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RESEARCH ARTICLE

A NOTE ON PROBABILISTIC REPRESENTATIONS OF OPERATOR SEMIGROUPS

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In the theory of strongly continuous semigroups of bounded linear operators $\{T(t) \mid t \geq 0\}$ on a Banach space $\mathscr{X}$ many representation theorems of the form

$$(1) \qquad T(\xi)f = \lim_{n\to\infty} \Psi_\xi^n(T(\tfrac{1}{n}))f$$

$$(2) \qquad T(\xi)f = \lim_{n\to\infty} \Psi_\xi^n(nR(n))f$$

$$(3) \qquad T(\xi)f = \lim_{n\to\infty} \Psi_\xi^n(I + \tfrac{1}{n}A)f \quad (\xi \geq 0,\ f \in \mathscr{X})$$

have been established by several authors ([2], [3], [5], [7], [8], [9]), where Ψ_ξ is a suitable function analytic in some interval $[0,\delta]$ with $\delta > 1$, $R(\lambda) = \int_0^\infty e^{-\lambda t}T(t)dt$ for sufficiently large λ denotes the resolvent of the semigroup, I denotes the identity operator and A denotes the corresponding infinitesimal generator. (Note that (3) is only meaningful if A is bounded.) The common background all of these representation theorems is probabilistic in that Ψ_ξ is the generating function of a non-negative integer-valued random variable N (see [4] for definitions) with expectation $E(N) = \xi$ ([3], [7], [8]); in fact, relations (1), (2) and (3) are in some sense a consequence of the famous law of large numbers in probability theory.

The aim of this note now is to prove that under mild conditions only these probabilistic representations are possible.

THEOREM 1. Let Ψ_ξ be analytic in some interval $[0,\delta]$, $\delta > 1$, with non-negative coefficients. Then if any of the three representations given by (1), (2) or (3) holds for an arbitrary strongly continuous non-periodic semigroup $\{T(t) \mid t \geq 0\}$ with $\|T(t)\| > 0$, Ψ_ξ is necessarily the generating function of a non-negative integer-valued random variable N with expectation $E(N) = \xi$. In this case, the representations (1) and (2) hold true for every strongly continuous semigroup, and (3) holds true in case that A is bounded.

PROOF. Let $\Psi_\xi(t) = \sum_{k=0}^{\infty} a_k(\xi)t^k$ for $0 \leq t \leq \delta$ with all $a_k(\xi) \geq 0$. Then $\Psi_\xi(1) = \sum_{k=0}^{\infty} a_k(\xi) > 0$ since otherwise $\Psi_\xi \equiv 0$, hence $\|T(\xi)\| = 0$ which is a contradiction. But then by $\{\frac{a_k(\xi)}{\Psi_\xi(1)}\}$ a probability distribution of some non-negative integer-valued random variable N is given, with

$$\Psi_\xi^* = \frac{\Psi_\xi}{\Psi_\xi(1)}$$

being its generating function.

By the inequality $x \leq \frac{1}{y} e^{xy}$ for arbitrary $x \geq 0$, $y > 0$, N is integrable with

$$\zeta = E(N) \leq \frac{1}{\ln \delta} E(e^{N \ln \delta}) = \frac{\Psi_\xi^*(\delta)}{\ln \delta} = \frac{\Psi_\xi(\delta)}{\Psi_\xi(1) \ln \delta} < \infty.$$

Also, since Ψ_ξ and hence Ψ_ξ^* are analytic in $[0,\delta]$ with $\delta > 1$, the characteristic function of N is analytic in some complex neighbourhood of the origin, hence all of the relations

$$T^*(\zeta)f = \lim_{n\to\infty} \{\Psi_\xi^*(T^*(\tfrac{1}{n}))\}^n f \tag{4}$$

$$T^*(\zeta)f = \lim_{n\to\infty} \{\Psi_\xi^*(nR^*(n))\}^n f \tag{5}$$

(6) $$T^*(\zeta)f = \lim_{n\to\infty} \{\Psi_\xi^*(I + \frac{1}{n}A^*)\}^n f, \quad f \in \mathscr{X}$$

hold true for every strongly continuous semigroup $\{T^*(t) \mid t \geq 0\}$ (the latter only, if A^* is bounded) ([7], Corollary 2; [8], (6)). Let S_n denote one of the operators $T(\frac{1}{n})$, $nR(n)$ or $I + \frac{1}{n}A$, corresponding to which of the relations (1), (2) or (3) holds. Choose $f \in \mathscr{X}$ such that $\|T(\xi)f\| > 0$, then

$$0 < \|T(\xi)f\| = \lim_{n\to\infty}\|\{\Psi_\xi(S_n)\}^n f\| = \lim_{n\to\infty}(\Psi_\xi(1))^n \|\{\Psi_\xi^*(S_n)\}^n f\|$$

with $\lim_{n\to\infty} \|\{\Psi_\xi^*(S_n)\}^n f\| = \|T(\zeta)f\| < \infty$ by (4), (5) and (6), implying $\Psi_\xi(1) \geq 1$.

Let further $g \in \mathscr{X}$ be such that $\|T(\zeta)g\| > 0$. Then

$$\infty > \|T(\xi)g\| = \lim_{n\to\infty} (\Psi_\xi(1))^n \|\{\Psi_\xi^*(S_n)\}^n g\|$$

with $\lim_{n\to\infty}\|\{\Psi_\xi^*(S_n)\}^n g\| = \|T(\zeta)g\| > 0$ by (4), (5) and (6), implying $\Psi_\xi(1) \leq 1$. That is, $\Psi_\xi(1) = 1$ and hence $\Psi_\xi = \Psi_\xi^*$. Again by (4), (5) and (6),

$$T(\zeta)f = \lim_{n\to\infty} \Psi_\xi^n(S_n)f = T(\xi)f \quad \text{for all } f \in \mathscr{X}$$

by assumption, hence $T(\zeta) = T(\xi)$.

Suppose now $\zeta \neq \xi$, say $\zeta < \xi$. Then for $\eta = \xi - \zeta$,

$$T(\xi + \eta) = T(\xi)T(\eta) = T(\zeta)T(\eta) = T(\zeta + \eta) = T(\xi),$$

hence the semigroup is periodic with period η which is a contradiction to the assumption. Hence $\zeta = \xi$, and the theorem is proved.

Note that the above theorem is a strong extension of the general convergence theorem 1.2.6 in [1] in the case of strongly continuous semigroups since only one (essentially) arbitrary "test function" is needed.

The assumption of non-periodicity in the above theorem is only necessary to guarantee that $E(N) = \xi$. Without this assumption, the theorem essentially remains valid in that the representations (1), (2) and (3) hold true in general if $T(\xi)f$ is replaced by $T(\zeta)f$.

Of course, the point in the above theorem is the assumption that the coefficients on Ψ_ξ are non-negative. One could ask now whether generally only probabilistic representations of the form (1), (2) or (3) are possible. But this is not true as can be seen by the following non-probabilistic extension of Kendall's formula [5].

THEOREM 2. Let $\Psi_\xi(t) = 1 - \xi + \xi t$ for $\xi > 1$. Then (1), (2) and (3) hold true for every uniformly continuous semigroup $\{T(t) \mid t \geq 0\}$.

PROOF. Let again S_n denote $T(\frac{1}{n})$, $nR(n)$ or $I + \frac{1}{n}A$, and let $V_n = n(S_n - I)$. Then $A = \lim_{n\to\infty} V_n$, and

$$\|\Psi_\xi^n(S_n) - e^{\xi V_n}\| = \|(I + \tfrac{\xi}{n}V_n)^n - (e^{\frac{\xi}{n}V_n})^n\|$$

$$\leq \|I + \tfrac{\xi}{n}V_n - e^{\frac{\xi}{n}V_n}\| \sum_{k=0}^{n-1} \|I + \tfrac{\xi}{n}V_n\|^k \, \|e^{\frac{\xi}{n}V_n}\|^{n-k-1}$$

$$\leq \{\sum_{k=2}^{\infty} (\tfrac{\xi}{n})^k \frac{\|V_n\|^k}{k!}\} \, n(1 + \tfrac{\xi}{n}\|V_n\|)^n \, e^{\xi\|V_n\|}$$

$$\leq \frac{1}{n} e^{3\xi\|V_n\|} \leq \frac{1}{n} e^{4\xi\|A\|}$$

for sufficiently large n.

Since by the exponential formula, $T(\xi) = \lim_{n\to\infty} e^{\xi V_n}$, the theorem is proved.

REMARK. In [7] it was implicitly assumed that the mapping $t \to T(t)$ is measurable and separably valued, which e.g. is true in the case of uniformly continuous semigroups. But the results in [6] remain also valid in the general strongly continuous case since for a suitable non-negative random variable X

$$\{E(T(X))\}f = E(T(X)f), \quad f \in \mathscr{X}$$

defines a bounded linear operator on $\mathscr{X}$ into $\mathscr{X}$ with the properties $\|E(T(X))\| \leq E(\|T(X)\|)$ and $E(T(X+Y)) = E(T(X)) \circ E(T(Y))$ for independent random variables X and Y, where $E(T(X)f)$ for $f \in \mathscr{X}$ is to be understood as a Pettis expectation in the sense of Mourier [6] (which exists since the mappings $t \to T(t)f$ are continuous, hence measurable and separably valued (see also [2])).

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