Probab. Th. Rel. Fields 75, 213–221 (1987)

Probability Theory and Related Fields

On a Joint Strong Approximation Theorem for Record and Inter-Record Times

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Summary. We present a simple joint strong approximation for the logarithms of record and inter-record times from an exchangeable sequence, including an exact estimation for the rate of convergence in terms of upper and lower class functions of a Wiener process. The approach chosen here allows for simple proofs of exact and asymptotic (joint) results for record and inter-record times, such as the Law of Large Numbers (LLN), Central Limit Theorem (CLT) and Law of the Iterated Logarithm (LIL), and others.

I. Introduction

Let $\{X_n; n \geqq 1\}$ be an i.i.d. sequence of random variables (r.v.'s) defined on some probability space $(\Omega, \mathscr{A}, P)$ with a c.d.f. F such that either $F(x) < 1$, all x, or the right end $x_\infty = \sup\{x \in \mathbb{R};\ F(x) < 1\}$ is not an atom of F. Then the associated record times $\{U_n; n \geqq 0\}$ and inter-record times $\{\Delta_n; n \geqq 0\}$ given by

$$\Delta_0 = 1, \quad \Delta_{n+1} = \inf\{k; X_{U_n+k} > X_{U_n}\} \text{ where } U_n = \sum_{k=0}^{n} \Delta_k, \quad n \geqq 0, \tag{1.1}$$

are a.s. well-defined (see Shorrock [10]). A large number of exact and asymptotic results for these sequences has been given in the literature, e.g. by Rényi [8] who stated that the record times form a homogeneous Markov chain with transition probabilities

$$P(U_{n+1} = k \mid U_n = j) = \frac{j}{k(k-1)}, \quad 1 \leqq j < k, \quad n \geqq 0, \tag{1.2}$$

independent of F, in case that F is continuous (which we shall assume in what follows). He also proved the LLN, CLT and LIL for record times, i.e.

$$\frac{1}{n} \log U_n \to 1 \quad \text{a.s.} \quad (n \to \infty) \tag{1.3}$$

⋆ Research supported by the Air Force Office of Scientific Research Contract No. F49620 85 C0144

$$\frac{1}{\sqrt{n}}(\log U_n - n) \xrightarrow{\mathscr{L}} \mathcal{N}(0,1) \quad (n \to \infty) \tag{1.4}$$

$$\limsup \pm \frac{\log U_n - n}{\sqrt{2n \log\log n}} = 1 \quad \text{a.s.} \tag{1.5}$$

For the inter-record time sequence it was shown by Shorrock [10] that given the record value sequence $\{X_{U_n}; n \geqq 0\}$, the inter-record times are conditionally independent and geometrically distributed, i.e.

$$P(\Delta_{n+1} = k \mid X_{U_n}) = \{1 - F(X_{U_n})\} F^{k-1}(X_{U_n}) \quad \text{a.s.,} \quad k \geqq 1, \; n \geqq 0, \tag{1.6}$$

which is intuitively clear by (1.1) since the record times U_n are stopping times for the original sequence $\{X_n; n \geqq 1\}$. From this observation he derived

$$\frac{1}{n} \log \Delta_n \to 1 \quad \text{a.s.} \quad (n \to \infty) \tag{1.7}$$

$$\frac{1}{\sqrt{n}}(\log \Delta_n - n) \xrightarrow{\mathscr{L}} \mathcal{N}(0,1) \quad (n \to \infty) \tag{1.8}$$

$$\limsup \pm \frac{\log \Delta_n - n}{\sqrt{2n \log\log n}} = 1 \quad \text{a.s.} \tag{1.9}$$

(these results have been obtained previously by Neuts [5] and Strawderman and Holmes [12], using different techniques.) The fact that although we have $U_n = \sum_{k=1}^{n} \Delta_k$, $n \geqq 0$, the same normalizing constants appear in these limit laws for both sequences makes it clear that e.g. the normal approximation (1.4) and (1.8) cannot be very powerful for small values of n. This was already pointed out by Neuts [5] who suggested that n should be at least as large as 1000 in order to obtain satisfactory results. This will also be discussed in this paper when asymptotic evaluations for $\log U_n - \log \Delta_n$ are investigated.

Shorrock [10] also pointed out that the sequence $\{\log U_n;\, n \geqq 0\}$ should be close to a Poisson point process in some sense (see also Resnick [9]), explaining for the limit relations (1.3) to (1.5). A constructive approach to this question was made by Williams [15] and Westcott [14] who showed that if $\{Y_n; n \geqq 1\}$ is an i.i.d. sequence of exponentially distributed r.v.'s with unit mean, then the sequence $\{U_n^*;\, n \geqq 0\}$ defined recursively by

$$U_0^* = 1, \qquad U_{n+1}^* =]U_n^* e^{Y_{n+1}}[, \quad n \geqq 0, \tag{1.10}$$

where $].[$ denotes the nearest integer not less than the real number specified, is a Markov chain with the same distribution as $\{U_n; n \geqq 0\}$, and

$$\log U_n^* = \sum_{k=1}^{n} Y_k + O(1) \quad \text{a.s.} \quad (n \to \infty) \tag{1.11}$$

For a refinement of this, see [7]. In [6] it was also shown that

$$E(\log U_n)=n+1-\gamma+O(2^{-n}), \quad \operatorname{Var}(\log U_n)=n+1-\frac{\pi^2}{6}+O(n^3 2^{-n})$$
$$E(\log \Delta_n)=n-\gamma+O(n2^{-n}), \quad \operatorname{Var}(\log \Delta_n)=n+\frac{\pi^2}{6}+O(n^2 2^{-n}), \quad n\to\infty, \tag{1.12}$$

where $\gamma=0.577216$ denotes Euler's constant and Var means variance. An intuitive explanation for the last two relations is due to the strong approximation approach by Deheuvels [1, 2] who proved, starting with relation (1.6), that on the same probability space $(\Omega, \mathscr{A}, P)$ (possibly after enlarging) there exists a Poisson point process $\{T_n; n\geqq 1\}$ with unit rate and an i.i.d. sequence $\{Y_n: n\geqq 1\}$ of unit mean exponential r.v.'s, independent from the Poisson process, such that

$$\log \Delta_n=\log Y_n+T_n+o(1) \quad \text{a.s.} \quad (n\to\infty) \tag{1.13}$$

Here, $-\log Y_n$ is doubly exponentially distributed with mean γ and variance $\frac{\pi^2}{6}$. In fact, $T_n=-\log(1-F(X_{U_{n-1}}))$, $n\geqq 1$, as can be seen from Deheuvels' construction. The rate of convergence in (1.13) was made precise in [7] where it was shown that

$$\log \Delta_n=\log Y_n+T_n+o\left(\exp\left(-n+nH\left(\frac{1}{n}\right)\right)\right) \quad \text{a.s.} \quad (n\to\infty) \tag{1.14}$$

where $tH\left(\frac{1}{t}\right)$ belongs to the upper class of a Wiener process, i.e. $H(t)$ is a positive function defined in some positive neighbourhood of the origin such that $H(t)\uparrow$ and $t^{-1/2}H(t)\downarrow$, and the integral

$$I=\int_{0+}^{\infty} t^{-3/2} H(t)\exp(-H^2(t)/2t)\,dt \tag{1.15}$$

converges. It was also shown that this rate result cannot be extended to lower class functions (i.e. H as above with I being divergent), not even if $o(.)$ is replaced by $O(.)$.

Deheuvels [1] also gave a strong approximation result for record times, based on the one derived for the inter-record times, i.e., with the notation of (1.13),

$$\limsup \pm\frac{U_n-\sum_{k=-\infty}^{n} Y_k e^{T_k}}{\sqrt{2n\log\log n}}=3^{-1/2} \quad \text{a.s.}, \tag{1.16}$$

where now the index range for the sequences involved is extended to $\mathbb{Z}$, and also a corresponding CLT. It was also proved that this strong approximation result is best possible if the construction based on (1.6) is used.

It is the aim of the present paper to develop a corresponding strong approximation (on the same probability space) based on William's approach

(1.10), even valid for exchangeable sequences $\{X_n, n \geqq 1\}$, which simplifies (1.16) and at the same time allows for a joint strong approximation of record and inter-record times. Besides the limit laws mentioned above, also Galambos' and Seneta's [3] results for the i.i.d. case are easily reobtained, and some others, including exact estimations for the rate of convergence as in (1.14).

In fact, it is easy to see that if $\{X_n; n \geqq 1\}$ is an exchangeable sequence and the probability of ties is zero (i.e. $P(X_1 = X_2) = 0$), then again record and inter-record times can be defined as in (1.1), likewise for the record values $\{X_{U_n}; n \geqq 0\}$. Namely, by de Finetti's theorem, we may assume that there exists a real r.v. Λ on $(\Omega, \mathscr{A}, P)$ such that conditionally on $\Lambda = \lambda \in \mathbb{R}$, $\{X_n; n \geqq 1\}$ is an i.i.d. sequence with c.d.f. F_λ, say. Now since ties occur with probability zero only we must have $P\{X_1 = X_2 | \Lambda = \lambda) = 0$ P^Λ-a.s. which in turn implies that F_λ is continuous for P^Λ-almost all λ, i.e. given $\Lambda = \lambda$, $\{U_n; n \geqq 0\}$ is a Markov chain with transition probabilities given by (1.2), independent of λ, for P^Λ-almost all λ. Hence under exchangeability, if ties occur with probability zero only, $\{U_n; n \geqq 0\}$ and $\{\Delta_n; n \geqq 0\}$ are a.s. well defined, the record times forming a homogeneous Markov chain with transition probabilities given by (1.2) as in the i.i.d. case.

Note that the condition of zero probabilities for ties can in general not be replaced by a continuity assumption on the marginal distribution (as in the i.i.d. case) as can be seen by the exchangeable sequence $\{X_n; n \geqq 1\}$ where $X_n \equiv X_1$ for all $n \geqq 2$. Of course, in the case of independence, these two conditions coincide.

As has become obvious from the preceding remarks, all relevant information on record and inter-record times is contained in the Markov chain with transition probabilities given by (1.2). It will therefore be necessary to deal with strong approximation techniques for Markov chains as developed in the following section.

II. Strong Approximation for Markov Chains

Theorem 1. *Let $\{S_n; n \geqq 0\}$ be a real-valued homogeneous Markov chain on the probability space $(\Omega, \mathscr{A}, P)$ with $S_0 =$ constant a.s. and regular transition probabilities (which here always exist). Let further denote $F(.|.)$ the corresponding conditional c.d.f. Then there exists an i.i.d. sequence $\{V_n; n \geqq 1\}$ of uniformly $\mathscr{U}[0,1]$-distributed r.v.'s on the same probability space (eventually after enlarging) such that*

$$S_{n+1} = F^{-1}(V_{n+1} | S_n) \qquad \text{a.s. for all } n \geqq 0 \tag{2.1}$$

where $F^{-1}(v|.) = \inf\{z | F(z|.) \geqq v\}$, $0 < v < 1$ denotes the pseudo-inverse of $F(.|.)$.

Proof. Let $\{W_n; n \geqq 1\}$ be an i.i.d. sequence of $\mathscr{U}[0,1]$-distributed r.v.'s on $(\Omega, \mathscr{A}, P)$ independent of $\{S_n; n \geqq 0\}$ (which eventually exists after enlarging the probability space). Let further denote

$$F_{-}(z|.) = \lim_{h \downarrow 0} F(z-h|.), \qquad z \in \mathbb{R}. \tag{2.2}$$

Define

$$V_{n+1}=(1-W_{n+1})F(S_{n+1}|S_n)+W_{n+1}F_-(S_{n+1}|S_n), \qquad n\geqq 0. \tag{2.3}$$

Then $\{V_n;\ n\geqq 1\}$ is an i.i.d. sequence of $\mathscr{U}[0,1]$-distributed r.v.'s on $(\Omega,\mathscr{A},P)$ since by construction, for all $n\geqq 0$,

$$P^{(V_1,\ldots,V_{n+1})}(.|S_1,\ldots,S_{n+1})=\bigotimes_{k=0}^{n}\mathscr{U}[F_-(S_{k+1}|S_k);F(S_{k+1}|S_k)] \text{ a.s.} \tag{2.4}$$

(where in the degenerate case, $\mathscr{U}[z,z]$ is to be interpreted as the Dirac measure concentrated in $z\in\mathbb{R}$), hence by integration,

$$P^{(V_1,\ldots,V_{n+1})}=\bigotimes_{k=0}^{n}\mathscr{U}[0,1] \qquad \text{for all } n\geqq 0. \tag{2.5}$$

Now

$$S_{n+1}=F^{-1}(V_{n+1}|S_n) \qquad \text{iff } F_-(S_{n+1}|S_n)<V_{n+1}\leqq F(S_{n+1}|S_n) \text{ a.s.} \tag{2.6}$$

for all $n\geqq 0$ which by (2.3) holds a.s. This completes the proof.

A straightforward generalization to non-homogeneous or higher-dimensional Markov chains is obvious from the preceding proof but will be omitted here. In the case of record times, we have, by (1.2),

$$F(k|j)=1-\frac{j}{k}, \qquad 1\leqq j\leqq k, \tag{2.7}$$

hence by (2.3),

$$V_{n+1}=(1-W_{n+1})\frac{U_n}{U_{n+1}}+W_{n+1}\frac{U_n}{U_{n+1}-1}, \qquad n\geqq 0, \tag{2.8}$$

forms an i.i.d. $\mathscr{U}[0,1]$-distributed sequence (here we have used the fact that with V also $1-V$ is $\mathscr{U}[0,1]$-distributed). Further, letting $Y_n=-\log V_n$, $n\geqq 1$, we obtain an i.i.d. sequence of unit mean exponential r.v.'s. Theorem 1 thus translates into the following result.

Corollary 1. *Without loss of generality, there exists an i.i.d. sequence* $\{Y_n;\ n\geqq 1\}$ *on the same probability space where* $\{U_n;\ n\geqq 0\}$ *is defined which is exponentially distributed with unit mean such that*

$$U_{n+1}=\,]U_n e^{Y_{n+1}}[\qquad \textit{a.s. for all } n\geqq 0 \tag{2.9}$$

$$U_{n+1}-U_n e^{Y_{n+1}}=W_{n+1}\frac{U_{n+1}}{U_{n+1}+W_{n+1}-1} \qquad \textit{a.s. for all } n\geqq 0 \tag{2.10}$$

where $Y_{n+1}=-\log V_{n+1}$ *and* V_{n+1}, W_{n+1} *as in* (2.8)

$$\log U_{n+1}-\log U_n-Y_{n+1}=\log\left(1+\frac{W_{n+1}}{U_{n+1}-1}\right) \qquad \textit{a.s. for all } n\geqq 0 \tag{2.11}$$

$$\left]\frac{U_{n+1}}{U_n}\right[=\,]e^{Y_{n+1}}[\qquad \textit{a.s. for all } n\geqq 0, \tag{2.12}$$

providing an i.i.d. sequence with c.d.f. given by $F(.|1)$ *with* $F(.|.)$ *as in* (2.7).

Relation (2.12) gives a simple proof for the main result in Galambos and Seneta [3]. It should be pointed out here that for all $n \geqq 0$, Y_{n+1} and V_{n+1} are independent from $(U_0, \ldots, U_n)$ as can be seen from the recursive structure in (2.9).

Relation (2.11) will be the key for a joint strong approximation for the logarithms of record and inter-record times. This will be worked out in more detail in the following section.

III. Joint Strong Approximation for Record and Inter-Record Times

Theorem 2. *Without loss of generality, there exists on the same probability space where $\{U_n;\ n \geqq 0\}$ is defined a unit-rate Poisson point process $\{T_n;\ n \geqq 1\}$ and a non-negative r.v. Z possessing all positive moments with mean $E(Z) = 1 - \gamma$ such that*

$$Z \text{ and } \{(T_n - n)/\sqrt{n};\, n \geqq 1\} \text{ are asymptotically independent;}$$
$$\log U_n = Z + T_n + o\left(\exp\left(-n + nH\left(\frac{1}{n}\right)\right)\right) \quad \text{a.s. for } n \to \infty, \tag{3.1}$$

$$\log \Delta_n = Z + T_n + \log(1 - \exp(T_{n-1} - T_n)) + o\left(\exp\left(-n + nH\left(\frac{1}{n}\right)\right)\right) \quad \text{a.s. for } n \to \infty, \tag{3.2}$$

where again $tH\left(\frac{1}{t}\right)$ belongs to the upper class of a Wiener process. The rate result cannot be improved to lower class functions, not even if $o(.)$ is replaced by $O(.)$.

Proof. Summing the equalities in (2.11) we obtain

$$\log U_{n+1} - \sum_{k=0}^{n} Y_{k+1} = \sum_{k=0}^{n} \log\left(1 + \frac{W_{k+1}}{U_{k+1} - 1}\right) \quad \text{a.s.,} \quad n \geqq 0. \tag{3.3}$$

Now let

$$T_n = \sum_{k=1}^{n} Y_k \quad \text{for } n \geqq 1, \qquad Z = \sum_{k=1}^{\infty} \log\left(1 + \frac{W_k}{U_k - 1}\right). \tag{3.4}$$

From relation (2.8) we see that

$$\sum_{k=0}^{\infty} \log\left(1 + \frac{W_{k+1}}{U_{k+1}}\right) \leqq Z \leqq \sum_{k=0}^{\infty} \frac{W_{k+1}}{U_k} \tag{3.5}$$

with $1/U_k \leqq \exp(-T_k)$ a.s. from which it follows that Z and the normalized Poisson process $\{(T_n - n)/\sqrt{n};\ n \geqq 1\}$ are asymptotically independent. Also, relation (3.5) shows that the rate of convergence in (3.2) is exactly determined by the tail series

$$\sum_{k=n}^{\infty} \frac{W_{k+1}}{U_k} \leqq \sum_{k=n}^{\infty} \exp(-T_k) \quad \text{a.s.} \tag{3.6}$$

from which the $o(.)$-result follows as in (7), Theorem 2. On the other hand,

$$\frac{W_n}{U_n}=\exp(\log W_n-\log U_n)=\exp(O(\log n)-T_n)) \quad \text{a.s.} \quad (n\to\infty) \tag{3.7}$$

which shows that the rate result cannot be improved to lower class functions (cf. the proof of Theorem 4 in Deheuvels [2]). For the proof of the second statement in (3.2) note that by (2.10), we have

$$\frac{\Delta_{n+1}}{U_n}=\frac{U_{n+1}}{U_n}-1=e^{Y_{n+1}}-1+\frac{W_{n+1}+o(1)}{U_n} \quad a.s. \quad (n\to\infty), \tag{3.8}$$

hence

$$\log \Delta_{n+1}-\log U_n=Y_{n+1}+\log(1-e^{-Y_{n+1}})+o(e^{-cn}) \quad a.s. \quad (n\to\infty) \tag{3.9}$$

with $1<c<2$ arbitrary (but fixed).

The fact that $E(Z)=1-\gamma$ follows from (1.12) and (3.5) by the Dominated Convergence Theorem. This proves Theorem 2 completely.

It is interesting to note that in (3.2), the sequence $\{-\log(1-\exp(T_{n-1}-T_n));\ n\geqq 2\}$ is i.i.d. following an exponential distribution with unit mean, which is of order $O(\log n)$ a.s. for $n\to\infty$.

Theorem 2 gives a unified proof of all the limit relations (1.3) to (1.5) and (1.7) to (1.9), even for the exchangeable case. It can also be used to give complete characterizations for the upper and lower class of the record and inter-record times as in Deheuvels (2), which in the light of (3.2) are the same as those for a unit rate Poisson process. Some other consequences are listed below.

Corollary 2. $\{\log U_n-\log \Delta_n;\ n\geqq 1\}$ *is asymptotically i.i.d. with unit-mean exponential distribution;* (3.10)

$\{\log \Delta_n-\log U_{n-1};\ n\geqq 1\}$ *is asymptotically i.i.d., the asymptotic distribution being the same as that of* $W=Y+\log(1-e^{-Y})$ *where* Y *(and hence also* $-\log(1-e^{-Y})$*) follows a unit-mean exponential distribution, giving* $E(W)=0$ *and* $\mathrm{Var}(W)=\frac{\pi^2}{3}$ *(in fact, also* $W=U-V$ *where* U *and* V *are independent doubly exponentially distributed r.v.'s, implying that* W *follows a logistic distribution with* $P(W\leqq x)=(1+e^{-x})^{-1}$, $x\in\mathbb{R}$*);* (3.11)

$P(U_n>n^{1+\varepsilon}\Delta_n \ i.o.)=0, \quad P(U_n>n^{1-\varepsilon}\Delta_n \ i.o.)=1$ *for every* $0<\varepsilon<1$; (3.12)

$P(\Delta_n>s\,U_{n-1})=P(\log U_n-\log \Delta_n<\log(1+1/s))\to\frac{1}{1+s}$ *for* $n\to\infty$ *and every* $s>0$ *(Galambos and Seneta [3]);* (3.13)

there is no sequence $\{K_n;\ n\geqq 1\}$ *of real constants such that* U_n/K_n *or* Δ_n/K_n *follows some non-degenerate limit law for* $n\to\infty$ *(Shorrock [11], Tata [13]);* (3.14)

$\{U_n/U_{n+1};\ n \geqq 0\}$ *is asymptotically i.i.d. with* $\mathscr{U}[0,1]$ *distribution (Shorrock* [11]; *Tata* [13]); (3.15)

$\left\{\log \frac{U_{n+k}}{U_n};\ k \geqq 1\right\}$ *is - up to an a.s. error of magnitude* $o\left(\exp\left(-n + nH\left(\frac{1}{n}\right)\right)\right)$ *- uniformly close to a unit rate Poisson point process.* (3.16)

Of course, many more of such limit relations can immediately be derived from Theorem 2.

Finally, relations (3.10) and (3.11) show that with respect to the normal approximation of record and inter-record times given in (1.4) and (1.8), we have

$$\frac{\log U_n - n}{\sqrt{n}} - \frac{\log \Delta_n - n}{\sqrt{n}} = O(n^{-1/2}\log n) \quad \text{a.s.} \quad (n \to \infty) \tag{3.17}$$

which cannot be improved upon. This explains why for small or moderate values of n this approximation cannot give satisfactory results.

Concluding remarks. Obviously, the Poisson processes used in Deheuvel's representation (1.13) and in our approach (3.2) are not the same, hence Theorem 2 does not give joint results for record times and record values as in (1.16). However, a possible connection between these two representations can be seen as follows.

Let $\{E_n;\ n \geqq 1\}$ be the i.i.d. sequence of unit-mean exponential r.v.'s forming the increments of the Poisson point process used in (3.2). Starting with (1.13), we obtain

$$\begin{aligned} \log \Delta_{n+1} - \log \Delta_n &\cong T_{n+1} - T_n + \{\log Y_{n+1} - \log Y_n\} \\ &\stackrel{\mathscr{L}}{=} -\log(1 - e^{-E_n}) + \{E_{n+1} + \log(1 - e^{-E_{n+1}})\} \end{aligned} \tag{3.18}$$

since $T_{n+1} - T_n$ and $\{\log Y_{n+1} - \log Y_n\}$ are independent and by (3.11),

$$\log Y_{n+1} - \log Y_n \stackrel{\mathscr{L}}{=} E_{n+1} + \log(1 - e^{-E_{n+1}}),$$

following a logistic distribution. But the right hand side of (3.18) is just (asymptotically) the increment of the logarithmic inter-record time sequence derived from the representation (3.2). In the light of relation (3.18) it seems impossible to establish a direct (strong) relationship between the different Poisson processes involved, unless constructions as in (1.16) are considered.

Acknowledgements. I would like to thank P. Deheuvels for some stimulating discussions in this area of extremal statistics.

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Received October 17, 1984; in revised form November 5, 1985