t \) and at the \( i \)th site on the lattice. To use a ring means to impose periodic boundary conditions, i.e. \( x_t(L+1) = x_t(1) \).

The parameter \( \varepsilon \) varies in the interval \( 0 \leq \varepsilon \leq 1 \) and measures the relative coupling strength among neighboring sites. As Eq. (1) shows, the dynamics is controlled by two different contributions: a local one, and another, nonlocal, containing all dependencies from nearby sites and collecting all terms depending on the coupling strength \( \varepsilon \).

As mentioned, here we consider rings of quadratic (logistic) maps

\[ x_{t+1} = f(x_t) = 1 - ax_t^2 \]

(2)
a controlling the local dynamics. For convenience, Fig. 2 shows the familiar bifurcation diagram characterizing the behavior of the local oscillators in the absence of coupling. This figure will be useful for the discussions to follow.

After a suitable transient time, the variable \( x_t(i) \) of each local oscillator \( i \) changes in its own characteristic way, depending on the values of \( a, \varepsilon \) and the initial conditions. For any given instant \( t \) after the transient, we call the set \( \{x_t(i)\} \) a pattern on the lattice. Then, to define periodicity, one needs to investigate whether such patterns reappear on the lattice or not as time evolves.

We say that a pattern \( \{x_t(i)\} \) “reappears” or “repeats” in the lattice if, for some fixed integer \( d \), there is an instant \( t' \) in the future such that

\[
x_t'(i + d) = x_t(i), \quad \text{for all } i, \quad (t' > t).
\]

A pattern is periodic with period \( \tau \) if Eq. (3) is satisfied for \( t' = t + m\tau \), for all \( m = 0, 1, \ldots \). Notice that the sum \( i + d \) in Eq. (3) is to be taken modulo \( L \) whenever one works with periodic boundary conditions on a lattice of finite size \( L \). Accordingly, \( d \) may only assume values lying in the interval \( 0, 1, \ldots, L-1 \). When \( d = 0 \) one has pattern repetition without displacement along the lattice. Any nonzero value of \( d \) signals the occurrence of a spatial displacement.

As a function of time, the asymptotic patterns seen in the ring may evolve either periodically or aperiodically. “Periodic” variation means that the same pattern repeats after \( \tau \) time steps. Thus, it is necessary to discriminate unambiguously where in the lattice the pattern reappears.

To this end, we start considering the period \( \tau \). First, we fix as a reference pattern the configuration \( \{x_0(i)\} \) emerging after the transient time. This reference pattern is then compared with all subsequent patterns \( \{x_t(i)\} \), \( t = 1, 2, \ldots \), producing an auxiliary quantity

\[
T(t) = \sum_{i=1}^{L} [x_t(i) - x_0(i)]^2.
\]

The utility of this function lies in the fact that \( T(t_\tau) \approx 0 \) for all instants \( t_\tau \) for which the pattern reappears. The reason for not having \( T(t_\tau) \) identically equal to zero here is that local values of moving patterns change very slowly, as time evolves. Only for class \( S \) time-evolutions one finds \( T(t_\tau) = 0 \) within numerical precision. We define the period \( \tau \) of a pattern as the first \( t_\tau \) that minimizes \( T(t) \).

Having determined \( \tau \), instead of considering the time-evolution on the “fast scale” \( t \), we then use the “slow scale” \( m \), comparing snapshots taken modulo \( \tau \), i.e. only between instants \( t = m\tau \) which are multiples of \( \tau \).

To determine \( d \) in Eq. (3) we introduce an auxiliary function \( S(j, m) \) defined by

\[
S(j, m) = \sum_{i=1}^{L} [x_{m\tau}(i + j) - x_0(i)]^2.
\]

Notice that for \( j = 0 \) the above \( S(0, m) \equiv T(m\tau) \). Additional details concerning \( T(t) \) and \( S(j, m) \) are given by Lind et al. [2001].

For each value of \( m \) we define \( d(m) \) as the first \( j \) that minimizes \( S(j, m) \). As it is not difficult to see, \( d(m) \) must satisfy the condition \( 1 \leq d(m) \leq L \) for all \( m \). Accordingly, the sum \( i + j \) in Eq. (5) is taken modulo \( L \). For nonmoving patterns (class \( S \)), one finds \( d(m) = 0 \) for all \( m = 1, 2, 3, \ldots \). Moving patterns might be easily recognized by checking when the first nonzero value of \( d(m) \), say \( D \), occurs. When this happens, we update the reference configuration \( x_0(i) \) in \( S(j, m) \) by then taking on the new configuration \( x_{m\tau}(i + D) \). For classes \( P \) and \( N \), the quantity \( D \) is always constant. We assume here that if \( 1 \leq D < L/2 \) then the pattern moves positively, i.e. it moves in the direction of increasing indices. Otherwise, for \( L/2 < D \leq L-1 \), the pattern moves negatively. Furthermore, assuming patterns to move with a maximum speed of 1 site per timestep, one finds that positive displacements imply \( 1 \leq D \leq \tau \) while negative ones imply \( L - \tau \leq D \leq L-1 \).

Table 1 summarizes the characterization described so far. Up to this point we used \( \tau \) and \( d \) only to characterize and discriminate the five classes of time-evolutions. However, \( \tau \) and \( d \) may be also used to compute the velocity \( v \) of moving patterns (classes \( P \) and \( N \)).
Table 1. The five classes of time-evolution classified according to the period $\tau$ and the displacement “indicator”. $L$ denotes the lattice size. (See text for details.)

<table>
<thead>
<tr>
<th>Time-evolution</th>
<th>Class</th>
<th>$\tau$</th>
<th>$d, D$</th>
<th>$s$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Periodic</td>
<td>$S$</td>
<td>Constant</td>
<td>$d = 0$</td>
<td>$s = 0$</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$P$</td>
<td>Constant</td>
<td>$1 \leq D \leq \tau$</td>
<td>Constant</td>
<td>$\sigma = 1$</td>
</tr>
<tr>
<td></td>
<td>$N$</td>
<td>Constant</td>
<td>$L - \tau \leq D \leq L - 1$</td>
<td>Constant</td>
<td>$\sigma = -1$</td>
</tr>
<tr>
<td>Aperiodic</td>
<td>$H$</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
</tr>
<tr>
<td></td>
<td>$C$</td>
<td>Undefined</td>
<td>Undefined</td>
<td>Undefined</td>
<td>Undefined</td>
</tr>
</tbody>
</table>

The magnitude of displacement between two consecutive values of $D$ is

$$s = \frac{L - |L - 2D|}{2},$$

from which we obtain the velocity

$$v = \sigma \frac{s}{\tau},$$

where the sign $\sigma$ is given by the sign of $L - 2D$.

We call attention to the fact that time-evolutions will be necessarily aperiodic whenever the value of $D$ varies with $m$. The distinction between class $H$ and $C$ has many subtleties and requires a detailed discussion that will be presented elsewhere.

3. Results

The results reported here were obtained for a lattice of $L = 64$ sites, the same lattice size considered in two earlier papers of Kaneko [1992, 1993a] where he investigates traveling patterns in CMLs. Following his lead, we also initialized our lattices randomly, using a uniform amplitude distribution in the interval $[-1, 1]$.

Before starting large-scale simulations one needs to determine two important auxiliary quantities: (i) a reasonable size for the “suitable transients” necessary for the dynamics to approach asymptotic attractors and, (ii) the quantity of initial conditions to be considered in order to have statistically meaningful results. In all computations we used a transient of 150,000 timesteps because, for a number of situations, the transient of 50,000 timesteps used by Kaneko [1992, 1993a, 1993b] was found to be too small.

To determine a statistically representative number of initial conditions, we computed the relative distribution of the five classes, $S, P, N, H, C$, for two different samplings, with 100 and 600 initial conditions, respectively. This was done for $\varepsilon = 0.5$ and for 100 values of $a$. As Fig. 3 shows, the relative distribution of the classes for 100 initial conditions is qualitatively similar to that obtained with the considerably larger sample of 600 initial conditions.

Figure 3, done for the same parameters and conditions previously used by Kaneko, corroborates his finding that static time-evolutions dominate in the interval $1.0 \leq a \leq 1.4$. In addition, Fig. 3 shows that traveling waves exist predominantly for $1.6 \leq a \leq 1.85$ and that waves moving positively (class $P$) and negatively (class $N$) arise with about the same frequency over the entire parameter range, facts more or less already implicit in the results of Kaneko. Additionally, we further observe a small quantity of traveling waves over an interval around $a = 1.4$ which, as might be recognized from Fig. 2, appears after the $2 \rightarrow 4$ bifurcation of the local map.

3.1. Distribution of time-evolutions in parameter space

Figure 3 contains the percentual distribution of aperiodic and periodic time-evolutions for just one single value of $\varepsilon$. A general view of the same distribution over the full interval $0 \leq \varepsilon \leq 1$ of the coupling strength is shown in Fig. 4.

In both graphs at the top, one easily distinguishes a plateau dominated by periodic time-evolutions, approximately for $a < a_s \approx 1.4$ for any $\varepsilon$. This plateau extends further down for $a < 1$, into the domain of period 1 of the quadratic map.

For $a > a_s$ there is a clear predominance of aperiodic time-evolutions, except for three roughly parallel “tongues of periodicity”, approximately delimited by $1.6 \leq a \leq 1.9$ and $0.4 \leq \varepsilon \leq 1.0$. 

Fig. 3. Percentual distribution of periodic and aperiodic time-evolutions as seen after 150,000 timesteps for a lattice with $L = 64$ sites, $\varepsilon = 0.5$, and classified according to the criteria in Sec. 2. The column on the left (right) shows the distribution for a sample of 100 (600) initial conditions. (a) Aperiodic time-evolutions (i.e. $H + C$); (b) Periodic time-evolutions (i.e. $S + P + N$); (c) Class $S$; (d) Class $P$; (e) Class $N$.

An additional tongue appears further down, around $0.15 \leq \varepsilon \leq 0.2$ and $a \geq 1.6$.

Apart from discriminating aperiodic and periodic behaviors, Fig. 4 also discriminates the three flavors of periodic time-evolutions, namely static, positive and negative evolutions. Noteworthy is the almost perfect coincidence of the histograms corresponding to the frequency and the location of both types of traveling waves (classes $P$ and $N$).

Comparing the three histograms displaying the relative distribution of the periodic classes we can observe two main features. First, in the periodicity plateau, one sees a subregion where moving patterns exist: $1.3 \leq a \leq a_s$ and $0.4 \leq \varepsilon \leq 1$. Moving patterns in this region travel with smaller velocities than those observed elsewhere (see discussion below, Sec. 3.2). Second, for $a > a_s$, all three periodicity classes exist within the tongues. Furthermore, in the region around the tongues, there is an overwhelming presence of static time-evolutions which only compete here with aperiodic evolutions. In other words, there are essentially no traveling patterns in such domain.

Kaneko says that there are no traveling waves for $\varepsilon < 0.4$. However, Fig. 4 clearly shows the existence of traveling patterns for $\varepsilon < 0.4$, in a relatively small domain. The existence of traveling waves in this domain were corroborated by space-time
Fig. 4. Distribution of classes in parameter space measured during 50,000 timesteps after a transient of 150,000 timesteps, for a lattice with \( L = 64 \). The \( a \times \varepsilon \) mesh is \( 51 \times 51 \) with 100 initial conditions computed for each point.

Diagrams like those presented in Fig. 1. Moreover, traveling waves in such a region were also observed by Oppo and Kapral, although for a particular case and for a slightly different diffusive coupling.

All in all, Fig. 4 summarizes what happens over practically all the range of parameters. It significantly complements the earlier results of Kaneko which were restricted mainly to the tongues of periodicity and to the behavior for \( \varepsilon = 0.5 \), essentially. The very thin “line” showing class \( P \) time-evolutions for \( a \sim 1.05 \) is neglected because it is formed by a few isolated cases. In fact, when
Fig. 5. Logarithm of the percentual distribution of velocities obtained for 100 values of \( a \) sampled over 100 different initial conditions for each \( a \). (a) \( \varepsilon = 0.5 \), (b) \( \varepsilon = 0.7 \), (c) \( \varepsilon = 0.9 \).
increasing the sampling from 100 to 600 initial conditions, the absolute frequency observed in that region remains the same.

3.2. **Velocity distribution of traveling patterns**

Figure 5 shows the velocity distribution of traveling patterns as a function of $a$ for three values of $\varepsilon$ larger than 0.4, namely $\varepsilon = 0.5, 0.7, 0.9$. These representative values were chosen because, as may be seen from Fig. 4, the domain $\varepsilon > 0.4$ is the one containing the majority of traveling patterns. Qualitatively similar results are obtained for the tongue of periodicity located at $\varepsilon \sim 0.1$.

From Fig. 5 one recognizes the existence of a regime, centered roughly around $a = 1.4$, where low velocities dominate. This behavior is characteristic of the aforementioned subregion of the periodicity plateau (see discussion of Fig. 4). On the other hand, in the interval $1.6 < a < 1.9$, lying within the tongues of periodicity, it is also possible to find velocities of relatively high magnitudes. Thus, one clearly recognizes the existence of two velocity regimes. For a given $a$, the quantity of traveling patterns in both regimes decreases as the magnitude of the velocity increases. Moreover, both velocity regimes are still dominated by static patterns ($v = 0$) and it is only the regime of high velocities that seems to display the structure of “bands” emphasized by Kaneko.

Figure 6 shows an illustrative example of the velocity distribution as a function of $\varepsilon$ for $a = 1.4$, i.e. within the low-velocity regime. The magnitude of the velocity increases with $\varepsilon$.

![Logarithm of the percentual distribution of velocities obtained for 100 values of $\varepsilon$ sampled over 100 different initial conditions, for $a = 1.4$.](image)

Fig. 6.
For six representative values of $a$, Fig. 7 shows on a logarithmic scale the velocity distribution as a function of $\varepsilon$. In all cases, the velocities shown fall into the high-velocity regime. In this regime, the abundance of traveling patterns goes down as the magnitude of the velocity increases. Furthermore, one sees a spread of the velocities when the local nonlinearity increases. This behavior may be more easily recognized from Fig. 8, which shows a $v \times \varepsilon$ projection of the tridimensional graphs. From these projections one sees that within the tongues of periodicity ($\varepsilon > 0.4$) there are several intervals containing ellipsoidal-shaped velocity distributions. As the local nonlinearity $a$ grows, one sees a significant increase of the velocity magnitude in each ellipsoid.
One interesting feature seen for all values of $a$ is the existence of particular values of $\varepsilon$ separating ellipsoids, $\varepsilon$-boundaries, at which one finds one of two behaviors: either a discontinuity in the velocities when passing from one ellipsoid to another, or a vanishing of the velocity.

Finally, for high nonlinearities ($a \gtrsim 1.9$) one observes the existence of characteristic "parameter thresholds" beyond which there is a sudden disappearance of traveling patterns. This threshold depends on the value of $\varepsilon$ (see Fig. 4). This phenomenon is different from the aforementioned vanishing among ellipsoids.

### 3.3. Histograms of periodicities

Figure 9 shows the relative abundance of periods $\tau$ as a function of $a$ for class $S$ time-evolution. For
Fig. 9. Distribution of the periodicity $\tau$ for class $S$ time-evolution showing period-doubling. Here, $\varepsilon = 0.5$ and sampling as before: 100 initial conditions for each of 100 values, after a transient of 150,000 time-steps. (a) $1 \leq a \leq 2$, (b) $1.35 \leq a \leq 1.5$, (c) $1.25 \leq a \leq 1.28$, (d) same as (c), but recorded after a transient of 600,000 time-steps.

Fig. 10. Period-doubling analogous to that shown in Figs. 9(a) and 9(b), but for class $N$, obtained for same conditions. A similar figure is obtained for class $P$. (a) $1 \leq a \leq 2$, (b) $1.35 \leq a \leq 1.5$. 
$1 < a < 1.25$ one only sees period $\tau = 2$. For $a \simeq 1.25$ there is a sudden appearence of period $\tau = 4$ with a simultaneous vanishing of period 2. As may be seen from Fig. 9(d), the transition is robust since it survives a four times longer transient. From the bifurcation diagram in Fig. 2 one sees that $a = 1.25$ is the value at which the local map bifurcates from period 2 to 4. Similar bifurcations also occur for doublings involving higher periodicities, as may be seen in Fig. 9(b). Therefore, one sees a somewhat surprising fact, namely that despite the relatively high coupling of $\varepsilon = 0.5$, the lattice seems to follow the behavior of the local oscillator as if uncoupled. Recall that aperiodic time-evolutions begin to appear in the uncoupled map only for $a > a_\infty$ where $a_\infty \simeq 1.401155189\ldots$

is the accumulation point of the $2^t$ cascade. For precise estimates of the accumulation point for this and other cascades see [Beims & Gallas, 1997].

The purpose of Fig. 10 is to illustrate the fact that period-doublings in lattices of coupled maps occur not only for static patterns, class $S$, but also for both types of traveling patterns, either $P$ or $N$.

Notice that the relative emptiness seen in Figs. 9 and 10 is a simple consequence of the fact that for those parameter values there are no traveling patterns (compare with Fig. 4).

4. Conclusions

This paper summarizes and quantifies basic aspects of the dynamical behavior of one-dimensional rings

![Graphs showing pattern distributions in CMLs](image)

Fig. 11. Relative abundance of aperiodic and periodic time-evolutions as a function of the lattice size, for $\varepsilon = 0.5$, $16 \leq L \leq 128$, sampled over 100 initial conditions. (a) Aperiodic, i.e. $H + C$, (b) Periodic, i.e. $S + P + N$, (c) $S$, (d) $P$, (e) $N$. 

of diffusively coupled quadratic maps, with particular emphasis on traveling patterns. Our Fig. 4 summarizes what happens over practically all the useful range of parameters. It significantly complements the earlier results of Kaneko which were restricted mainly to the tongues of periodicity and to the behavior for $\varepsilon = 0.5$, essentially.

In addition, for representative parameters, we investigated the distribution of velocities. For fixed values of the coupling strength $\varepsilon$ our Fig. 5 corroborates an earlier observation of Kaneko concerning the existence of a structure of bands as the local nonlinearity grows. A particularly interesting new result are the “pulsations” and $\varepsilon$-boundaries observed in Fig. 8, in the band structures of the velocity, when considered along the coupling strength axis, for a given value of $a$.

Figures 9 and 10 provide illustrative examples of period-doubling bifurcations in one-dimensional coupled map lattices. Such bifurcations exist for any class of periodic time-evolution, i.e. either for static or moving patterns.

As discussed elsewhere [Lind et al., 2001], it is important to realize that while the classification of periodic motions into $S$, $P$, $N$ classes is complete, classes $H$ and $C$ may be further subdivided almost ad infinitum. For instance, there is a class of waves that while traveling always in the same direction, may change their speed of propagation. Such patterns belong to class $H$ because of the aperiodic evolution suffered by the system during the relatively short time intervals while the velocity changes. See [Lind et al., 2001] for additional details. One may think of more elaborate time-evolutions, e.g. the propagation of fronts moving simultaneously along negative and positive directions. However, such phenomena were not observed yet and, therefore, are supposed to be relatively rare if they indeed exist in CMLs.

All simulations presented in this paper were obtained for $L = 64$, the same lattice size considered earlier by Kaneko. A natural question is to wonder what might happen for different sizes. To answer this question, Fig. 11 shows the relative distribution of aperiodic and periodic time-evolutions, inclusive of the sub-distributions for static, positive and negative patterns. These distributions were computed for parameter values located in the tongues of periodicity, the region where traveling patterns are abundant. Surprisingly, it is possible to recognize a quite regularly-spaced structure of peaks of varying amplitudes. Furthermore, while periodic behaviors dominate for most values of $L$, there are specific lattice sizes which seem to make it more difficult for traveling patterns to grow and propagate. Despite our attempts of correlating such lattice sizes with the “wavelength” of patterns observed on the lattice, this interesting question remains open.

An additional open problem is the detailed investigation of the structuring of the aperiodic classes, in particular, to see whether transient times alter or not the relative abundance of dynamical behaviors. The discussion of these questions will be presented elsewhere, along with simulations of several sinoptic atmospheric systems.

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