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POISSON APPROXIMATIONS IN SELECTED METRICS BY COUPLING AND SEMIGROUP METHODS WITH APPLICATIONS

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Abstract: Let $S_n = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are independent Bernoulli random variables. In this paper, we evaluate probability metrics of the Wasserstein type between the distribution of S_n and a Poisson distribution. Our results show that, if $E(S_n) = \mathrm{O}(1)$ and if the individual probabilities of success of the X_i's tend uniformly to zero, then the general rate of convergence of the above mentioned metrics to zero is $\mathrm{O}(\sum_{i=1}^n p_i^2)$. We also show that this rate is sharp and discuss applications of these results.

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1. Introduction

There has been in the last decades a continued interest for the approximation of the distribution of $S_n = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are independent Bernoulli random variables with probabilities of success $p_1, \ldots, p_n$ by a Poisson random variable T_n whose expectation is a function of $p_1, \ldots, p_n$.

The main justification which has been given for such investigations is that the exact distribution of S_n is in general extremely involved for unequal p_i's and large values of n. Simple examples show in fact that exact evaluations may require an excessive computing time in practice. Therefore it is logical to seek a simple approximant. The most reasonable candidate for such an approximation has been found to be the Poisson distribution.

From the poineering work of LeCam (1960) to the recent papers of Barbour and Hall (1984) and Deheuvels and Pfeifer (1986a) a great effort has been made in orde to evaluate the error of these approximations by measuring the fit via the total variation distance

$$d_0(L(S_n), L(T_n)) = \sup_A |P(S_n \in A) - P(T_n \in A)|$$
$$= \frac{1}{2} \sum_{k=0}^{\infty} |P(S_n = k) - P(T_n = k)|. \tag{1.1}$$

Several precise evaluations of $d_0(L(S_n), L(T_n))$ are now known, so that one may consider that this problem is basically solved, except for second order refinements.

The motivation for this paper is that, in spite of the fact that a precise knowledge of the magnitude of $d_0(L(S_n), L(T_n))$ is of great practical interest, it does not answer all questions.

Consider for instance the following simple example. Suppose that one knows the exact values of $p_1, \ldots, p_n$ and that the aim is to compute from these values a and b such that $P(a \le S_n \le b) \ge 95\%$. Clearly a Poisson approximation will be of no use for this sake, if we only know that (say) $d_0(L(S_n), L(T_n)) \simeq 10\%$.

It turns out that one may give answers to such a problem by computing other measures of fit that the total variation distance. Aside from the Kolmogorov distance, this leads to the general question of evaluating the distance between the distributions of S_n and T_n for an arbitrary distance in the space of probability distributions.

The aim of this paper is precisely to bring solutions to this problem in the general setting of *minimal metrics* of the form

$$d^{(\delta)}(L(S_n), L(T_n)) = \inf E(\delta(|S_n - T_n|)), \tag{1.2}$$

where $\delta(\cdot)$ is a non-negative function, and where the infimum in (1.2) is taken over all possible joint distributions of (S_n, T_n) with the given marginals.

In Section 2, we give a general review of the properties of minimal metrics (we use this term in spite of the fact that further assumptions are requested on $\delta(\cdot)$ for (1.2) to define a metric in the usual sense) which will be needed in our work. We also show that d_0 is a minimal metric for an appropriate δ.

In Section 3, we use *coupling techniques* for the evaluation of $d^{(\delta)}(L(S_n), L(T_n))$. Such methods consist in building explicit joint distributions of S_n and T_n to obtain bounds for $d^{(\delta)}$. We also give in this section lower bounds based on direct methods.

In Section 4, we use semigroup arguments which enable us to obtain further upper bounds for $d^{(\delta)}$ under very weak assumptions on $\delta(\cdot)$.

Section 5 contains some examples of applications of this work, with emphasis on confidence intervals.

Our main result, captured in a series of theorems, is that, whenever $p_1 = p_{1,n}, \ldots, p_n = p_{n,n}$ vary in such a way that $\sum_{i=1}^{n} p_i = \mathrm{O}(1)$ and $\max_{1 \le i \le n} p_i \to 0$, then, under very weak assumptions on $\delta(\cdot)$, there exist constants $0 < C_1 \le C_2 < \infty$ such that, as $n \to \infty$,

$$C_1 \sum_{i=1}^{n} p_i^2 \le d^{(\delta)}(L(S_n), L(T_n)) \le C_2 \sum_{i=1}^{n} p_i^2. \tag{1.3}$$

The interesting feature of (1.3) is that (with the exception of the values of the normalizing constants C_1 and C_2) the rate $\mathrm{O}(\sum_{i=1}^{n} p_i^2)$ for $d^{(\delta)}(L(S_n)L(T_n))$ does not depend upon δ.

Asymptotic evaluations for the total variation distance and for the Fortet–Mourier metric (which corresponds to $\delta(u) = |u|$) given in Deheuvels and Pfeifer (1986a,b) show that such a result fails to be true in the range where $\sum_{i=1}^{n} p_i \to \infty$, so that (1.3) gives an optimal result in the case we consider.

2. Probability metrics

Let (Ω, A, P) be a probability space, and let $\mathbf{X}$ denote the set of all random variables taking values in $\mathbf{N} = \{0, 1, \ldots\}$ and defined on (Ω, A, P). We shall denote by $\mathscr{P} = \mathscr{P}_1 = \{L(\xi)\}$, $\mathscr{P}_2 = \{L(\xi, \zeta)\}, \ldots$ the sets of all probability distributions $L(\xi)$ of ξ, $L(\xi, \zeta)$ of $(\xi, \zeta), \ldots$, where $\xi, \zeta, \ldots \in \mathbf{X}$. Throughout, we shall make the assumption that (Ω, A, P) is rich enough to carry all possible distributions in $\mathscr{P}_k$, where $k \ge 1$ is an arbitrary integer. In other words, for any multi-indexed non-negative sequence $\{p_{i_1, \ldots, i_k}\}_{i_1 \ge 0, \ldots, i_k \ge 0}$ with sum equal to one, there exist random variables $\xi_1, \ldots, \xi_k \in \mathbf{X}$ such that, for all $i_1 \ge 0, \ldots, i_k \ge 0$,

$$P(\xi_1 = i_1, \ldots, \xi_k = i_k) = p_{i_1, \ldots, i_k}. \tag{2.1}$$

The existence of probability spaces which fulfill this condition can be proved by Kolmogorov's theorem. Notice (see e.g. Bauer (1981) pp. 168–169) that there exists an (Ω, A, P) which carries all distributions in $\mathscr{P}_k$ for an arbitrary $k \ge 1$. The same arguments by taking the product spaces for all values of $k = 1, 2, \ldots$ gives the result.

The following lemma will be useful in the sequel.

Lemma 1. *Let A, B and C be complete separable metric spaces, let P_1 be a probability measure on $A \times B$ and P_2 on $B \times C$. If the marginals $P_1|_B$ and $P_2|_B$ coincide, then there exists a probability measure P on $A \times B \times C$ such that $P|_{A \times B} = P_1$ and $P|_{B \times C} = P_2$.*

Proof. See Berkes and Philipp (1979). Note here that $\mathbf{N}, \mathbf{N}^2, \ldots$ are separable complete metric spaces with the discrete metric $\varrho_d(p, q) = 1_{\{p \ne q\}}$, the corresponding Borel sets being all subsets of $\mathbf{N}, \mathbf{N}^2, \ldots$. $\square$

Following Zolotarev (1984), we shall say that a function μ defined on $\mathscr{P}_2$ and assuming values in the extended interval $[0, \infty]$ defines a *probability metric* on $\mathbf{X}$ if the following conditions hold for all ξ, ζ and $\eta \in \mathbf{X}$:

(D1) $P(\xi = \zeta) = 1 \Rightarrow \mu(L(\xi, \zeta)) = 0$;

(D2) $\mu(L(\xi, \zeta)) = \mu(L(\zeta, \xi))$;

(D3) $\mu(L(\xi, \eta)) \leq \mu(L(\xi, \zeta)) + \mu(L(\zeta, \eta))$.

Let ϱ be a function defined on $\mathbf{N}^2$ and taking values in $[0, \infty]$. The following lemma gives a simple construction of probability metrics based on ϱ:

Lemma 2. *Assume that* $\varrho : \mathbf{N}^2 \to [0, \infty]$ *satisfies for all* p, q *and* $r \in \mathbf{N}$:

(d1) $p = q \Rightarrow \varrho(p, q) = 0$;

(d2) $\varrho(p, q) = \varrho(q, p)$;

(d3) $\varrho(p, r) \leq \varrho(p, q) + \varrho(q, r)$.

Then, for all $1 \leq \gamma < \infty$, $\mu_{\gamma, \varrho}(L(\xi, \zeta)) = E^{1/\gamma}(\varrho^{\gamma}(\xi, \zeta))$ *defines a probability metric on* $\mathbf{X}$.

Proof. (D1) and (D2) are straightforward from (d1) and (d2). For (D3), we use (d3) jointly with the convexity inequality $E(\psi(U, V)) \leq \psi(E(U), E(V))$, where $\psi(u, v) = (u^{1/\gamma} + v^{1/\gamma})^{\gamma}$, $U = \varrho^{\gamma}(\xi, \zeta)$ and $V = \varrho^{\gamma}(\zeta, \eta)$. □

In the sequel, we will consider the following main examples.

$$\varrho(p, q) = A(|p - q|), \tag{2.2}$$

where $A(0) = 0$, $A(n) = \sum_{i=1}^{n} a_i$, and where $\{a_n, n \geq 1\}$ is a non-negative non-increasing sequence.

$$\varrho(p, q) = \theta_s(p - q) = |p - q|^s, \quad 0 < s \leq 1. \tag{2.3}$$

$$\varrho(p, q) = \theta_0(p - q) = 1_{\{p \neq q\}}, \tag{2.4}$$

where we denote by 1_E the indicator function of E.

We will now introduce the notion of *minimal metric* (Zolotarev (1976)) as follows. Let μ be a probability metric on $\mathbf{X}$. The corresponding minimal metric $\hat{\mu}$ is defined for all $\xi, \zeta \in \mathbf{X}$ by

$$\hat{\mu}(\xi, \zeta) = \inf \mu(L(\xi, \zeta)), \tag{2.5}$$

where the infimum is taken over all joint distributions $L(\xi, \zeta)$ with fixed margins $L(\xi)$ and $L(\zeta)$. We shall call such a joint distribution a *coupling* of ξ and ζ, and denote by $C(\xi, \zeta)$ the set of all couplings of ξ and ζ.

Lemma 3. *If* μ *is a probability metric on* $\mathbf{X}$, *then* $\hat{\mu}$ *is also a probability metric on* $\mathbf{X}$.

Proof. Here and in the sequel, we shall let $\mu(\xi, \zeta) = \mu(L(\xi, \zeta))$ when μ is a probability metric. The proof that $\hat{\mu}$ satisfies (D1) and (D2) is straightforward. For (D3), we limit ourselves to the case where $\hat{\mu}(\xi,\zeta)<\infty$ and $\hat{\mu}(\zeta,\eta)<\infty$. Let $\varepsilon>0$ be fixed. There exist couplings $L(\xi,\zeta)$ and $L(\zeta,\eta)$ such that

$$\mu(L(\xi,\zeta))+\mu(L(\zeta,\eta))\leq\hat{\mu}(\xi,\zeta)+\hat{\mu}(\zeta,\eta)+\varepsilon. \tag{2.6}$$

By Lemma 1, we see that there exists a joint distribution $L(\xi,\zeta,\eta)$ with margins $L(\xi,\zeta)$ and $L(\zeta,\eta)$. By (D3) and (2.6), this implies that

$$\mu(L(\xi,\zeta))\leq\mu(L(\xi,\zeta))+\mu(L(\zeta,\eta))\leq\hat{\mu}(\xi,\zeta)+\hat{\mu}(\zeta,\eta)+\varepsilon. \tag{2.7}$$

Since (2.7) holds for all $\varepsilon>0$, this implies that $\hat{\mu}(\xi,\zeta)\leq\hat{\mu}(\xi,\zeta)+\hat{\mu}(\zeta,\eta)$. □

Notice that, with our definitions, if $L(\xi)=L(\zeta)$, then there exists a coupling $L(\xi,\zeta)$ of ξ and ζ such that $P(\xi=\zeta)=1$. It follows that $\hat{\mu}(\xi,\zeta)=0$. This shows that $\hat{\mu}(\xi,\zeta)=\hat{\mu}(L(\xi),L(\zeta))$ depends only upon the marginal distributions $L(\xi)$ of ζ and $L(\zeta)$ of ζ. Such a probability metric (which defines an unbounded semi-distance on $\mathscr{P}$) is usually called *simple*.

We will now specialize in the so-called *Wasserstein metrics* (Wasserstein (1969)), originally introduced by Kantorovitch and Rubinstein (1958), defined for $\xi,\zeta\in\mathbf{X}$ by

$$\begin{aligned} d_{\gamma,\varrho}(\xi,\zeta)&=d_{\gamma,\varrho}(L(\xi),L(\zeta))=\hat{\mu}_{\gamma,\varrho}(\xi,\zeta)\\ &=\inf_{C(\xi,\zeta)}E^{1/\gamma}(\varrho^{\gamma}(\xi,\zeta)),\quad 1\leq\gamma<\infty, \end{aligned} \tag{2.8}$$

where the infimum is taken as in (2.5), and where ϱ satisfies (d1)–(d3).

In 1982, Szulga proved the following result, in the case of separable metric spaces.

Lemma 4. *Let ϱ satisfy* (d1)–(d3). *Then*

$$d_{1,\varrho}(\xi,\zeta)=\sup\{E(f(\xi)-f(\zeta))\}, \tag{2.9}$$

where the supremum is taken over all functions f such that, for all $p,q\in\mathbf{N}$,

$$|f(p)-f(q)|\leq\varrho(p,q). \tag{2.10}$$

Further extensions of Szulga's result have been given by Kellerer (1982) who showed among other theorems that, for $\gamma=1$, the infimum in (1.8) is reached. In fact, his argument directly extends for a general $1\leq\gamma<\infty$ so that we have:

Lemma 5. *Let ϱ satisfy* (d1)–(d3). *Then, for any $L(\xi)$ and $L(\zeta)\in\mathscr{P}$, and for any $1\leq\gamma<\infty$, there exists a joint distribution $L(\xi,\zeta)\in\mathscr{P}_2$ such that*

$$d_{\gamma,\varrho}(\xi,\zeta)=E^{1/\gamma}(\varrho^{\gamma}(\xi,\zeta)). \tag{2.11}$$

The joint distribution $L(\xi,\zeta)$ of ξ and ζ such that (1.11) holds is called a *maximal coupling* of ξ and ζ. In the sequel, we shall give explicit descriptions of maximal couplings in several examples (see e.g. Proposition 3 and Lemma 6).

Example 1. The *total variation metric* is a Wasserstein metric corresponding to (2.4) and can be defined by (see e.g. Dobrushin (1970))

$$d_0(\xi,\zeta)=\inf_{C(\xi,\zeta)} P(\xi\neq\zeta)=\sup_{A\subset\mathbf{N}}|P(\xi\in A)-P(\zeta\in A)|. \tag{2.12}$$

Observe that the probability metric associated to the discrete distance by $\mu(\xi,\zeta)=P(\xi\neq\zeta)=E(1_{\{\xi\neq\zeta\}})$ coincides on $\mathbf{N}$ with the Ky–Fan metric (distance in probability) $K(\xi,\zeta)=\inf\{\varepsilon\colon 0\le\varepsilon\le 1,\ P(|\xi-\zeta|>\varepsilon)\le\varepsilon\}$. It follows that (2.12) is a special case of a result of Strassen (1965) who showed that the Prohorov distance $\pi(\xi,\zeta)=\inf\{\varepsilon\colon 0<\varepsilon<1\colon P(\xi\in C)\le P(\zeta\in C^{\varepsilon})+\varepsilon,\ P(\zeta\in C)\le P(\xi\in C^{\varepsilon})+\varepsilon$ for all closed sets $C\}$ coincides with the minimal metric associated to K (see also Dudley (1968)), namely

$$\pi(\xi,\zeta)=\inf_{C(\xi,\zeta)} K(\xi,\zeta). \tag{2.13}$$

It is straightforward on $\mathbf{N}$ that $\pi(\xi,\zeta)=d_0(\xi,\zeta)$.

Example 2. The Fortet–Mourier (1953) metric (see e.g. Vallender (1973)) can be defined by

$$\begin{aligned} d_1(\xi,\zeta)&=\inf_{C(\xi,\zeta)} E(|\xi-\zeta|)=\sum_{n=0}^{\infty}|P(\xi\le n)-P(\zeta\le n)| \\ &=\sup E(f(\xi)-f(\xi)), \end{aligned} \tag{2.14}$$

where the supremum is taken over all functions f such that, for all $p,q\in\mathbf{N}$, $|f(p)-f(q)|\le|p-q|$.

Example 3. For any $0<\gamma<\infty$, define

$$d_\gamma(\xi,\zeta)=\inf_{C(\xi,\zeta)} E^{1/\max(\gamma,1)}(|\xi-\zeta|^{\gamma}). \tag{2.15}$$

Observe by (2.3) that for $0<\gamma/s\le 1$, $\varrho(p,q)=|p-q|^{\gamma/s}$ satisfies (d1)-(d3), so that by Lemma 2, $E^{1/s}(\varrho^{s}(p,q))$ is a probability metric for all $s\ge\max(\gamma,1)$. It follows from Lemma 3 that d_γ is always a probability metric. Here, the definition of d_γ coincides with that of d_1 for $\gamma=1$. Furthermore, for any $0\le\gamma'\le\le\gamma''<\infty$, we always have

$$d_{\gamma'}(\xi,\zeta)\le d_{\gamma''}(\xi,\zeta). \tag{2.16}$$

Finally, we see that $|\xi-\zeta|^{\gamma}\downarrow 1_{\{\xi\neq\zeta\}}$ as $\gamma\downarrow 0$, which justifies the notation for d_0.

It is noteworthy that the metric d_0 induces on $\mathscr{P}$ the topology of weak convergence (see e.g. Billingsley (1969)). Aside from this case and the case $\gamma=1$, very little is known concerning d_γ (see Rüschendorf (1985) and the references therein). For $\gamma\ge 1$, Rachev (1982) showed that the following statements are equivalent:

$$d_\gamma(\xi_n,\xi)\to 0 \quad \text{as } n\to\infty, \tag{2.17a}$$

$$\xi_n \xrightarrow{\mathrm{w}} \xi \text{ and } E(|\xi_n|^\gamma) \to E(|\xi|^\gamma) \quad \text{as } n \to \infty, \tag{2.17b}$$

where $\xrightarrow{\mathrm{w}}$ denotes weak convergence and where $\xi_1, \xi_2, \ldots, \xi \in L^\gamma(\Omega, A, P)$.

Further references on probability metrics are to be found in Dudley (1968) and Zolotarev (1976, 1984).

3. Poisson approximations by couplings

In this section, we consider a non-negative function $\delta : \mathbf{N} \to [0, \infty)$ and evaluate $d^{(\delta)}(\xi, \zeta) = \inf_{C(\xi,\zeta)} E(\delta(|\xi - \zeta|))$ under additional assumptions on $\xi, \zeta \in \mathbf{X}$ and δ. First, we consider the simple case where ξ follows a Bernoulli distribution.

3.1. Individual couplings

We assume here that $\xi = X$ follows a Bernoulli $B(p)$ distribution (i.e. $P(X=1) = 1 - P(X=0) = p \in (0,1)$) and that Y follows a Poisson $P(\Lambda)$ distribution (with expectation Λ). In our first result, we compute the exact value of $d^{(\delta)}(X, Y)$.

Proposition 1. *Assume that $\delta(j)$ is non-decreasing in $j = 0, 1, \ldots$. Suppose that $X \sim B(p)$ and $Y \sim P(\Lambda)$, where $\Lambda \le \lambda = -\log(1-p)$. Then*

$$d^{(\delta)}(X, Y) = (1-p)\delta(0) + (\mathrm{e}^{-\Lambda} - 1 + p)\delta(1) + \sum_{j=1}^{\infty} \frac{\Lambda^j}{j!} \mathrm{e}^{-\Lambda} \delta(j-1). \tag{3.1}$$

Proof. Let $p_{ij} = P(X=i, Y=j)$. Notice that, for all $j = 0, 1, \ldots$, $p_{0j} = (\Lambda^j / j!)\mathrm{e}^{-\Lambda} - p_{1j}$. Hence, by replacing p_{0j} by this expression in

$$E(\delta(|X-Y|)) = \sum_{j=0}^{\infty} p_{1j} \delta(|j-1|) + \sum_{j=0}^{\infty} p_{0j} \delta(j),$$

we obtain

$$\begin{aligned} E(\delta(|X-Y|)) = {} & p_{10}(\delta(1) - \delta(0)) - \sum_{j=1}^{\infty} p_{1j}(\delta(j) - \delta(j-1)) \\ & + \mathrm{e}^{-\Lambda}\delta(0) + \sum_{j=1}^{\infty} \frac{\Lambda^j}{j!} \mathrm{e}^{-\Lambda} \delta(j). \end{aligned} \tag{3.2}$$

Clearly the $p_{ij} \ge 0$ form a contingency table subject to the constraints

$$p_{10} + \sum_{j=1}^{\infty} p_{1j} = p, \tag{3.3a}$$

$$\max(0, \mathrm{e}^{-\Lambda} + p - 1) \le p_{10} \le \min(p, \mathrm{e}^{-\Lambda}), \tag{3.3b}$$

and

$$0 \le p_{1j} \le \frac{\Lambda^j}{j!} \mathrm{e}^{-\Lambda}. \tag{3.3c}$$

It follows from (3.2) and (3.3) that the minimum possible value of $E(\delta(|X-Y|))$ is reached when simultaneously $p_{10}=\max(0, \mathrm{e}^{-\Lambda}+p-1)=\mathrm{e}^{-\Lambda}-1+p$ is minimum, while for $j=1,2,\ldots$, $p_{1j}=(\Lambda^j/j!)\mathrm{e}^{-\Lambda}$ is maximum. Such a choice fulfills $p_{10}+\sum_{j=1}^{\infty}p_{1j}=p$ and implies (3.1) □

Observe for further use, that if $\Lambda\geq\lambda=-\log(1-p)$, we have the lower bound (let in (3.2) $p_{10}=0$ and $p_{1j}=(\Lambda^j/j!)\mathrm{e}^{-\Lambda}$, $j=1,2,\ldots$)

$$d^{(\delta)}(X,Y)\geq \mathrm{e}^{-\Lambda}\delta(0)+\sum_{j=1}^{\infty}\frac{\Lambda^j}{j!}\mathrm{e}^{-\Lambda}\delta(j-1). \tag{3.4}$$

In the course of the proof of Proposition 1, we have obtained the following result.

Proposition 2. *Under the assumptions of Proposition* 1, *a coupling of X and Y such that* $d^{(\delta)}(X,Y)=E(\delta(|X-Y|))$ *is defined by* $p_{ij}=P(X=i, Y=j)$, *where*

$$p_{00}=1-p,\quad p_{0j}=0,\quad p_{10}=\mathrm{e}^{-\Lambda}-1+p,\quad p_{1j}=\frac{\Lambda^j}{j!}\mathrm{e}^{-\Lambda},\ j=1,2,\ldots. \tag{3.5}$$

This coupling is the unique one such that $d^{(\delta)}(X,Y)=E(\delta(|X-Y|))$ *if we assume that* $\delta\neq 0$. *An explicit construction may be obtained by letting* $Y=X\eta$, *where X and* η *are independent, and* η *follows the distribution given by*

$$P(\eta=0)=p^{-1}(\mathrm{e}^{-\Lambda}-1+p),\qquad P(\eta=j)=\frac{\Lambda^j}{pj!}\mathrm{e}^{-\Lambda},\quad j=1,2,\ldots. \tag{3.6}$$

We will now concentrate in minimizing $d^{(\delta)}(X,Y)$ with respect to Λ. Our main result is as follows.

Proposition 3. *If, in addition to the hypotheses of Proposition* 1, *we assume that* $\delta(0)=0$, $\delta(1)>0$, *and that* $\delta(j)-\delta(j-1)$ *is non-increasing in* $j=1,2,\ldots$, *then, for all* $0<p\leq\frac{1}{2}$, *the minimum of* $d^{(\delta)}(X,Y)$ *is reached for* $\Lambda=\lambda=-\log(1-p)$.

Proof. Let $a_j=\delta(j)-\delta(j-1)\geq 0$. Straightforward computations show that

$$\frac{\mathrm{d}}{\mathrm{d}\Lambda}\left(\sum_{j=2}^{\infty}\frac{\Lambda^j}{j!}\mathrm{e}^{-\Lambda}\delta(j-1)\right)=\sum_{j=1}^{\infty}\frac{\Lambda^j}{j!}\mathrm{e}^{-\Lambda}a_j>0, \tag{3.7}$$

while

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\Lambda}\left(\mathrm{e}^{-\Lambda}\delta(1)+\sum_{j=2}^{\infty}\frac{\Lambda^j}{j!}\mathrm{e}^{-\Lambda}\delta(j-1)\right)&=\mathrm{e}^{-\Lambda}\left(-a_1+\sum_{j=1}^{\infty}\frac{\Lambda^j}{j!}a_j\right)\\ &\leq a_1\mathrm{e}^{-\Lambda}(\mathrm{e}^{\Lambda}-2)\leq 0,\end{aligned} \tag{3.8}$$

whenever $\Lambda\leq\log 2$. □

If $0<p\leq\frac{1}{2}$, then $\lambda=-\log(1-p)\leq\log 2$. It follows from (3.1), (3.4), (3.7) and (3.8) that $d^{(\delta)}(X,Y)$ is a function of Λ, non-increasing for $0<\Lambda\leq\lambda$ and increasing for $\lambda \le \Lambda < \infty$. A close look to (3.8) shows that Proposition 3 can be extended as follows.

Proposition 4. *If, in addition to the hypotheses of Proposition* 1, *we assume that* $\delta(0)=0$ *and* $\delta(1)>0$, *there exists a* p_δ *depending upon the sequence* $\delta(\cdot)$ *only, such that, for all* $0<p\le p_\delta$, *the minimum of* $d^{(\delta)}(X, Y)$ *is reached for* $\Lambda=\lambda=-\log(1-p)$.

Proof. Take $\lambda_\delta=-\log(1-p_\delta)$ to be the unique root of the equation in λ

$$\delta(1)=\sum_{j=1}^{\infty}\frac{\lambda^j}{j!}(\delta(j)-\delta(j-1)). \qquad \square \tag{3.9}$$

Remark 1. Among the possible values of $\Lambda>0$, we see that two cases deserve a special interest:

(i) $\Lambda=-\log(1-p)$ which achieves minimization of $d^{(\delta)}(X, Y)$ independently of δ for $p\to 0$. This choice has been introduced by Serfling (1975). If we consider the construction of the maximal coupling $Y=X\eta$ given in (3.6), we see that for such a choice $Y\ge X$ with probability one.

(ii) $\Lambda=p$ which amounts to take $E(X)=E(Y)$. If $a_0\simeq 1.596$ is the root of the equation x^3+3x+2, Deheuvels and Pfeifer (1986a) have shown that if $\sum_{i=1}^n p_i\to a$ with $0\le a<\infty$ and $\max(p_1,\dots,p_n)\to 0$ as $n\to\infty$, then $d_0(S_n, P(\sum_{i=1}^n p_i))$ becomes asymptotically smaller than $d_0(S_n, P(\sum_{i=1}^n \lambda_i))$ as $n\to\infty$ if $a>a_0$, where $\lambda_i=-\log(1-p_i)$, $i=1,\dots,n$.

3.2. *Upper bounds based on individual couplings*

From now on, we assume that $S_n=X_1+\cdots+X_n$, where $X_1,\dots,X_n$ are independent random variables following Bernoulli $B(p_1),\dots,B(p_n)$ distribution. We will also assume that

$$\begin{aligned}&\delta(0)=0,\ \delta(1)>0, \text{ and for some } 0<r\le 1,\\ &\delta^{1/r}(|p-q|) \text{ satisfies (d1)–(d3),}\end{aligned} \tag{3.10}$$

so that $D(\xi,\zeta)=\{d^{(\delta)}(\xi,\zeta)\}^r$ defines a probability metric on $\mathbf{X}$. Typical examples are:

$\delta(m)=m^\gamma$, $0<\gamma<\infty$, $r=1/\max(\gamma,1)$ (see (2.15));

$\delta(m)=1_{\{m\ne 0\}}$, $r=1$ (see (2.12));

$\delta(m)=\sum_{j=1}^m a_j$, where $\{a_j, j>1\}$ is non-negative and non-increasing, $r=1$ (see (2.2)).

Our first result is as follows.

Theorem 1. *Assume that* (3.10) *holds, and that* $\max_{1\le i\le n} p_i\to 0$. *Then, uniformly over* $n\ge 1$,

$$d^{(\delta)}\left(L(S_n), P\left(\sum_{i=1}^n p_i\right)\right)\le(1+\mathrm{o}(1))\delta(1)\sum_{i=1}^n p_i^2, \tag{3.11}$$

and

$$d^{(\delta)}\left(L(S_n), P\left(\sum_{i=1}^n \lambda_i\right)\right) \le (1+\mathrm{o}(1))\frac{\delta(1)}{2}\sum_{i=1}^n p_i^2, \tag{3.12}$$

where $\lambda_i = -\log(1-p_i)$, $i=1,\ldots,n$.

Proof. Define $T_n = Y_1 + \cdots + Y_n$, where $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent pairs of random variables such that for $i=1,\ldots,n$, $d^{(\delta)}(X_i, Y_i) = E(\delta(|X_i - Y_i|))$. The feasability of such a construction follows from the general results of Section 2, and from Propositions 2, 3. By taking successively $\Lambda = p_i$ and $\Lambda = \lambda_i$ in (3.1), we see that

$$d^{(\delta)}(X_i, Y_i) = (1+\mathrm{o}(1))\delta(1)p_i^2 \quad (\Lambda = p_i), \tag{3.13}$$

and

$$d^{(\delta)}(X_i, Y_i) = (1+\mathrm{o}(1))\frac{\delta(1)}{2}p_i^2 \quad (\Lambda = \lambda_i), \tag{3.14}$$

where the 'o(1)' terms are uniform over $i=1,\ldots,n$, when $\max_{1\le i\le n} p_i \to 0$.

The results follow from (3.13) and (3.14), jointly with the inequalities

$$\begin{aligned}\{d^{(\delta)}(S_n, T_n)) &\le E\left(\delta\left(\left|\sum_{i=1}^n (X_i - Y_i)\right|\right)\right) \le E\left(\left\{\sum_{i=1}^n \delta^{1/r}(|X_i - Y_i|)\right\}^r\right) \\ &\le E\left(\sum_{i=1}^n \delta(|X_i - Y_i|)\right) = \sum_{i=1}^n d^{(\delta)}(X_i, Y_i),\end{aligned} \tag{3.15}$$

where we have made use of the triangle inequality $\delta^{1/r}(p+q) \le \delta^{1/r}(p) + \delta^{1/r}(q)$ and of the inequality $(a_1^{1/r} + \cdots + a_n^{1/r})^r \le a_1 + \cdots + a_n$ for $a_1, \ldots, a_n \ge 0$ and $0 < r \le 1$. □

By (3.15) and Proposition 1, we have the following result.

Theorem 2. *Assume that* (3.10) *holds. Then*

$$\begin{aligned}&d^{(\delta)}\left(L(S_n), P\left(\sum_{i=1}^n \Lambda_i\right)\right) \\ &\quad\le \sum_{i=1}^n \left\{(\mathrm{e}^{-\Lambda_i} - 1 + p_i)\delta(1) + \sum_{j=2}^\infty \frac{\Lambda_i^j}{j!}\mathrm{e}^{-\Lambda_i}\delta(j-1)\right\},\end{aligned} \tag{3.16}$$

where $\Lambda_1, \ldots, \Lambda_n$ *are arbitrary positive numbers.*

Example 4. (a) Take $\delta(m) = 1_{\{m\neq 0\}}$. By (3.16), we obtain the bounds for the total variation distance:

$$d_0\left(L(S_n), P\left(\sum_{i=1}^n p_i\right)\right) \le \sum_{i=1}^n p_i(1-\mathrm{e}^{-p_i}) \le \sum_{i=1}^n p_i^2 \tag{3.17}$$

(Le Cam (1960)), and

$$d_0\left(L(S_n), P\left(\sum_{i=1}^{n} \lambda_i\right)\right) \le \sum_{i=1}^{n} (1-e^{-\lambda_i}(1+\lambda_i)) \le \frac{1}{2}\sum_{i=1}^{n} \lambda_i^2 \tag{3.18}$$

(Serfling (1978)), where $\lambda_i = -\log(1-p_i)$, $i=1,\ldots,n$.

(b) Take $\delta(m)=|m|$. By (3.16), we obtain the bounds for the Fortet–Mourier metric:

$$d_1\left(L(S_n), P\left(\sum_{i=1}^{p} p_i\right)\right) \le 2\sum_{i=1}^{n} (e^{-p_i}-1+p_i) \le \sum_{i=1}^{n} p_i^2, \tag{3.19}$$

and

$$d_1\left(L(S_n), P\left(\sum_{i=1}^{n} \lambda_i\right)\right) \le \sum_{i=1}^{n} (e^{-\lambda_i}-1+\lambda_i) \le \frac{1}{2}\sum_{i=1}^{n} \lambda_i^2. \tag{3.20}$$

(c) Take $\delta(m)=|m|^2$. By (3.16), we obtain likewise

$$d_2^2\left(L(S_n), P\left(\sum_{i=1}^{n} p_i\right)\right) \le \sum_{i=1}^{n} p_i^2, \tag{3.21}$$

and

$$d_2^2\left(L(S_n), P\left(\sum_{i=1}^{n} \lambda_i\right)\right) \le \sum_{i=1}^{n} (-e^{-\lambda_i}+1-\lambda_i+\lambda_i^2). \tag{3.22}$$

Similar results could be obtained for $\delta(m)=|m|^j$, $j=3,4,\ldots$.

An interesting application of (3.21) and the arguments above is as follows.

Corollary 1. *Let $X_1, X_2, \ldots$ be a sequence of Bernoulli random variables with $P(X_i=1)=p_i\in[0,1]$, $i=1,2,\ldots$. Without loss of generality, we may assume that $X_1, X_2, \ldots$ are defined on a probability space jointly with a sequence $Y_1, Y_2, \ldots$ of independent Poisson random variables with $E(Y_i)=p_i$, $i=1,2,\ldots$, and such that, if $S_n = X_1+\cdots X_n$ and $T_n = Y_1+\cdots+Y_n$, we have*

$$E(|S_n-T_n|^2) = \sum_{i=1}^{n} p_i^2 \quad \text{for all } n=1,2,\ldots. \tag{3.23}$$

Furthermore, we may assume that $(X_1,Y_1),(X_2,Y_2),\ldots$ are independent and such that

$$E(|X_i-Y_i|^2) = p_i^2 \quad \text{for all } i=1,2,\ldots. \tag{3.24}$$

Proof. We use the construction in the proof of Theorem 1, jointly with (3.1). □

3.3. Lower bounds

In this section, we discuss the sharpness of the upper bounds obtained in Theorems 1, 2. We start with the straightforward lower bound:

Theorem 3. *Assume that* (3.10) *holds. We have for all* $\Lambda>0$,

$$d^{(\delta)}(L(S_n), P(\Lambda)) \geq \delta(1) d_0(L(S_n), P(\Lambda)). \tag{3.25}$$

Proof. By (3.10), we assume that for some $r>0$, $\delta^{1/r}(|p-q|)$ satisfies (d1)–(d3), so that we always have for all $s, t \geq 0$,

$$\delta^{1/r}(s) - \delta^{1/r}(t) \leq \delta^{1/r}(s+t), \tag{3.26}$$

so that $\delta(m)$ is non-decreasing in $m \geq 0$. Assume now that S_n and $T_n \sim P(\Lambda)$ are defined on the same probability space. We have

$$E(\delta(|S_n - T_n|)) = \sum_{n=1}^{\infty} \delta(n) P(|S_n - T_n| = n) \geq \delta(1) P(S_n \neq T_n).$$

This, jointly with (2.12), suffices for proof. □

By Theorem 3, we can prove the sharpness of the bounds in Theorem 1. This follows from the following evaluations of $d_0(L(S_n), P(\Lambda))$.

Proposition 5. *Let* $\lambda_i = -\log(1-p_i)$, $i = 1, 2, \ldots$.

(1) *If* $\sum_{i=1}^n p_i \to 0$, *then*

$$d_0\left(L(S_n), P\left(\sum_{i=1}^n p_i\right)\right) = (1+\mathrm{o}(1)) \sum_{i=1}^n p_i^2, \tag{3.27}$$

and

$$\begin{aligned} d_0\left(L(S_n), P\left(\sum_{i=1}^n \lambda_i\right)\right) &= (1+\mathrm{o}(1)) \inf_{\Lambda>0} d_0(L(S_n), P(\Lambda)) \\ &= (1+\mathrm{o}(1)) \frac{1}{2} \sum_{i=1}^n p_i^2. \end{aligned} \tag{3.28}$$

(2) *If* $\sum_{i=1}^n p_i \to \infty$ *and* $\{\sum_{i=1}^n p_i^2\}/\{\sum_{i=1}^n p_i\} \to 0$, *then*

$$\begin{aligned} d_0\left(L(S_n), P\left(\sum_{i=1}^n p_i\right)\right) &= (1+\mathrm{o}(1)) \inf_{\Lambda>0} d_0(L(S_n), P(\Lambda)) \\ &= \frac{1+\mathrm{o}(1)}{\sqrt{2\pi \mathrm{e}}} \left\{\sum_{i=1}^n p_i^2\right\} \Big/ \left\{\sum_{i=1}^n p_i\right\}. \end{aligned} \tag{3.29}$$

(4) *If* $\sum_{i=1}^n p_i \to a \in (0,\infty)$ *and* $\max_{1\le i\le n} p_i \to 0$, *then there exists positive functions* $A_0(a)$, $A_0'(a)$ *and* $A_0''(a)$ *such that*

$$d_0\left(L(S_n), P\left(\sum_{i=1}^n p_i\right)\right) = (1+\mathrm{o}(1))A_0(a)\sum_{i=1}^n p_i^2, \tag{3.30}$$

$$d_0\left(L(S_n), P\left(\sum_{i=1}^n \lambda_i\right)\right) = (1+\mathrm{o}(1))A_0'(a)\sum_{i=1}^n p_i^2, \tag{3.31}$$

and

$$\inf_{\Lambda>0} d_0(L(S_n), P(\Lambda)) = (1+\mathrm{o}(1))A_0''(a)\sum_{i=1}^n p_i^2. \tag{3.32}$$

Proof. See Deheuvels and Pfeifer (1986a) where explicit closed forms are given for $A_0(a)$ and $A_0'(a)$. Note that in the above reference (3.29) is proved with the restriction that $\sum_{i=1}^n p_i^2 = \mathrm{O}(1)$. The extension of this result to the conditions in (2) is given in Deheuvels and Pfeifer (1986b). □

Corollary 2. *Assume that* (3.10) *holds and that* $\sum_{i=1}^n p_i^2 \to 0$. *Then*

$$d^{(\delta)}\left(L(S_n), P\left(\sum_{i=1}^n p_i\right)\right) = (1+\mathrm{o}(1))\delta(1)\sum_{i=1}^n p_i^2, \tag{3.33}$$

and

$$\begin{aligned} d^{(\delta)}\left(L(S_n), P\left(\sum_{i=1}^n \lambda_i\right)\right) &= (1+\mathrm{o}(1)) \inf_{\Lambda>0} d(L(S_n), P(\Lambda)) \\ &= \frac{1+\mathrm{o}(1)}{2}\delta(1)\sum_{i=1}^n p_i^2. \end{aligned} \tag{3.34}$$

Proof. It follows from Theorems 1, 3 and Proposition 5. □

We also see from Theorem 3 and Proposition 5 that, whenever $\sum_{i=1}^n p_i = \mathrm{O}(1)$ and $\max_{1\le i\le n} p_i \to 0$, then the bounds given in (3.11) and (3.12) are sharp up to a multiplicative constant. This result will be extended in the next section, under weaker hypotheses upon δ.

4. The semigroup approach

In this section, we shall provide a general upper bound for $d^{(\delta)}(L(S_n), P(\sum_{i=1}^n p_i))$. Throughout, we will use the notation of Section 3. We will make the following assumptions on $\delta(\cdot)$ which are implied by (3.10):

(M1) $\delta(k-j) \le \delta(k) + \delta(j)$, $0 \le j \le k$;
(M2) $\delta(k) > 0$, $k \ge 1$;
(M3) $\sup_{k\ge 1}\{\delta(k+1)/\delta(k)\} = M < \infty$.

Observe that if (3.10) holds, then (see (3.26)) $\delta(\cdot)$ is non-decreasing and $\delta(j)>0$, so that (M1)–(M2) hold (in fact we have $\delta(k-j)\le\delta(k)+\delta(j)$). Also, by the triangle inequality,

$$\delta(k+1)\le\{\delta^{1/r}(k)+\delta^{1/r}(1)\}^r\le\{2\delta^{1/r}(k)\}^r\le 2^r\delta(k),$$

so that $M\le 2^r$.

To see that (M1)–(M3) covers a larger family of $\delta(\cdot)$ functions than given by (3.10), we can use the following example.

$$\delta(m)=a^m,\quad a\ge 1, \tag{4.1}$$

for which $M=a$.

The main result of this section is as follows.

Theorem 4. *Under the conditions* (M1)–(M3), *we have*

$$d^{(\delta)}\left(L(S_n),P\left(\sum_{i=1}^n p_i\right)\right)$$
$$\le\frac{\delta(1)}{2}\left(M+2+\frac{1}{M}\right)\left\{\sum_{i=1}^n p_i^2\right\}\exp\left((M-1)\sum_{i=1}^n p_i\right). \tag{4.2}$$

The proof of Theorem 4 is based on the following auxiliary results.

Lemma 6. *Under* (M1), *we have*

$$d^{(\delta)}\left(L(S_n),P\left(\sum_{i=1}^n p_i\right)\right)\le\Delta_\delta(S_n,T_n)=\sum_{k=0}^\infty\delta(k)|P(S_n=k)-P(T_n=k)|, \tag{4.3}$$

where T_n denotes a random variable with a Poisson $P(\sum_{i=1}^n p_i)$ distribution.

Proof. We define a joint distribution for S_n and T_n such that $P(S_n\ne T_n)=d_0(S_n,T_n)$, which is actually a maximal coupling for the total variation distance.

Denote by $\mathbf{N}^+$ and $\mathbf{N}^-$ the following subsets of $\mathbf{N}$:

$$\mathbf{N}^+=\{k\ge 0: P(S_n=k)\ge P(T_n=k)\},$$

$$\mathbf{N}^-=\{k\ge 0: P(S_n=k)<P(T_n=k)\}.$$

If we further let

$$P(S_n=k,T_n=j)$$
$$=\begin{cases}\min\{P(S_n=k),P(T_n=k)\}, & k=j\ge 0,\\ \dfrac{1}{d}|P(S_n=k)-P(T_n=k)|\,|P(S_n=j)-P(T_n=j)|, & k\in\mathbf{N}^+,\ j\in\mathbf{N}^-,\\ 0, & \text{all other cases},\end{cases}$$

where

$$d = d_0(S_n, T_n) = \sum_{k \in \mathbf{N}^+} (P(S_n = k) - P(T_n = k))$$
$$= \sum_{j \in \mathbf{N}^-} (P(T_n = j) - P(S_n = j)) > 0,$$

then Q defines the desired coupling.

If (S_n, T_n) is distributed as above, it is easily verified that $d_0(S_n, T_n) = P(S_n \neq T_n)$, and also

$$d^{(\delta)}(S_n, T_n) \leq E(\delta(|S_n - T_n|)) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \delta(|k-j|) P(S_n = k, T_n = j)$$
$$= \frac{1}{d} \sum_{k \in \mathbf{N}^+} \sum_{j \in \mathbf{N}^-} \delta(|k-j|) |P(S_n = k) - P(T_n = k)| \, |P(S_n = j) - P(T_n = j)|, \tag{4.4}$$

from which, by (M1), (4.3) follows immediately. $\square$

The form of Δ_δ in (4.3) suggests that a semigroup approach as in the case of the total variation distance or the Kolmogorov distance (Deheuvels and Pfeifer (1986a,b, 1988)) should also be possible here. Unfortunately Δ_δ is not a metric in the strict sense since $\delta(0) = 0$. To overcome this difficulties, we shall therefore deal with δ^* instead of δ, which is defined by

$$\delta^*(k) = \begin{cases} \delta(1)/M, & k = 0, \\ \delta(k), & k > 1, \end{cases} \tag{4.5}$$

so that $\sup_{k \geq 0} \delta^*(k+1)/\delta^*(k) = M$.

Let $\ell^1_{\delta^*}$ denote the Banach space of all δ^*-summable sequences $g = (g(0), g(1), \ldots)$, i.e. such that $|g|_{\delta^*} = \sum_{k=0}^{\infty} g(k)\delta^*(k) < \infty$. For $f \in \ell^1$ (the usual Banach space of summable sequences) and $g \in \ell^1_{\delta^*}$, define the convolution $f * g$ by

$$f * g(n) = \sum_{k=0}^{\infty} f(k) g(n-k), \quad n = 0, 1, \ldots .$$

In general, $f * g$ need not belong to $\ell^1_{\delta^*}$. This is however always the case for $f = \varepsilon_1$, where ε_1 denotes the sequence corresponding to a Dirac measure concentrated in 1. We also have

$$\|\varepsilon_1 * g\|_{\delta^*} \leq \sum_{k=0}^{\infty} |g(k)| \delta^*(k+1) \leq M \|g\|_{\delta^*}. \tag{4.6}$$

Let us now define the operator $B : \ell^1_{\delta^*} \to \ell^1_{\delta^*}$ by

$$Bg = \varepsilon_1 * g, \quad g \in \ell^1_{\delta^*}. \tag{4.7}$$

Then, by (4.6), $\|B\| = \sup_{g\neq 0}\{\|Bg\|_{\delta^*}/\|g\|_{\delta^*}\} \le M$, and hence $A = B - I$ (where I denotes the identity operator) generates the Poisson convolution semigroup $\{e^{tA}: t\ge 0\}$, i.e.

$$e^{tA}g = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}B^k g = P(t)*g, \quad g\in \ell^1_{\delta^*},\ t\ge 0, \tag{4.8}$$

with

$$\|e^{tA}\| \le e^{-t}e^{\|B\|t} \le e^{(M-1)t}, \quad t\ge 0. \tag{4.9}$$

By (4.3), we therefore have

$$\Delta_\delta(S_n, T_n) \le \Delta_{\delta^*}(S_n, T_n) = \left\| \prod_{k=1}^{n}(I+p_kA)\varepsilon_0 - \exp\left(\sum_{k=1}^{n}p_kA\right)\varepsilon_0\right\|_{\delta^*}, \tag{4.10}$$

where $\varepsilon_0 = (1,0,0,\dots)\in \ell^1_{\delta^*}$.

Lemma 7. *For any $g\in\ell^1_{\delta^*}$, we have*

$$\left\|\prod_{k=1}^{n}(I+p_kA)g - \exp\left(\sum_{k=1}^{n}p_kA\right)g\right\|_{\delta^*}$$
$$\le \frac{1}{2}\|A^2g\|_{\delta^*}\exp\left((M-1)\sum_{k=1}^{n}p_k\right)\sum_{k=1}^{n}p_k^2. \tag{4.11}$$

Proof. Let $U_k = I + p_kA$ and $V_k = e^{p_kA}$. Since

$$\prod_{k=1}^{n}U_k - \prod_{k=1}^{n}V_k = \sum_{k=1}^{n}\left\{\prod_{i=k+1}^{n}U_i(U_k - V_k)\prod_{j=1}^{k-1}V_j\right\}, \tag{4.12}$$

with $\|V_j\|\le e^{(M-1)p_j}$, $\|U_i\|\le 1+(M-1)p_i\le e^{(M-1)p_i}$, we have

$$\left\|\prod_{k=1}^{n}(I+p_kA)g - \exp\left(\sum_{k=1}^{n}p_kA\right)g\right\|_{\delta^*}$$
$$\le \sum_{k=1}^{n}\exp\left((M-1)\sum_{i\neq k}p_i\right)\|e^{p_kA}g - (I+p_kA)g\|_{\delta^*}$$
$$\le \exp\left((M-1)\sum_{k=1}^{n}p_k\right)\frac{1}{2}\|A^2g\|_{\delta^*}\sum_{k=1}^{n}p_k^2$$

by standard arguments from semigroup theory (cf. Butzer and Behrens (1967)). □

Since here $\|A^2\varepsilon_0\|_{\delta^*} = \delta^*(0) + 2\delta^*(1) + \delta^*(2) \le M^{-1}\delta(1) + 2\delta(1) + M\delta(1)$, the proof of Theorem 4 now follows from Lemmas 6 and 7. □

Theorem 4 says basically that, as long as $\delta(k)$ does not grow faster than geometrically in k, and $\sum_{j=1}^{n}p_j$ remains bounded, we always have

$$d^{(\delta)}\left(L(S_n), P\left(\sum_{i=1}^{n} p_i\right)\right) = \mathrm{O}\left(\sum_{i=1}^{n} p_i^2\right) \quad \text{as } n \to \infty.$$

From the lower bound estimations in Section 3.3, we see that this is indeed the exact rate which cannot be improved upon, except for the constants involved.

Example 5. (a) Let $\delta(u) = |u|^r$, $r > 0$. Then $M = 2^r$ and hence, by Theorem 4,

$$d^{(\delta)}\left(L(S_n), P\left(\sum_{i=1}^{n} p_i\right)\right) \le \tfrac{1}{2}(2^r + 2 + 2^{-r}) \exp\left((2^r - 1) \sum_{j=1}^{n} p_j\right) \sum_{i=1}^{n} p_i^2. \quad (4.13)$$

Observe that the bound given in (4.13) is far from being sharp (see e.g. (3.11)).

(b) Let $\delta(u) = a^{|u|}$, $a > 1$. Notice that in this case $d^{(\delta)}$ does not define a probability metric since $\delta(p, q) = \delta(|p - q|)$ violates (d1) and (d3). However Theorem 4 still applies and gives the bound:

$$d^{(\delta)}\left(L(S_n), P\left(\sum_{i=1}^{n} p_i\right)\right) \le \tfrac{1}{2}\left(a + 2 + \frac{1}{a}\right) \exp\left((a - 1) \sum_{j=1}^{n} p_j\right) \sum_{i=1}^{n} p_i^2. \quad (4.14)$$

5. Statistical application for confidence intervals

In this section, we consider several applications of the results of Section 2–4.

5.1. Upper confidence bounds for the number of failures in an equipment

Consider an equipment whose components have each a small probability of failure, and denote by X_i, $i = 1, \ldots, n$, the random variable taking value one if the i-th component has failed and zero otherwise. Let $0 < \theta < 1$ be a given confidence level. We want to evaluate N such that $P(S_n \ge N) \le \theta$, where $S_n = X_1 + \cdots + X_n$, and under the assumptions that $p_i = P(X_i = 1)$ is known for $i = 1, \ldots, n$.

In the following, we give a simple solution of this problem, based on coupling arguments.

The first idea which comes in mind is to choose N to be the smallest integer such that $P(N, \Lambda) \le \theta - \varepsilon$, where

$$P(N, \Lambda) = \sum_{m=N}^{\infty} \frac{\Lambda^m}{m!} \mathrm{e}^{-\Lambda}. \quad (5.1)$$

We will not discuss further approximations of $P(N, \Lambda)$ and refer to Johnson and Kotz (1969) pp. 98–102 for a review of the topic.

In (5.1), $\Lambda = \Lambda(p_1, \ldots, p_n)$ is a function of $p_1, \ldots, p_n$, and ε is the approximation error.

It turns out that the following simple choices of Λ and $\varepsilon = 0$ give a sharp solution.

Proposition 6. *Let* $\Lambda = -\sum_{i=1}^{n} \log(1 - p_i)$. *Then* $P(S_n \ge N) \le P(N, \Lambda)$ *for all* $N \ge 0$.

Proof. Observe that Proposition 6 says that if $0<\theta<1$ is fixed and if N is the smallest integer such that $P(N,\Lambda)\le\theta$, then $P(S_n\ge N)\le\theta$.

The proof of Proposition 6 is based on the fact (see Remark 1) that if $Y_1,\dots,Y_n$ are independent Poisson random variables with $E(Y_i)=-\log(1-p_i)$ and such that, for $i=1,\dots,n$, $d_0(X_i,Y_i)=P(X_i\ne Y_i)$, then $Y_i\ge X_i$ a.s., and hence $T_n=Y_1+\cdots+Y_n\ge S_n$ a.s. □

5.2. Lower confidence bounds for the number of failures in an equipment

We use the same notation as in 5.1 and seek an integer M which guarantees that $P(S_n\le M)\le\theta$ for a fixed $0<\theta<1$. Since $P(S_n=0)=\prod_{i=1}^n(1-p_i)\ge 1-\sum_{i=1}^n p_i$, we see that the problem is trivial for small values of $\sum_{i=1}^n p_i$. Therefore we shall limit our investigations to the case where $\sum_{i=1}^n p_i\not\to 0$. Deheuvels and Pfeifer (1986a) have shown that if $\sum_{i=1}^n p_i\to a\in(0,\infty)$ and $\max_{1\le i\le n}p_i\to 0$ as $n\to\infty$, then, asymptotically as $n\to\infty$, we have

$$d_0\left(L(S_n),P\left(\sum_{i=1}^n p_i\right)\right)<d_0\left(L(S_n),P\left(-\sum_{i=1}^n\log(1-p_i)\right)\right)$$

if $a>x_0\simeq 1.59$, where x_0 is the root of the equation $x^3+3x-2=0$. Because of this, we will limit ourselves in the sequel to the approximation of $L(S_n)$ by $P(\sum_{i=1}^n p_i)$.

Next, we mention the results of Deheuvels and Pfeifer (1986b) concerning the evaluation of Kolmogorov's metric $\sup_k|P(S_n\le k)-P(T_n\le k)|$ and of the Fortet–Mourier metric $d_1(\cdot,\cdot)$. It turns out that (as far as leading terms are concerned) the value of Kolmogorov's metric for $L(S_n)$ and $P(\sum_{i=1}^n p_i)$ is one half of $d_0(L(S_n),P(\sum_{i=1}^n p_i))$. For the Fortet–Mourier metric, if $\sum_{i=1}^n p_i\to a\in(0,\infty)$ and $\max_{1\le i\le n}p_i\to 0$ as $n\to\infty$, they have proved also that

$$d_1\left(L(S_n),P\left(\sum_{i=1}^n p_i\right)\right)\sim\frac{a^{[a]}}{[a]!}\mathrm{e}^{-a}\sum_{i=1}^n p_i^2\quad\text{as } n\to\infty.$$

Such evaluations can be used precise the bounds presented in the sequel. For sake of concision, we shall limit ourselves to simple estimates. Let

$$Q(M,\Lambda)=\sum_{m=0}^M\frac{\Lambda^m}{m!}\mathrm{e}^{-\Lambda},\quad M\ge 0,\qquad Q(M,\Lambda)=0,\quad M<0. \tag{5.2}$$

Proposition 7. *Let* $\Lambda=\sum_{i=1}^n p_i$ *and let, for some* $\gamma\ge 0$, $d_\gamma=d_\gamma(L(S_n),P(\sum_{i=1}^n p_i))$ *be defined as in* (2.12) *and* (2.15). *Let* $0<\theta<1$ *be fixed, and assume that* $k\ge 1$ *is an integer such that*

$$k^{-\gamma}d_\gamma^{\max(\gamma,1)}<\theta. \tag{5.3}$$

Furthermore, let M be the largest integer such that

$$Q(M,\Lambda)\le\theta-k^{-\gamma}d_r^{\max(\gamma,1)}. \tag{5.4}$$

Then, we have

$$P(S_n \le M-k+1) \le \theta. \tag{5.5}$$

Proof. By Lemma 5, there exists T_n following a Poisson $P(\sum_{i=1}^{n} p_i)$ distribution, and such that $E(|S_n - T_n|^\gamma) = d_\gamma^{\max(\gamma,1)}$. This and Markov's inequality imply

$$P(|S_n - T_n| > k) = P(|S_n - T_n| \ge k+1) \le k^{-\gamma} d_\gamma^{\max(\gamma,1)}. \tag{5.6}$$

By (5.6), we have

$$\begin{aligned} P(S_n \le M-k+1) &\le P(T_n \le M) + P(|S_n - T_n| \ge k+1) \\ &\le Q(M,\Lambda) + k^{-\gamma} d_\gamma^{\max(\gamma,1)}, \end{aligned}$$

which suffices for proof. □

Example 6. The aim of this example is to show that the bounds given in (5.5) may improve upon the case $\gamma = 0$, corresponding to the total variation distance. For this sake, we shall assume that n is so large and $\max_{1 \le i \le n} p_i$ is so small, that, for $\sum_{i=1}^{n} p_i = a$, we may use the asymptotic approximations:

$$d_0(L(S_n), P(a)) \simeq \frac{1}{2} \left\{ \frac{a^{\alpha-1}(\alpha - a)}{\alpha!} - \frac{a^{\beta-1}(\beta - a)}{\beta!} \right\} e^{-a} \sum_{i=1}^{n} p_i^2 = A_0(a) \sum_{i=1}^{n} p_i^2,$$

where $\alpha = [a + \frac{1}{2} + (a + \frac{1}{4})^{1/2}]$ and $\beta = [a + \frac{1}{2} - (a + \frac{1}{4})^{1/2}]$, and

$$d_1(L(S_n), P(a)) \simeq \frac{a^{[a]}}{[a]!} e^{-a} \sum_{i=1}^{n} p_i^2 = A_1(a) \sum_{i=1}^{n} p_i^2.$$

It is straightforward from Proposition 7 that for $\gamma = 0$, we need assume that

$$\theta > d_0(L(S_n), P(A)) = A_0(a) \sum_{i=1}^{n} p_i^2. \tag{5.7}$$

Take $a = 4$. We have $A_0(a) \simeq 0.1254$, and $A_1(a) \simeq 0.1954$. For $k = 2$, we have evidently

$$k^{-1} d_1 \simeq 0.0977 \sum_{i=1}^{n} p_i^2 < d_0 \simeq 0.1254 \sum_{i=1}^{n} p_i^2.$$

It follows that by using $\gamma = 1$, $k = 2$, we may obtain confidence intervals for S_n for smaller values of θ than those which satisfy (5.7).

Example 7. Consider the case where $a = \sum_{i=1}^{n} p_i$ is large, for instance $a = 500$, and $\sum_{i=1}^{n} p_i^2$ is small (say 25).

Using again the asymptotic evaluations in Deheuvels and Pfeifer (1986a,b), we see that, for $a \to \infty$,

$$d_0(L(S_n), P(a)) \sim \frac{1}{\sqrt{2\pi e}} \left\{ \sum_{i=1}^{n} p_i \right\}^{-1} \sum_{i=1}^{n} p_i^2, \tag{5.8}$$

$$d_1(L(S_n), P(a)) \sim \frac{1}{\sqrt{2\pi}} \left\{ \sum_{i=1}^{n} p_i \right\}^{-1/2} \sum_{i=1}^{n} p_i^2, \tag{5.9}$$

while (3.21) gives an upper bound

$$d_2^2(L(S_n), P(a)) \le \sum_{i=1}^{n} p_i^2. \tag{5.10}$$

Suppose that our problem consists in finding an M such that $P(S_n \le M) \le 1\%$. We see that, if we take (5.8) as an equality (as a first order approximation), we get

$$d_0(L(S_n), P(a)) \sim 1.21\%,$$

which renders the problem unsolvable by the sole use of d_0. On the other hand, we see from (5.10) that (for instance) if we take $k = 75$, we have

$$k^{-2} d_2^2(L(S_n), P(a)) \le 0.45\%. \tag{5.11}$$

We will obtain a solution to our problem by choosing an M such that $Q(M, 500) \le 0.55\%$ and using the inequality $P(S_n \le M - 75) \le 1\%$.

In spite of the fact that bounds such as above may appear as very crude, the only alternative (aside from explicit computations) which can be offered to match such evaluations are moment inequalities based on the p_i's which yield even weaker estimates. We will not discuss here normal approximations noting only that in the range of interest, the approximation of $L(S_n)$ by a normal distribution is usually worse than what is obtained by Poisson approximations.

5.3. Two-sided bounds

We offer here a general statement enabling to use any of the estimates obtained in the preceding sections.

Proposition 8. *Let $\delta(n)$ be non-negative and non-decreasing in $n = 0, 1, \ldots$. For any $\xi, \zeta \in \mathbf{X}$, define $d^{(\delta)}(\xi, \zeta) = d^{(\delta)}(L(\xi), L(\zeta)) = \inf_{C(\xi,\zeta)} E(\delta(|\xi - \zeta|)$. Let $0 < \theta < 1$ be fixed, and assume that $k \ge 1$ is an integer such that*

$$\varepsilon = \{\delta(k)\}^{-1} d^{(\delta)}\left(L(S_n), P\left(\sum_{i=1}^{n} p_i\right)\right) < \theta. \tag{5.12}$$

Furthermore, let $P(N, \Lambda)$ and $Q(N, \Lambda)$ be as in (5.1) and (5.2). Let $\Lambda = \sum_{i=1}^{n} p_i$, and define N (resp. M) to be the largest (resp. smallest) integer such that

$$P(N, \Lambda) \le \tfrac{1}{2}\{\theta - \varepsilon\} \quad \text{and} \quad Q(M, \Lambda) \le \tfrac{1}{2}\{\theta - \varepsilon\}. \tag{5.13}$$

Then

$$P(M - k + 1 < S_n < N + k - 1) \le 1 - \theta. \tag{5.14}$$

Proof. The proof is identical to the proof of Proposition 7. Details are omitted. □

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