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A MARTINGALE CHARACTERIZATION OF MIXED POISSON PROCESSES

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Abstract

It is shown that an elementary pure birth process is a mixed Poisson process iff the sequence of post-jump intensities forms a martingale with respect to the σ-fields generated by the jump times of the process. In this case, the post-jump intensities converge almost surely to the mixing random variable of the process.

INTENSITIES; COMPLETELY MONOTONIC FUNCTIONS; BERNSTEIN'S THEOREM

1. Introduction

Mixed Poisson processes play an important role in many branches of applied probability, for instance in insurance mathematics and physics (see Albrecht (1985) and Pfeifer (1986) for recent surveys). They belong to the class of elementary pure birth processes $\{N(t);\ t \geqq 0\}$ with standard transition probabilities

$$(1)\qquad p_{nm}(s,t) = P(N(t) = m \mid N(s) = n), \qquad 0 \leqq n \leqq m, \quad 0 \leqq s \leqq t,$$

possessing right-continuous paths and positive and finite birth rates

$$(2)\qquad \lambda_n(t) = \lim_{h \downarrow 0} \frac{1}{h}\, p_{n,n+1}(t, t+h), \qquad n, t \geqq 0,$$

and all finite-dimensional marginals of the jump-time sequence $\{T_n\,;\ n \geqq 0\}$ are absolutely continuous with respect to Lebesgue measure (see Pfeifer (1982)). For such processes, the jump times form a Markov chain with transition probabilities

$$(3)\qquad P(T_n > t \mid T_{n-1} = s) = \frac{1 - F_n(t)}{1 - F_n(s)}, \qquad 0 \leqq s \leqq t, \quad n \geqq 1$$

and initial distribution function F_0 where

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(4) $$1-F_n(x)=\exp\left(-\int_0^x \lambda_n(u)du\right), \qquad x\geqq 0, \quad n\geqq 0$$

(the F_n are in fact all cumulative distribution functions), hence all F_n are absolutely continuous with densities f_n (say), and the conditional densities for the transition probabilities can be represented as

(5) $$f_n(s\mid t)=\frac{f_n(s)}{1-F_n(t)}, \qquad 0\leqq t\leqq s, \quad n\geqq 1.$$

Moreover, the birth rates coincide with the hazard rates

(6) $$\lambda_n(t)=\frac{f_n(t)}{1-F_n(t)} \quad \text{a.e., } n,t\geqq 0.$$

If especially $\{N(t); t\geqq 0\}$ is a mixed Poisson process, then also

(7) $$\lambda_n(t)=\int_0^\infty x^{n+1}\exp(-xt)dG(x)\Big/\int_0^\infty x^n\exp(-xt)dG(x), \qquad n,t\geqq 0,$$

where G is the c.d.f. of the mixing random variable Λ (say). In fact, Lundberg (1940) has proved that such a representation characterizes the intensities of a mixed Poisson process.

In terms of random variables, a mixed Poisson process behaves like a homogeneous Poisson process with rate λ given $\Lambda=\lambda$, from which it also follows that

(8) $$\lambda_n(t)=E(\Lambda\mid N(t)=n), \qquad n,t\geqq 0.$$

The following result completes some of Lundberg's (1940) results on the asymptotic behaviour of the intensities for mixed Poisson processes.

Lemma. Let $\{t_n; n\geqq 1\}$ be a sequence of positive real numbers converging to $t>0$ such that $|t_n/t-1|=o(n^{-1/3})$, $n\to\infty$. Then, if $1/t$ is a point of increase of G, we have

(9) $$\lim_{n\to\infty}\lambda_n(nt_n)=\frac{1}{t}.$$

Proof. Let $\varepsilon_n>0$ be chosen in such a way that $n\varepsilon_n^3\to 0$, $n\varepsilon_n^2\to\infty$ and $(t_n/t-1)/\varepsilon_n\to 0$ for $n\to\infty$. From relation (7) it follows that

$$\lambda_n(nt_n)=\frac{1}{t_n}\int_0^\infty e^n(xt_n)^{n+1}\exp(-nxt_n)dG(x)\Big/\int_0^\infty e^n(xt_n)^n\exp(-nxt_n)dG(x)$$

$$\sim\frac{\dfrac{1}{t_n}\displaystyle\int_{(1-\varepsilon_n)/t_n}^{(1+\varepsilon_n)/t_n} xt_n\exp\left(-\frac{n}{2}(1-xt_n)^2\right)dG(x)}{\displaystyle\int_{(1-\varepsilon_n)/t_n}^{(1+\varepsilon_n)/t_n}\exp\left(-\frac{n}{2}(1-xt_n)^2\right)dG(x)}$$

$$\sim\frac{1}{t_n} \quad \text{for } n\to\infty.$$

This proves the lemma.

The above result is in general not true without further conditions on G as can be seen for example by mixing distributions concentrated in a single point $\lambda > 0$; here $\lambda_n(t) = \lambda$ for all n and t, λ being the only point of increase of G. As an example, consider a Pólya–Lundberg process where Λ follows a gamma distribution with mean $\mu > 0$ and variance $\alpha\mu^2$, $\alpha > 0$. Here

$$\lambda_n(t) = \mu \frac{1+\alpha n}{1+\mu\alpha t}, \qquad n, t \geqq 0, \tag{10}$$

from which the validity of (9) can be seen explicitly, for all $t > 0$.

2. The martingale characterization

Let $\{\lambda_n(T_{n-1});\ n \geqq 1\}$ denote the sequence of post-jump intensities. In the light of (2), the post-jump intensities describe the transition behaviour of the process immediately after a jump has occurred. The following result gives a characterization of mixed Poisson processes by a martingale property of this sequence.

Theorem 1. Let $\{N(t);\ t \geqq 0\}$ be an elementary pure birth process with intensities $\{\lambda_n(t);\ n, t \geqq 0\}$ and jump times $\{T_n;\ n \geqq 0\}$. For $n \geqq 1$ let $\mathcal{A}_n$ denote the σ-field generated by $T_0, \cdots, T_{n-1}$. Then $\{N(t); t \geqq 0\}$ is a mixed Poisson process iff the post-jump intensities $\{\lambda_n(T_{n-1});\ n \geqq 1\}$ form a martingale with respect to $\{\mathcal{A}_n;\ n \geqq 1\}$, and $E(\lambda_1(T_0) \mid T_0 \geqq t) = \lambda_0(t)$ a.s., $t \geqq 0$.

Proof. Due to the Markov structure of jump times the martingale property of the post-jump intensities is equivalent to

$$E(\lambda_{n+1}(T_n) \mid T_{n-1} = t) = \lambda_n(t) \quad \text{a.s. for all } n \geqq 1 \tag{11}$$

which by (5) and (6) is in turn equivalent to

$$\int_t^\infty \frac{f_{n+1}(s)}{1-F_{n+1}(s)} \frac{f_n(s)}{1-F_n(t)}\, ds = \frac{f_n(t)}{1-F_n(t)} \quad \text{a.e.} \tag{12}$$

saying that (a suitable version of) f_n is almost everywhere differentiable with

$$f_n'(t) = -\frac{f_{n+1}(t)}{1-F_{n+1}(t)} f_n(t) \quad \text{a.e.} \tag{13}$$

or equivalently

$$\frac{d}{dt} \log(1-F_{n+1}(t)) = -\frac{f_{n+1}(t)}{1-F_{n+1}(t)} = \frac{f_n'(t)}{f_n(t)} = \frac{d}{dt} \log f_n(t) \quad \text{a.e.} \tag{14}$$

Integration of this last relation shows that there are constants $c_n > 0$ such that

$$1 - F_{n+1}(t) = c_n f_n(t), \qquad t \geqq 0, \tag{15}$$

which in turn implies that f_n is absolutely continuous and the recursive formula

(16) $$f_{n+1}(t) = -c_n f'_n(t)$$

holds everywhere on $[0,\infty)$. By induction, we see that all derivatives of f_n exist on $[0,\infty)$, and that for all $n \geqq 1$,

(17) $$f_n(t) = (-1)^n \prod_{k=0}^{n-1} c_k f_0^{(n)}(t), \qquad t \geqq 0.$$

Since by assumption, the intensities (and hence all f_n) are positive and finite, we have

(18) $$(-1)^n f_0^{(n)}(t) \geqq 0, \qquad n, t \geqq 0.$$

The density f_0 thus is completely monotonic on $[0,\infty)$, hence by Bernstein's (1928) theorem there is a bounded and non-decreasing right-continuous function H such that

(19) $$f_0(t) = \int_0^\infty \exp(-xt)dH(x), \qquad t \geqq 0.$$

In fact, since f_0 is a density, we have that $(1/x)dH(x) = dG(x)$ is a probability measure from which it follows that

(20) $$f_n(t) = \prod_{k=0}^{n-1} c_k \int_0^\infty x^{n+1} \exp(-xt)dG(x)$$
$$1 - F_n(t) = \prod_{k=0}^{n-1} c_k \int_0^\infty x^n \exp(-xt)dG(x), \qquad n, t \geqq 0.$$

Hence relation (7) is satisfied, saying that $\{N(t); t \geqq 0\}$ must be a mixed Poisson process with mixing distribution $dG(x)$. Conversely, since every mixed Poisson process has intensities of the form (7), it is easily seen that relation (12) holds, hence the post-jump intensities possess the martingale property, which proves the theorem.

It should be pointed out that since $f_0(0) < \infty$ by our assumptions, the mixing random variable must be integrable with

(21) $$E(\Lambda) = f_0(0).$$

A simple application of the martingale convergence theorem (see e.g. Billingsley (1979)) then shows that the post-jump intensities converge almost surely to some integrable random variable since also

(22) $$E(\lambda_1(T_0)) = \int_0^\infty \frac{f_1(t)f_0(t)}{1-F_1(t)}\,dt = \int_0^\infty -\frac{f'_0(t)}{f_0(t)} f_0(t)dt = f_0(0).$$

The question now is what the possible limits of the post-jump intensities are. The following result gives an answer to this.

Theorem 2. If Λ is the mixing random variable of the process, then the post-jump intensities converge almost surely to Λ.

Proof. For any mixed Poisson process, we have $(n+1)/T_n \to \Lambda$ a.s. by the strong law of large numbers, applied to the Poisson process with rate λ, conditionally on $\Lambda = \lambda$, and by the law of the iterated logarithm,

$$\left| \lambda \frac{T_n}{n+1} - 1 \right| = O\left(\sqrt{\frac{1}{n} \log\log n} \right) = o(n^{-1/3}) \quad \text{a.s. for } n \to \infty.$$

Since also Λ is almost surely concentrated on the points of increase of G, the c.d.f. of Λ, we have by the above lemma

$$\lambda_{n+1}(T_n) = \lambda_{n+1}((n+1)(T_n/n+1)) \sim (n+1)/T_n \to \Lambda \quad \text{a.s.,} \tag{23}$$

which proves the theorem.

For instance, if Λ is concentrated on two points $\nu_1 < \nu_2$ with mass α and $1-\alpha$ each ($\alpha > 0$), then

$$\lambda_n(nt) \to \begin{cases} \nu_2, & t < h(\nu_1, \nu_2) \\ \alpha\nu_1 + (1-\alpha)\nu_2, & t = h(\nu_1, \nu_2) \\ \nu_1, & t > h(\nu_1, \nu_2) \end{cases} \tag{24}$$

where $h(\nu_1, \nu_2) = (\log \nu_2 - \log \nu_1)/(\nu_2 - \nu_1)$, as can be seen from Lundberg (1940), relation (108). Since $1/\nu_2 < h(\nu_1, \nu_2) < 1/\nu_1$ always, it can explicitly be seen that

$$\lambda_{n+1}(T_n) \to \begin{cases} \nu_1 & \text{with probability } \alpha \\ \nu_2 & \text{with probability } 1-\alpha, \end{cases} \tag{25}$$

i.e.

$$\lambda_{n+1}(T_n) \to \Lambda \quad \text{a.s.}$$

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Note added in proof. A direct proof of Theorem 2 can be obtained by the fact that analogously to (8), we also have

$$\lambda_n(t) = E(\Lambda \mid T_{n-1} = t) \quad \text{a.s.,} \ t \geqq 0, \ n \geqq 1$$

and hence $\lambda_n(T_{n-1}) = E(\Lambda \mid T_{n-1})$ a.s., $n \geqq 1$. The statement then follows from the fact that the mixing random variable is measurable with respect to the terminal σ-field generated by the sequence $\{T_n ; n \geqq 1\}$.

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