

Structured Hazard Regression

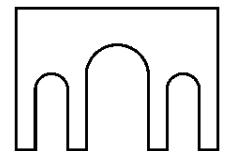
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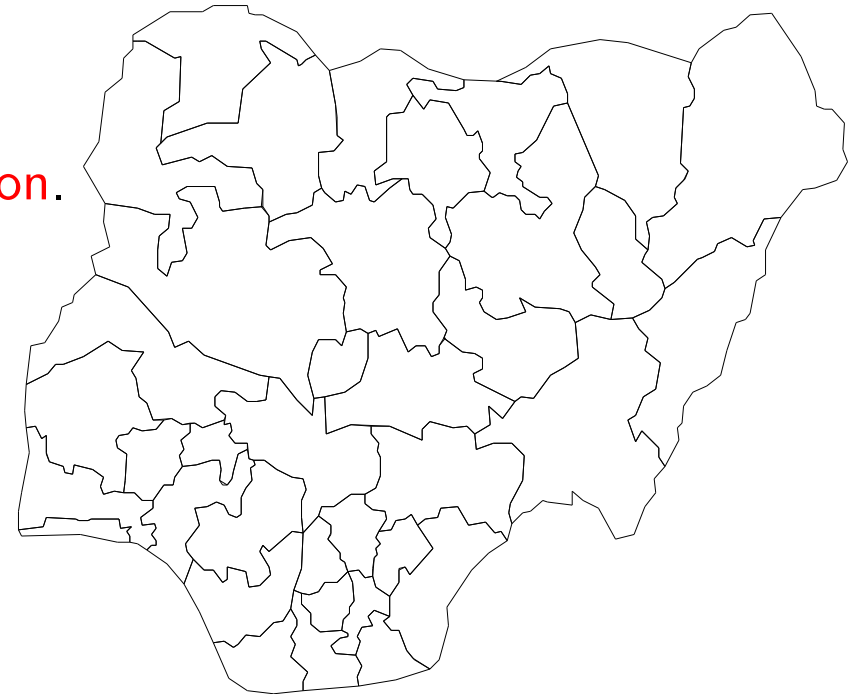
Outline

1. Geoadditive Modelling of Continuous Survival Times.
2. Inferential Concepts: Empirical Bayes vs. Full Bayes.
3. Continuous Time Multi-State Models.

Childhood mortality in Nigeria

- Data from the 2003 Demographic and Health Survey (DHS) in Nigeria.
- **Retrospective questionnaire** on the health status of women in reproductive age and their children.
- Survival time of $n = 5323$ children.
- Numerous covariates including **spatial information**.
- Analysis based on the **Cox model**:

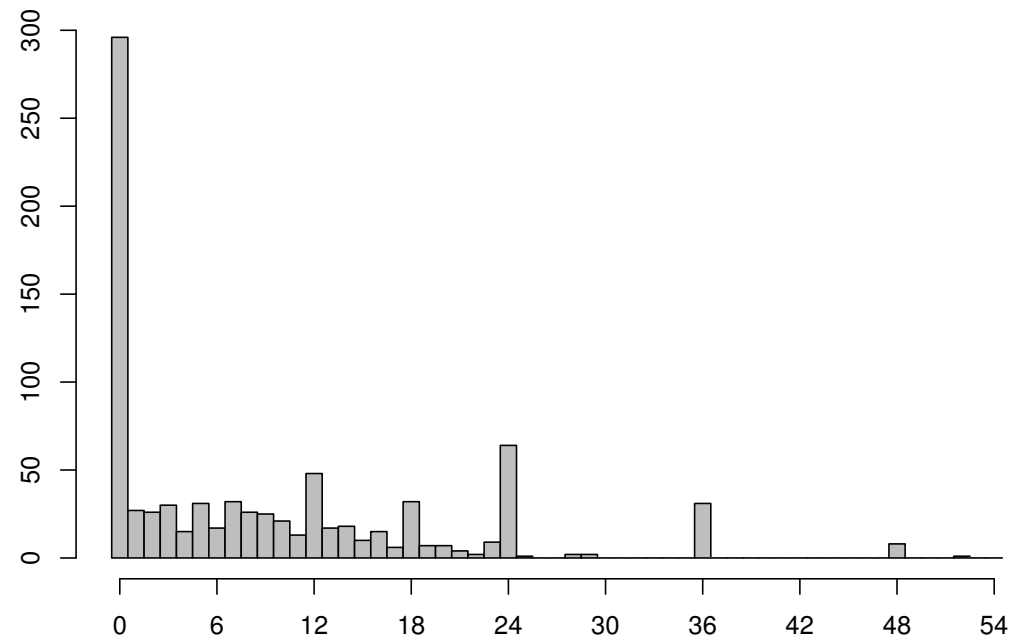
$$\lambda(t; u) = \lambda_0(t) \exp(u'\gamma).$$



- **Limitations** of the classical Cox model:
 - Restricted to right censored observations.
 - Post-estimation of the baseline hazard.
 - Proportional hazards assumption.
 - Parametric form of the predictor.
 - No spatial correlations.
- ⇒ **Structured hazard regression.**

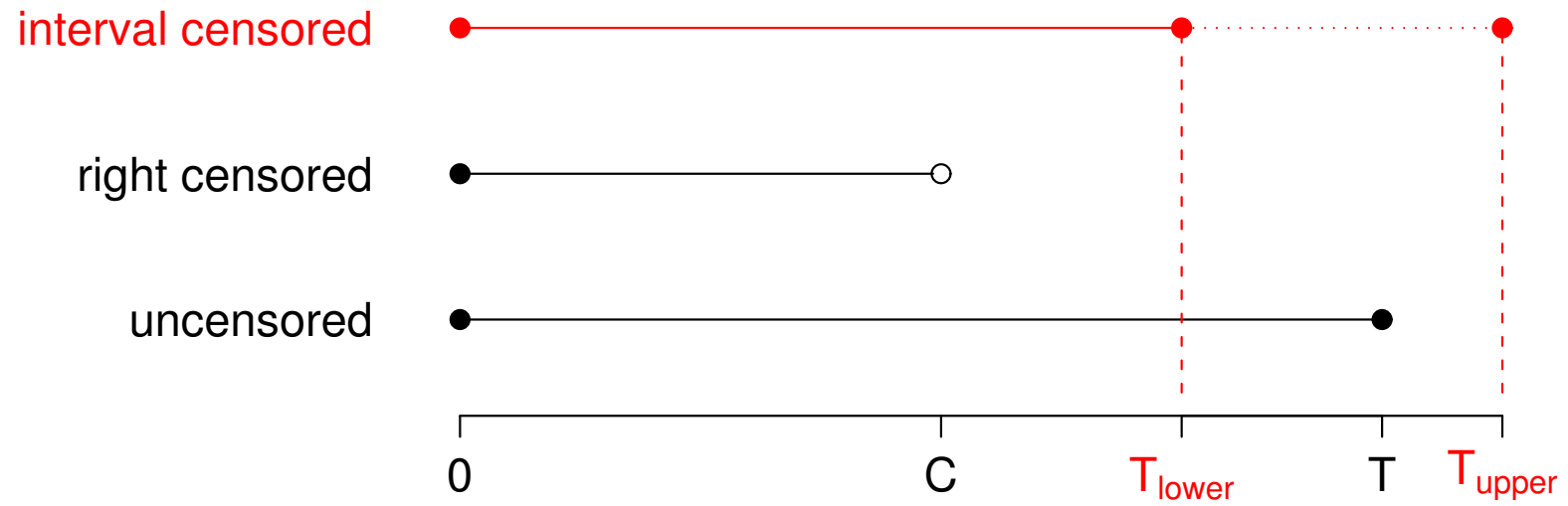
Interval censored survival times

- In theory, survival times should be available in days.
- Retrospective questionnaire \Rightarrow **most uncensored survival times are rounded** (Heaping).



- In contrast: censoring times are given in days.

\Rightarrow Treat survival times as **interval censored**.



- **Likelihood contributions:**

$$\begin{aligned} P(T > C) &= S(C) \\ &= \exp \left[- \int_0^C \lambda(t) dt \right]. \end{aligned}$$

$$\begin{aligned} P(T \in [T_{lower}, T_{upper}]) &= S(T_{lower}) - S(T_{upper}) \\ &= \exp \left[- \int_0^{T_{lower}} \lambda(t) dt \right] - \exp \left[- \int_0^{T_{upper}} \lambda(t) dt \right]. \end{aligned}$$

- Derivatives of the log-likelihood become much more complicated for interval censored survival times.
- **Numerical integration techniques** have to be used in both cases.
- Piecewise constant **time-varying covariates** and **left truncation** can easily be included.

Structured hazard regression

- Introduce a more flexible, **semiparametric hazard rate model**

$$\lambda(t; \cdot) = \exp \left[g_0(t) + \sum_{j=1}^q g_j(t) z_j(t) + \sum_{k=1}^p f_k(x_k(t)) + f_{spat}(s) + u(t)' \gamma \right]$$

where

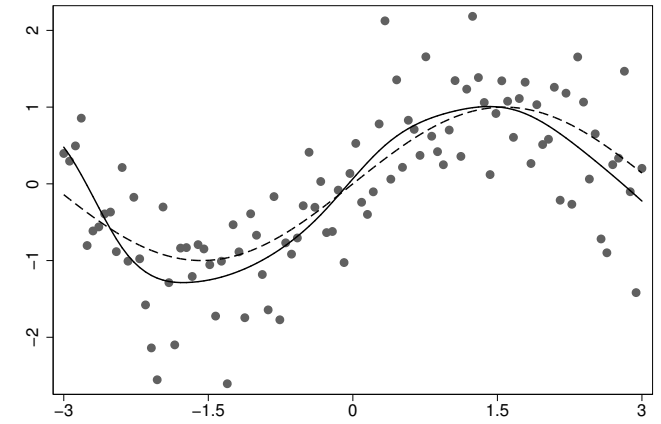
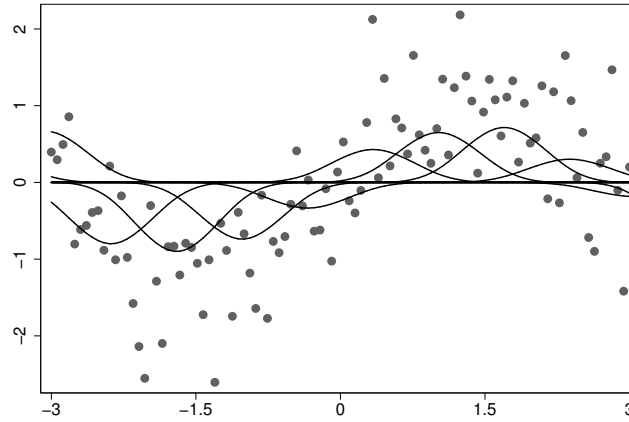
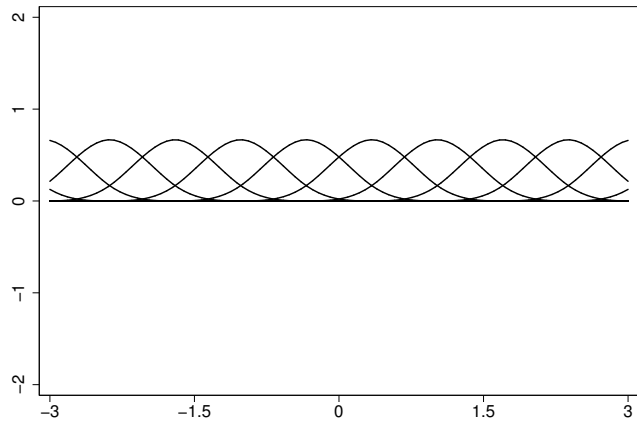
- $g_0(t) = \log(\lambda_0(t))$ is the **log-baseline-hazard**,
- g_j are **time varying effects** of covariates $z_j(t)$,
- f_k are **nonparametric** functions of continuous covariates $x_k(t)$,
- f_{spat} is a **spatial** function,
- $u(t)' \gamma$ are parametric effects.

Model Components and Priors

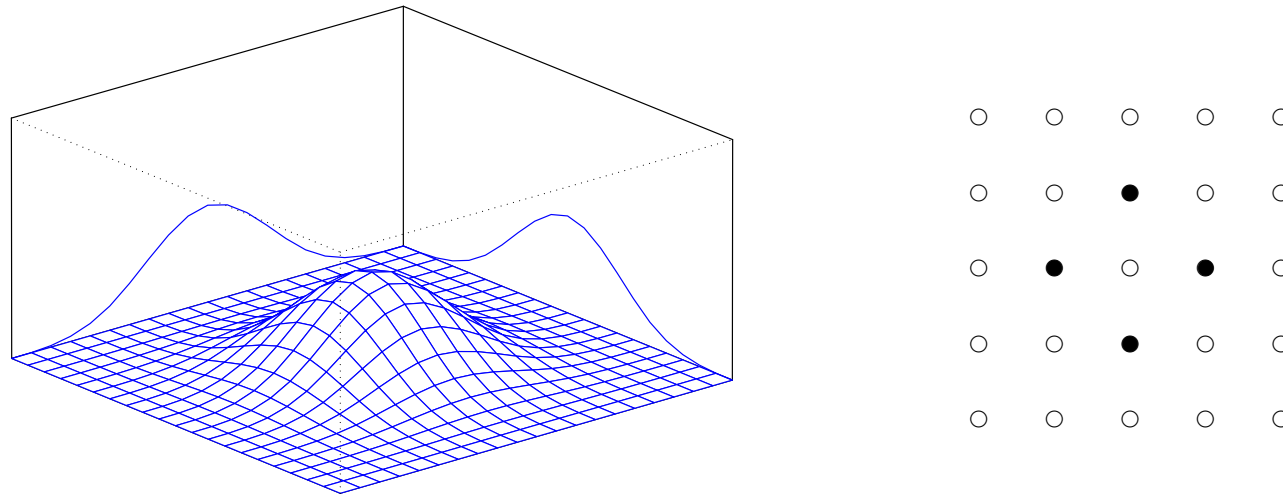
- **Penalised splines** for log-baseline, time-varying effects and nonparametric effects.
 - Approximate g_j (or f_k) by a weighted sum of **B-spline basis** functions

$$f(x) = \sum \xi_j B_j(x).$$

- Employ a large number of basis functions to enable flexibility.
- **Penalise differences** between parameters of adjacent basis functions to ensure smoothness.



- **Bivariate** penalised splines.



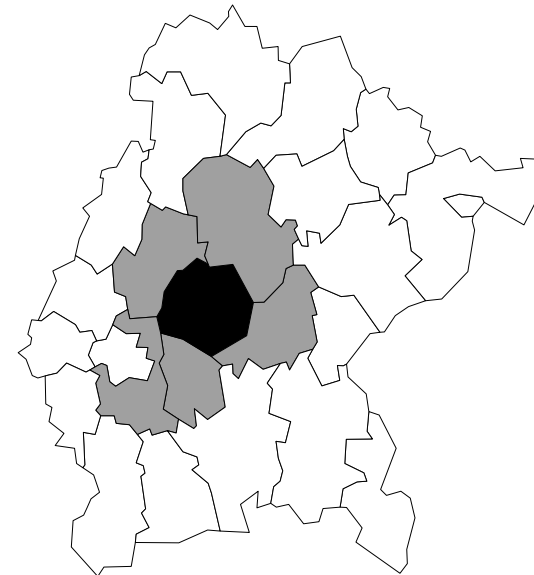
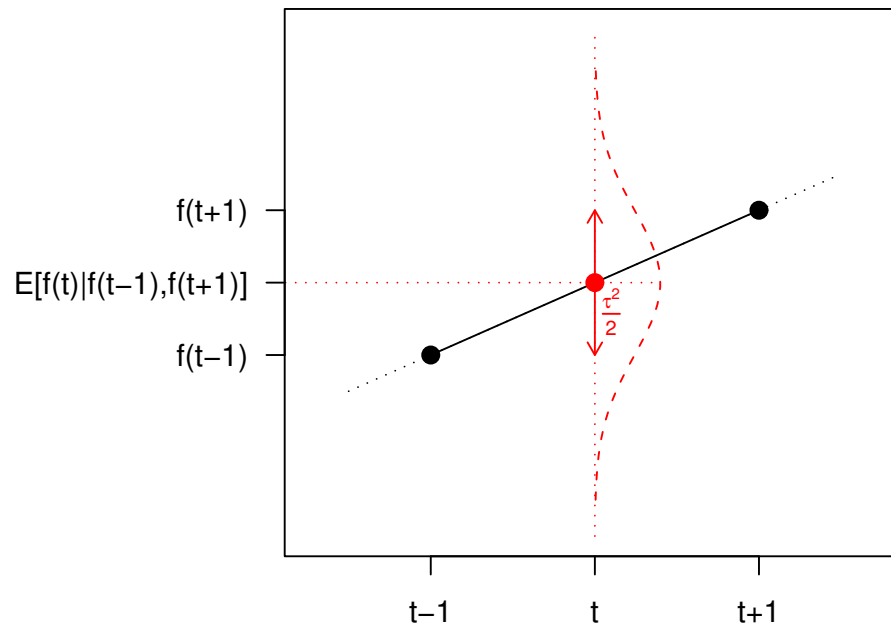
- **Varying coefficient models.**

- Effect of covariate x varies smoothly over the domain of a second covariate z :

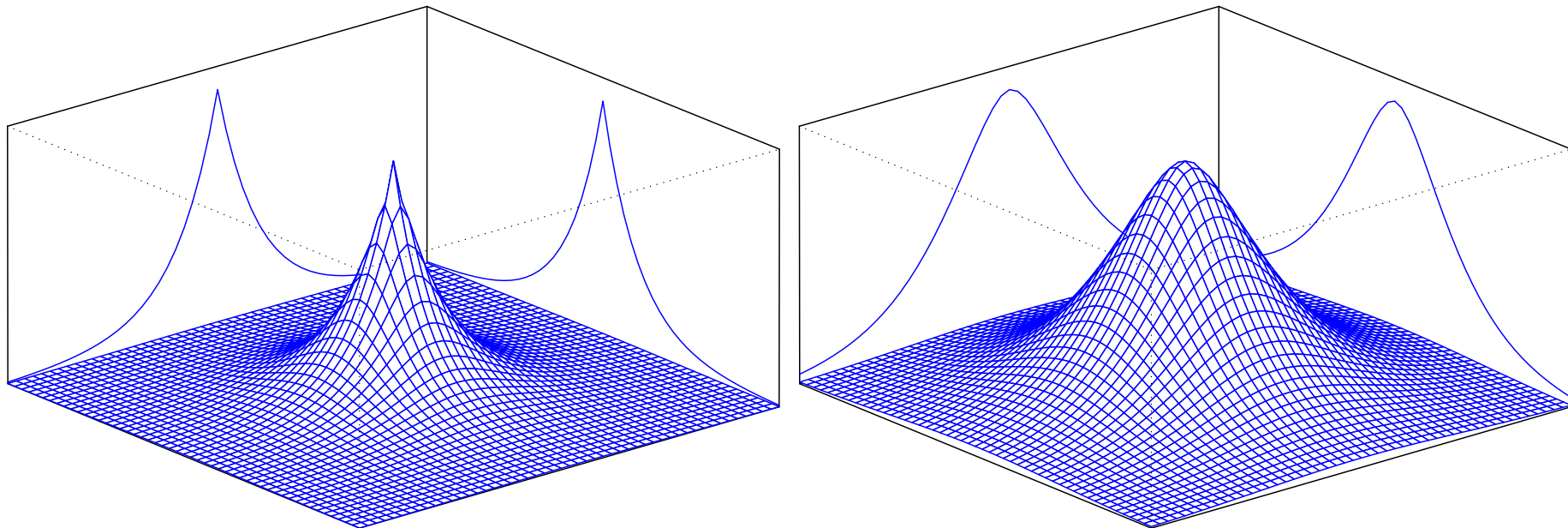
$$f(x, z) = x \cdot g(z)$$

- Survival time as effect modifier \Rightarrow **Time-varying effects** $x \cdot g(t)$.

- Spatial effect for regional data: **Markov random fields**.
 - Bivariate extension of a first order random walk on the real line.
 - Define appropriate **neighbourhoods** for the regions.
 - Assume that the expected value of $f_{spat}(s)$ is the **average of the function evaluations of adjacent sites**.



- Spatial effect for point-referenced data: **Stationary Gaussian random fields**.
 - Well-known as **Kriging** in the geostatistics literature.
 - Spatial effect follows a zero mean stationary Gaussian stochastic process.
 - Correlation of two arbitrary sites is defined by an **intrinsic correlation function**.
 - Can be interpreted as a basis function approach with **radial basis functions**.



- All effects can be cast into one **general framework**.
- All vectors of function evaluations f_j can be expressed as

$$f_j = Z_j \xi_j$$

with design matrix Z_j and regression coefficients ξ_j .

- **Generic form of the prior** for ξ_j :

$$p(\xi_j | \tau_j^2) \propto (\tau_j^2)^{-\frac{k_j}{2}} \exp \left(-\frac{1}{2\tau_j^2} \xi_j' K_j \xi_j \right).$$

- $K_j \geq 0$ acts as a **penalty matrix**, $\text{rank}(K_j) = k_j \leq d_j = \dim(\xi_j)$.
- $\tau_j^2 \geq 0$ can be interpreted as a **variance** or (inverse) **smoothness parameter**.

Bayesian Inference

- **Fully Bayesian inference:**
 - All parameters (including the variance parameters τ^2) are assigned suitable prior distributions.
 - Typically, estimation is based on **MCMC simulation techniques**.
 - Usual estimates: **Posterior expectation**, posterior median (easily obtained from the samples).
- **Empirical Bayes inference:**
 - Differentiate between **parameters of primary interest** (regression coefficients) and **hyperparameters** (variances).
 - Assign priors only to the former.
 - Estimate the hyperparameters by maximising their **marginal posterior**.
 - Plugging these estimates into the joint posterior and maximising with respect to the parameters of primary interest yields **posterior mode estimates**.

- MCMC-based inference:
 - Assign **inverse gamma prior** to τ_j^2 :

$$p(\tau_j^2) \propto \frac{1}{(\tau_j^2)^{a_j+1}} \exp\left(-\frac{b_j}{\tau_j^2}\right).$$

Proper for $a_j > 0, b_j > 0$ Common choice: $a_j = b_j = \varepsilon$ small.

Improper for $b_j = 0, a_j = -1$ Flat prior for variance τ_j^2 ,

$b_j = 0, a_j = -\frac{1}{2}$ Flat prior for standard deviation τ_j .

- **Conditions for proper posteriors** in structured additive regression are available.
- **Gibbs sampler** for $\tau_j^2 | \cdot$:

Sample from an inverse Gamma distribution with parameters

$$a'_j = a_j + \frac{1}{2} \text{rank}(K_j) \quad \text{and} \quad b'_j = b_j + \frac{1}{2} \xi_j' K_j \xi_j.$$

- **Metropolis-Hastings** update for $\xi_j|\cdot$:

Propose new state from a multivariate Gaussian distribution with precision matrix and mean

$$P_j = Z_j' W Z_j + \frac{1}{\tau_j^2} K_j \quad \text{and} \quad m_j = P_j^{-1} Z_j' W (\tilde{y} - \eta_{-j}).$$

IWLS-Proposal with appropriately defined working weights W and working observations \tilde{y} .

- Efficient algorithms make use of the sparse matrix structure of P_j and K_j .

- Empirical Bayes inference.
 - Consider the variances τ_j^2 as **unknown constants** to be estimated from their marginal posterior.
 - Consider the regression coefficients ξ_j as **correlated random effects** with multivariate Gaussian distribution
 - ⇒ Use mixed model methodology for estimation.
- Problem: In most cases **partially improper random effects distribution**.
- Mixed model representation: Decompose

$$\xi_j = X_j\beta_j + V_j b_j,$$

where

$$p(\beta_j) \propto \text{const} \quad \text{and} \quad b_j \sim N(0, \tau_j^2 I_{k_j}).$$

⇒ β_j is a **fixed effect** and b_j is an **i.i.d. random effect**.

- This yields a **variance components model** with predictor

$$\eta = X\beta + Vb$$

where in turn

$$p(\beta) \propto \text{const} \quad \text{and} \quad b \sim N(0, Q).$$

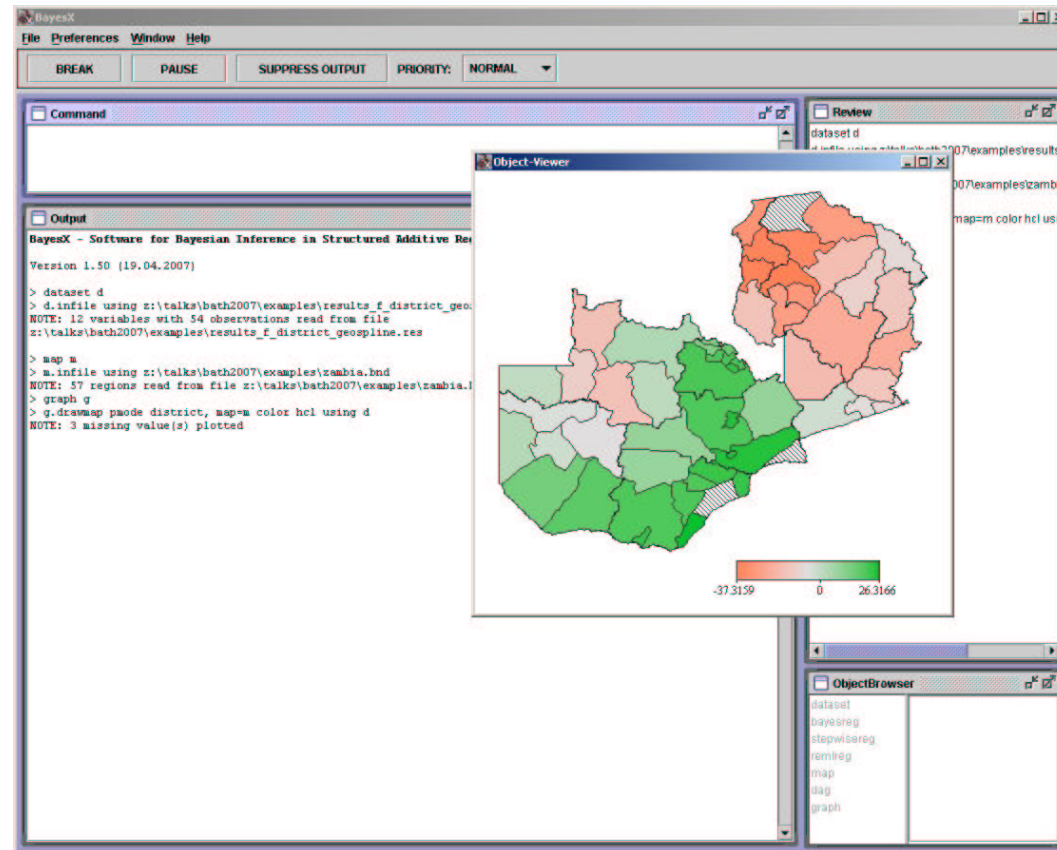
- Obtain **empirical Bayes estimates** / **penalized likelihood estimates** via iterating
 - Penalized maximum likelihood for the regression coefficients β and b .
 - Restricted Maximum / Marginal likelihood for the variance parameters in Q :

$$L(Q) = \int L(\beta, b, Q)p(b)d\beta db \rightarrow \max_Q.$$

- Involves a Laplace approximation to the marginal likelihood (corresponding to REML estimation of variances in Gaussian mixed models).

BayesX

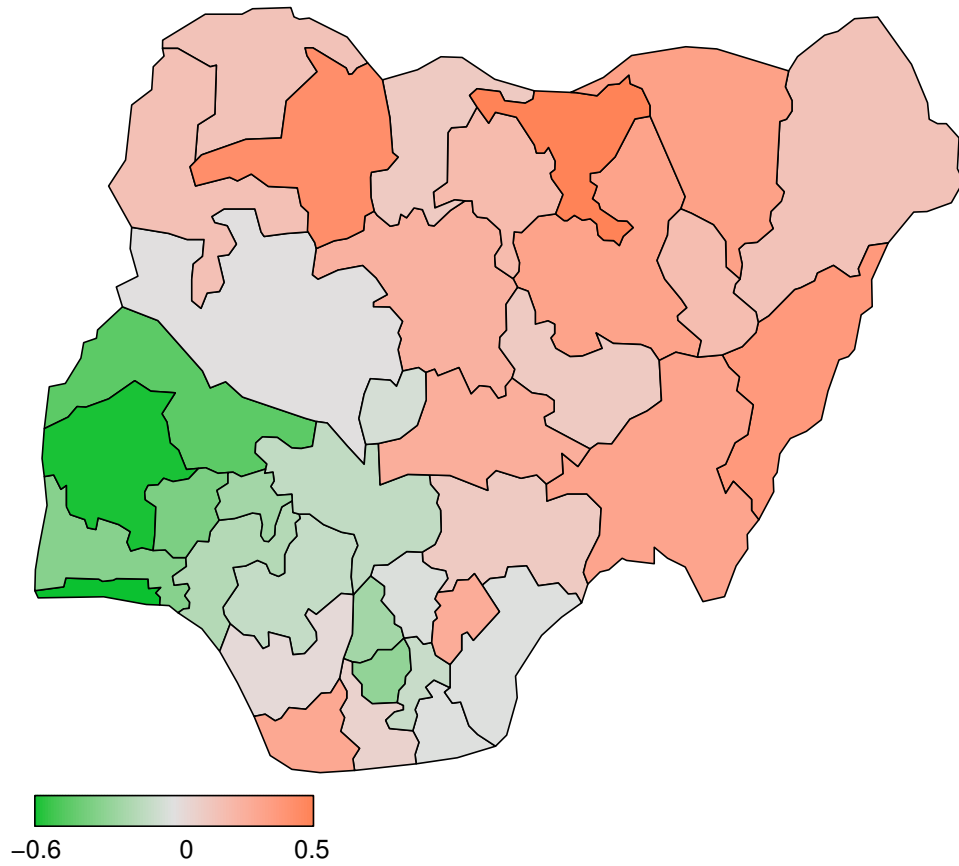
- BayesX is a software tool for estimating structured additive regression models.



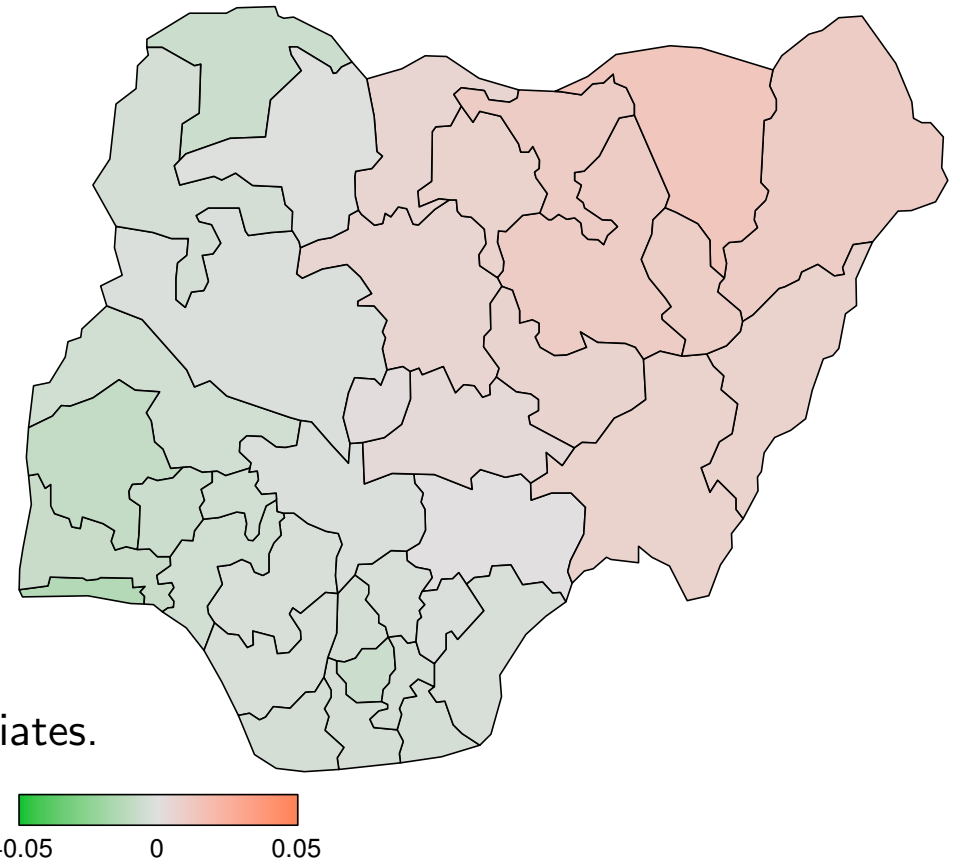
- Available from

<http://www.stat.uni-muenchen.de/~bayesx>

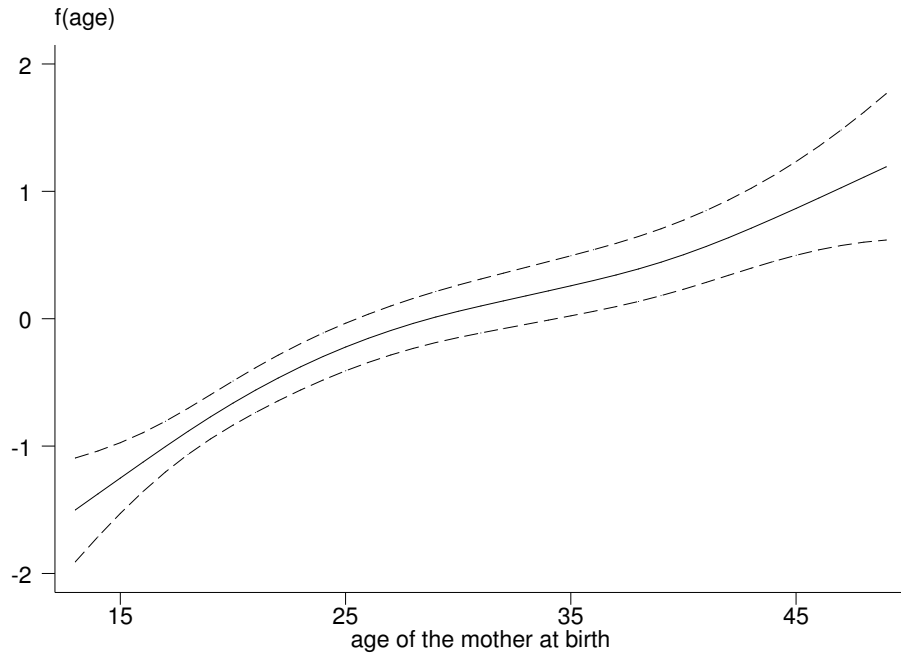
Childhood mortality in Nigeria II



Spatial effect without covariates.

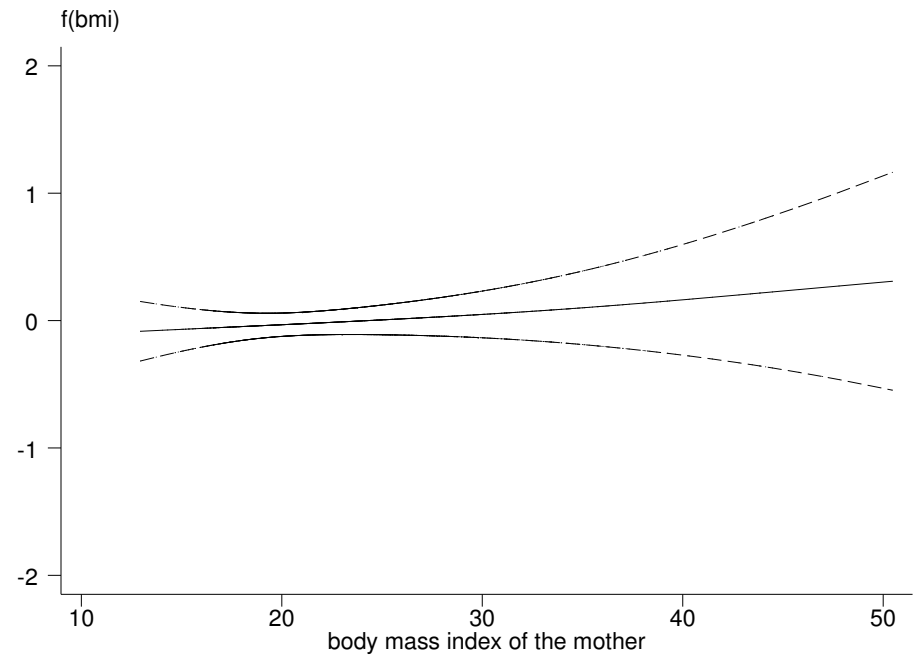


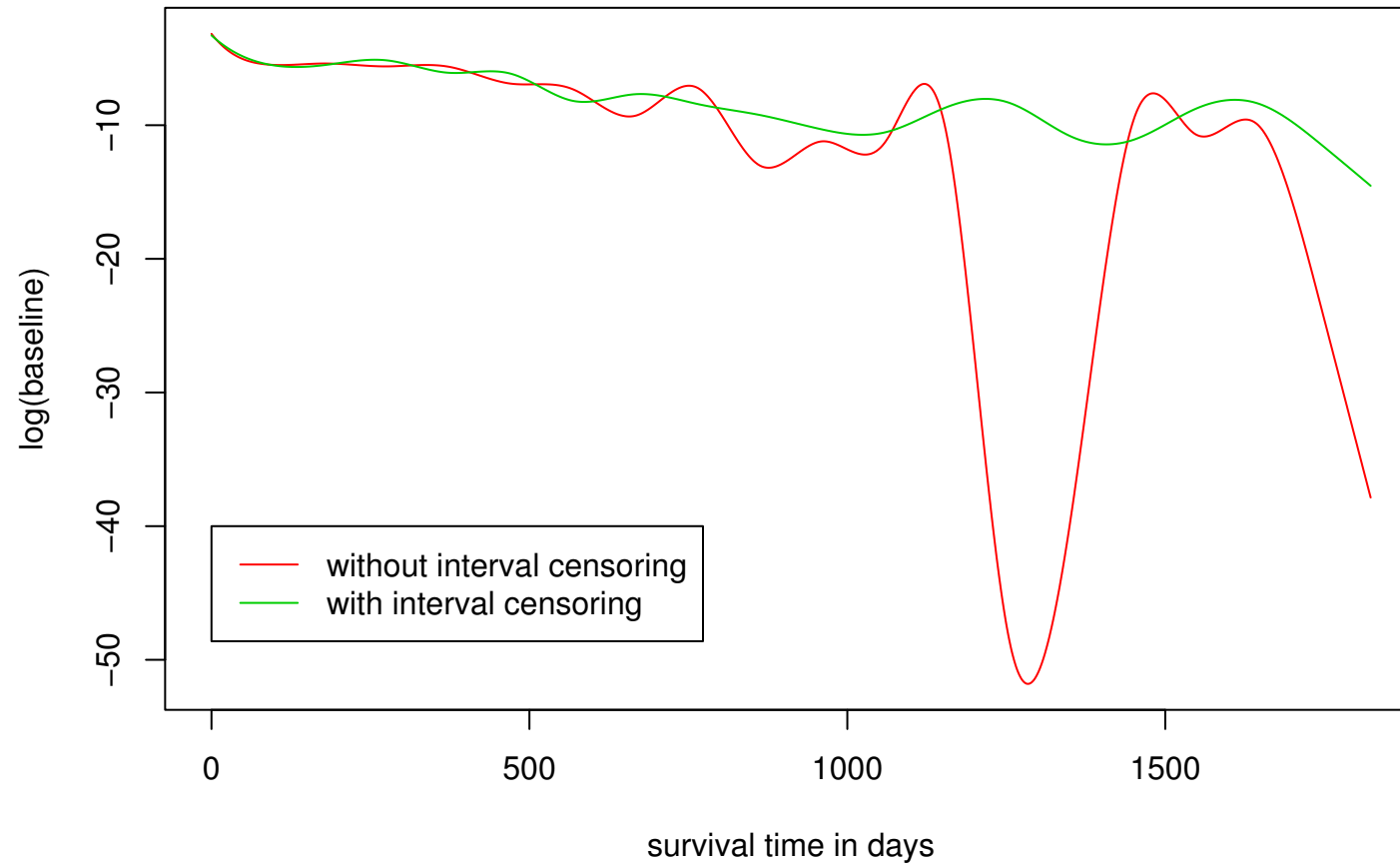
Spatial effect including covariates.



Age of the mother at birth.

Body mass index of the mother



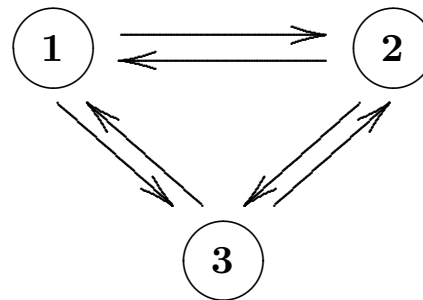


Multi-State Models

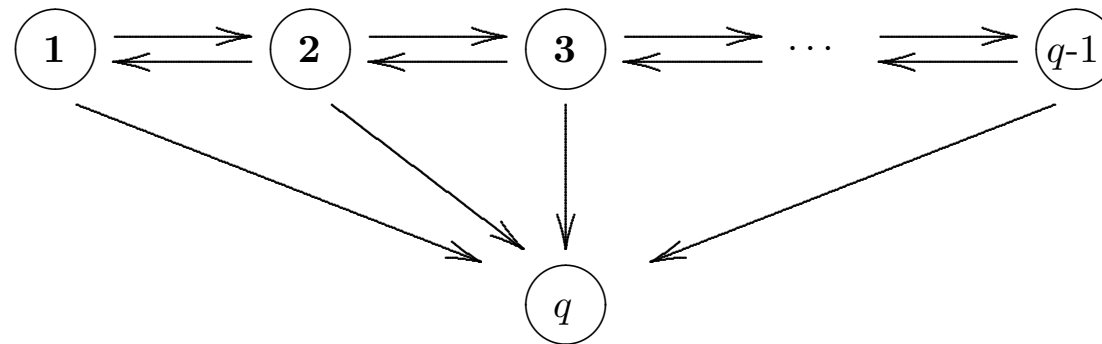
- Multi-state models form a general class for the description of the **evolution of discrete phenomena in continuous time** (i.e. event history analysis).
- We observe paths of a process

$$X = \{X(t), t \geq 0\} \quad \text{with} \quad X(t) \in \{1, \dots, q\}.$$

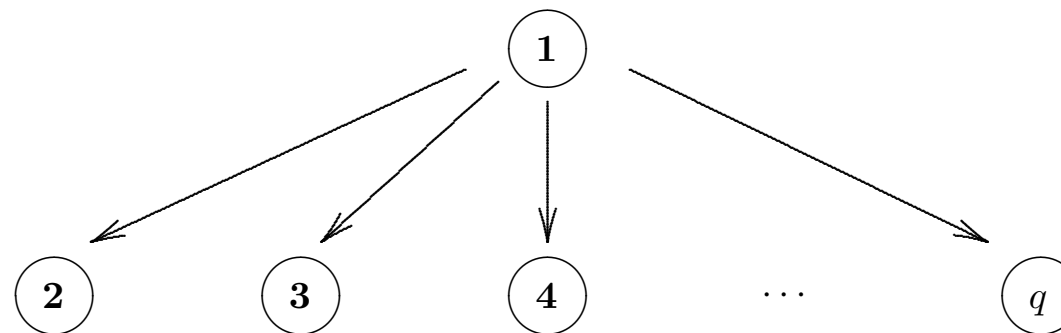
- Yields a similar data structure as for Markov processes.
- Examples:
 - Recurrent events:



– Disease progression:



– Competing risks:



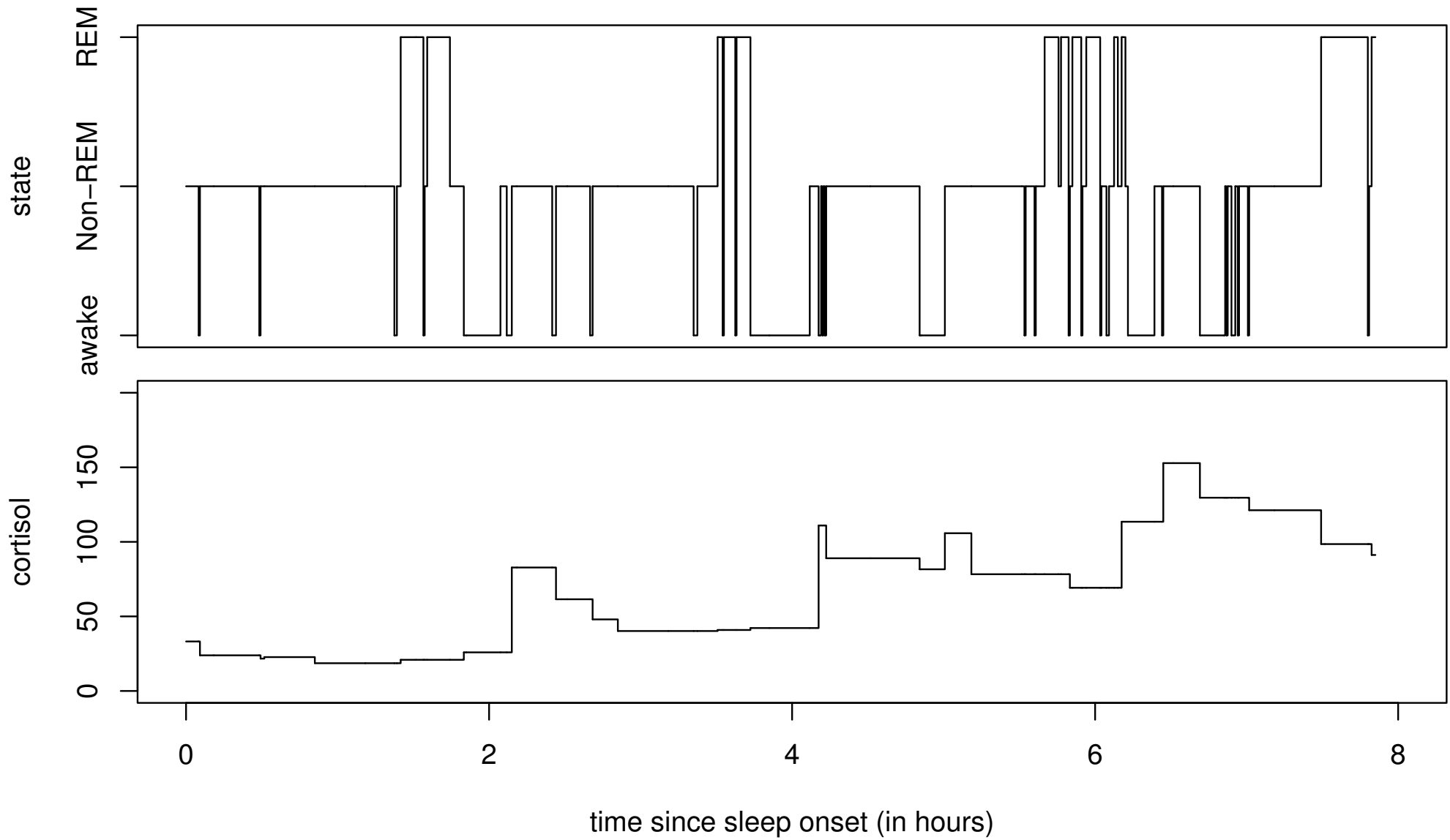
- (Homogenous) Markov processes can be compactly described in terms of the **transition intensities**

$$\lambda_{ij} = \lim_{\Delta t \rightarrow 0} \frac{P(X(t + \Delta t) = j | X(t) = i)}{\Delta t}$$

Human Sleep Data

- Human sleep can be considered an example of a recurrent event type multi-state model.
- State Space:

Awake	Phases of wakefulness
REM	Rapid eye movement phase (dream phase)
Non-REM	Non-REM phases (may be further differentiated)
- **Aims of sleep research:**
 - Describe the dynamics underlying the human sleep process.
 - Analyse associations between the sleep process and nocturnal hormonal secretion.
 - (Compare the sleep process of healthy and diseased persons.)



- **Data generation:**
 - Sleep recording based on electroencephalographic (EEG) measures every 30 seconds (afterwards classified into the three sleep stages).
 - Measurement of hormonal secretion based on blood samples taken every 10 minutes.
 - A training night familiarizes the participants of the study with the experimental environment.

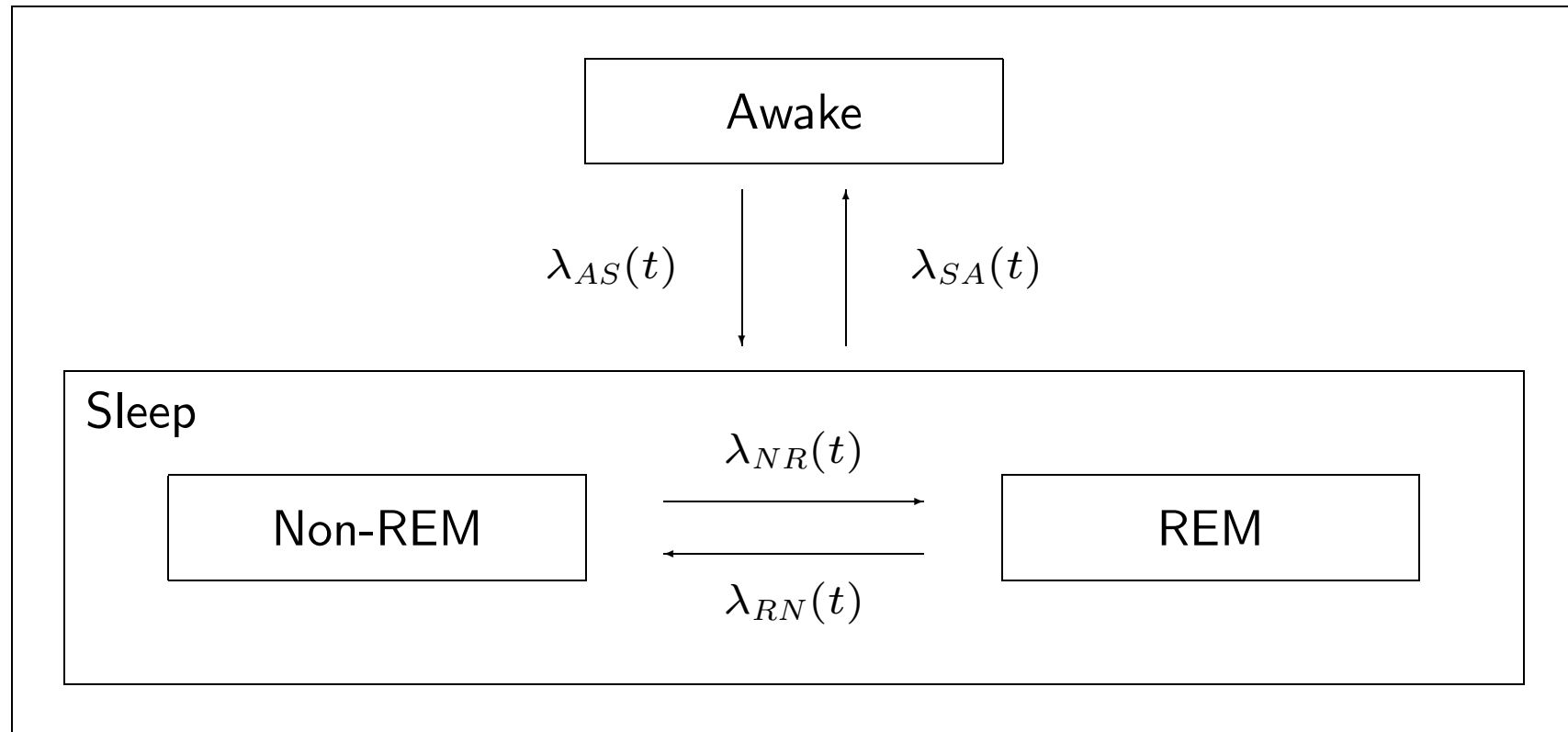
⇒ Sleep processes of 70 participants.

- Simple parametric approaches are not appropriate in this application due to
 - **Changing dynamics** of human sleep over night.
 - The **time-varying influence** of the hormonal concentration on the transition intensities.
 - **Unobserved heterogeneity**.

⇒ **Model transition intensities nonparametrically.**

Specification of Transition Intensities

- To reduce complexity, we consider a simplified transition space:



- Model specification:

$$\lambda_{AS,i}(t) = \exp \left[\gamma_0^{(AS)}(t) + b_i^{(AS)} \right]$$

$$\lambda_{SA,i}(t) = \exp \left[\gamma_0^{(SA)}(t) + b_i^{(SA)} \right]$$

$$\lambda_{NR,i}(t) = \exp \left[\gamma_0^{(NR)}(t) + c_i(t)\gamma_1^{(NR)}(t) + b_i^{(NR)} \right]$$

$$\lambda_{RN,i}(t) = \exp \left[\gamma_0^{(RN)}(t) + c_i(t)\gamma_1^{(RN)}(t) + b_i^{(RN)} \right]$$

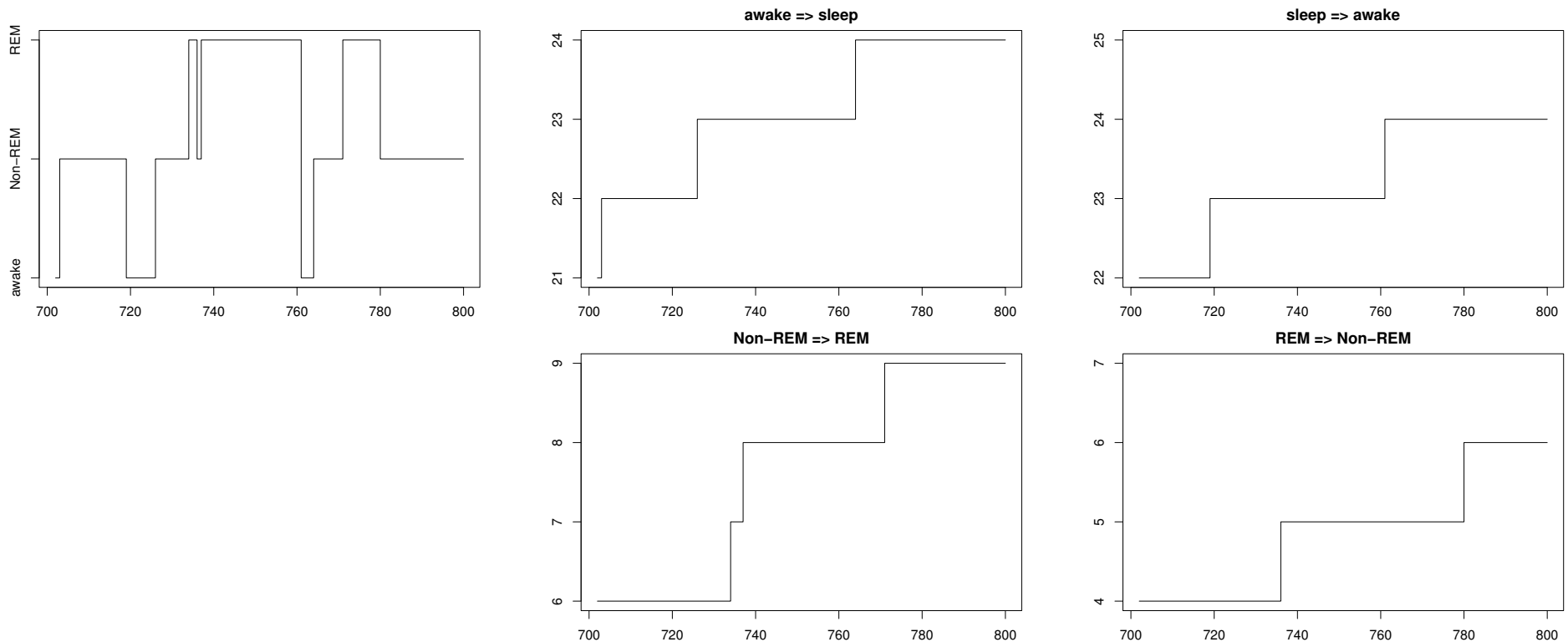
where

$$c_i(t) = \begin{cases} 1 & \text{cortisol} > 60 \text{ n mol/l at time } t \\ 0 & \text{cortisol} \leq 60 \text{ n mol/l at time } t, \end{cases}$$

$$b_i^{(j)} \sim N(0, \tau_j^2) = \text{transition- and individual-specific frailty terms.}$$

Counting Process Representation

- A multi-state model with k different types of transitions can be equivalently expressed in terms of k counting processes $N_h(t)$, $h = 1, \dots, k$ counting these transitions.



- From the counting process representation we can derive the likelihood contributions.
- The counting process representation also provides a possibility for model validation based on **martingale residuals**.
- Every counting process is a submartingale and can therefore be (Doob-Meyer-) decomposed as

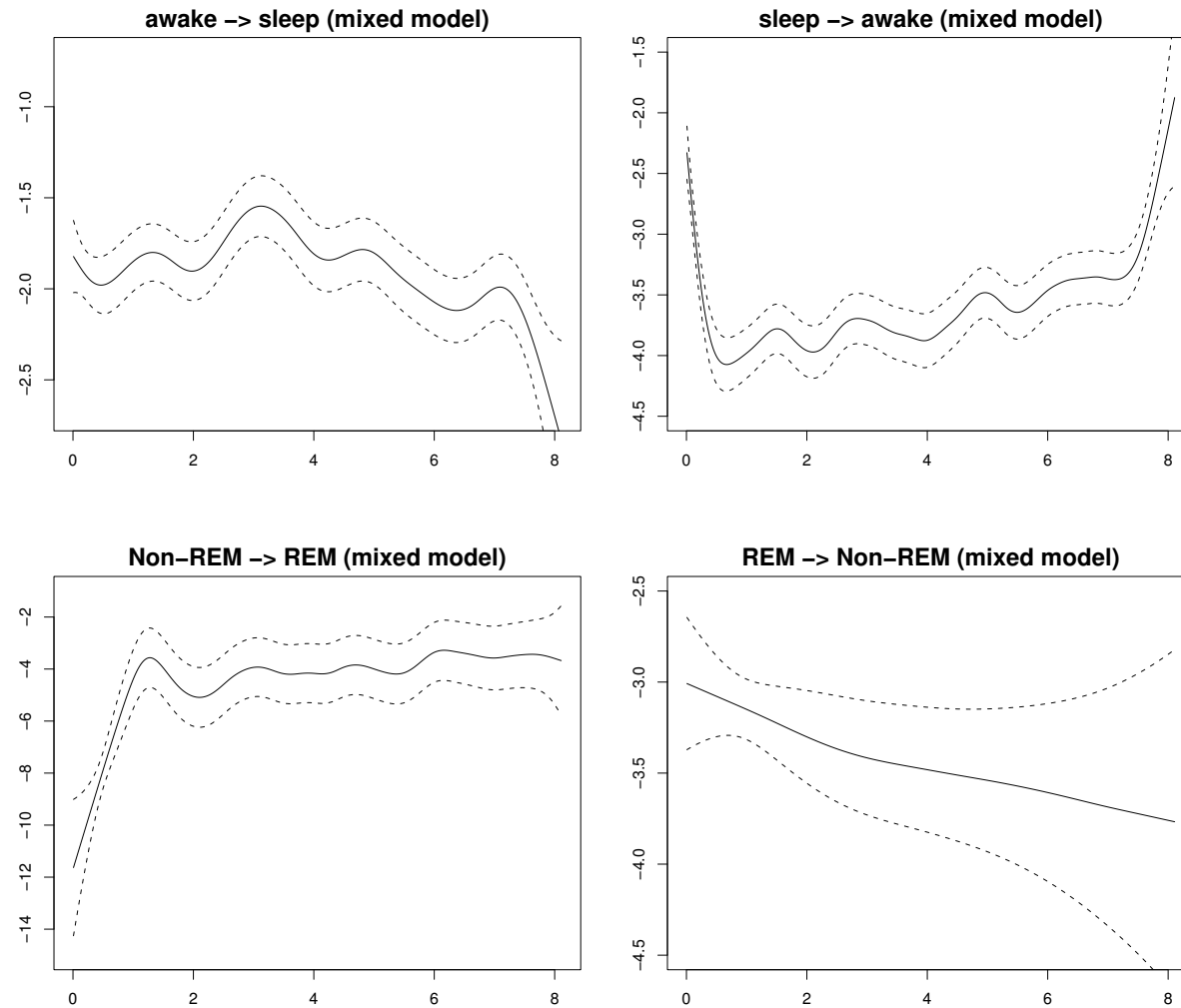
$$\begin{aligned} N_{hi}(t) &= A_{hi}(t) + M_{hi}(t) \\ &= \int_0^t \lambda_{hi}(u) Y_{hi}(u) du + M_{hi}(t), \end{aligned}$$

where $M_{hi}(t)$ is a martingale and $A_{hi}(t)$ is the (predictable) compensator process of $N_{hi}(t)$.

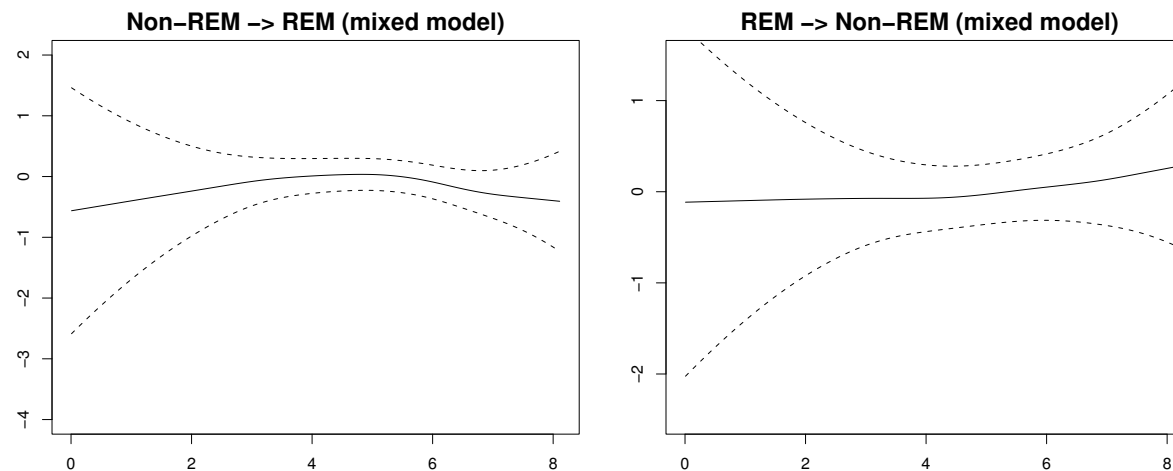
- The martingales $M_{hi}(t)$ can be interpreted as **continuous-time residuals**.
- Plots of $M_{hi}(t)$ against t can be used to compare models, evaluate the model fit, etc.

Human Sleep Data II

- Baseline effects:

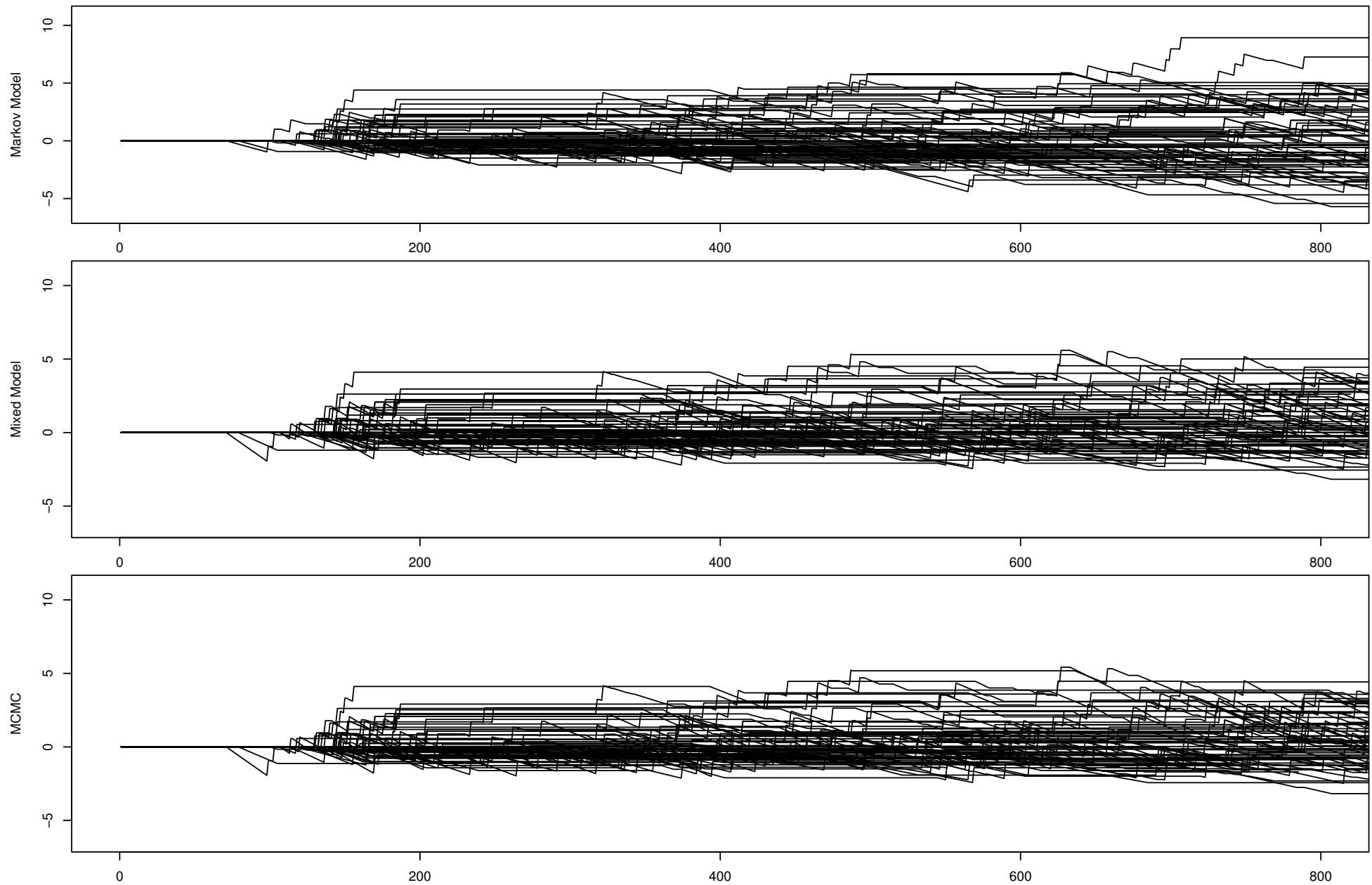


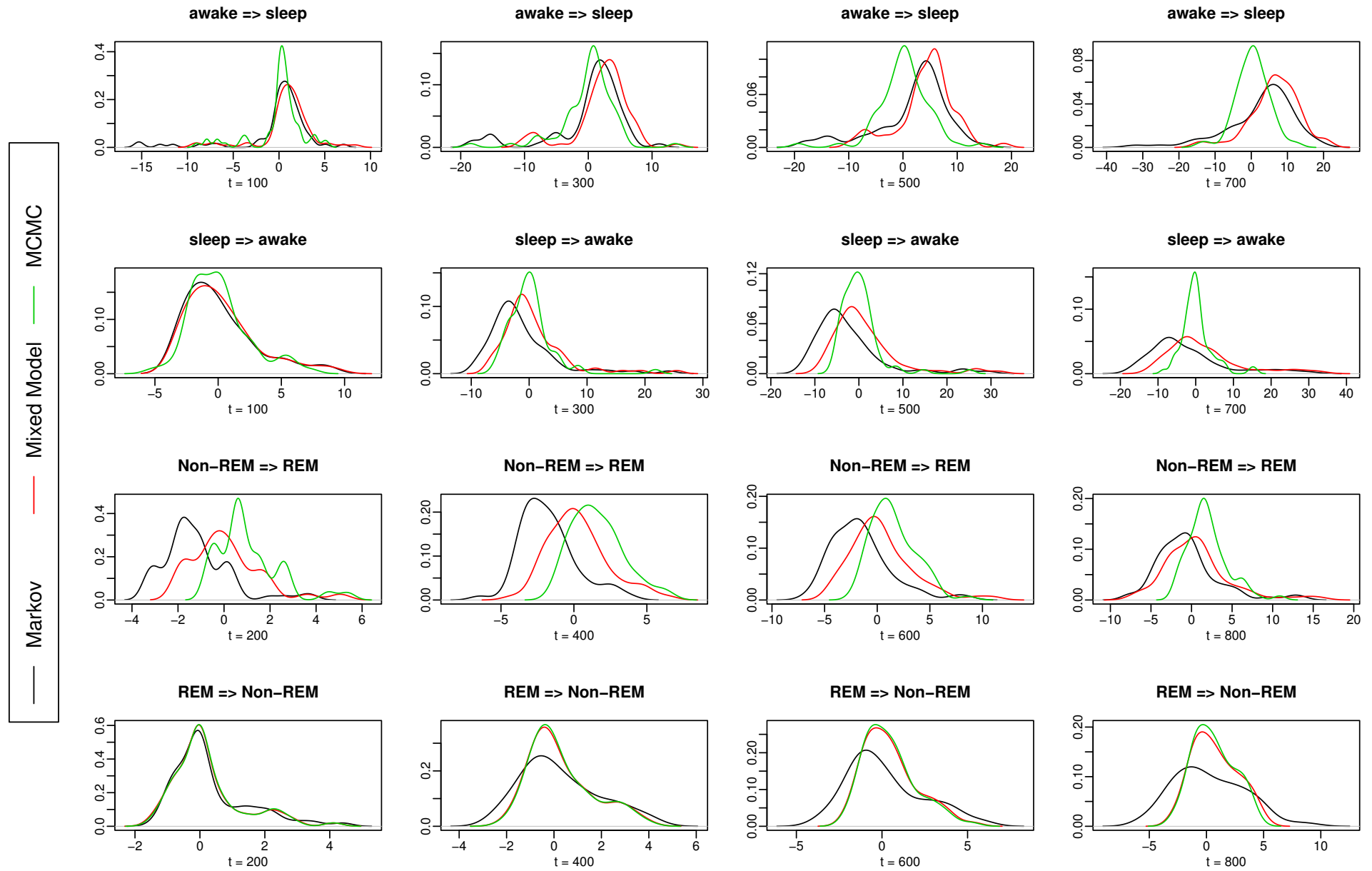
- Time-varying effects for a high level of cortisol:



- The fully Bayesian approach detects individual-specific variation for all transitions.
- The empirical Bayes approach only detects individual-specific variation for the transition between REM and Non-REM.

Martingale residuals REM => Non-REM





Summary and Outlook

- Computationally feasible semiparametric models for hazard rates / transition intensities.
- Fully Bayesian and empirical Bayes inference.
- General censoring mechanisms for analysing survival times.
- Model validation of multi-state models via martingale residuals.
- Future work:
 - Interval censored multi-state models.
 - Idea: MCMC-based imputation of unobserved path information.

References

- HENNERFEIND, BREZGER & FAHRMEIR (2006): Geoadditive survival models. *Journal of the American Statistical Association*, **101**, 1065-1075.
- KNEIB (2006): Geoadditive hazard regression for interval censored survival times. *Computational Statistics and Data Analysis*, **51**, 777-792.
- KNEIB & FAHRMEIR (2007): A mixed model approach for geoadditive hazard regression. *Scandinavian Journal of Statistics*, **34**, 207-228.
- KNEIB & HENNERFEIND (2007): Bayesian semiparametric multi-state models. *Statistical Modelling*, to appear.
- A place called home:

<http://www.stat.uni-muenchen.de/~kneib>