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THE STRUCTURE OF ELEMENTARY PURE BIRTH PROCESSES

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Abstract

A complete characterization of elementary pure birth processes is given by means of record counting processes from independent (non-identically) distributed random variables.

PURE BIRTH PROCESS; RECORD VALUES; COUNTING PROCESS

1. Introduction

Call a pure birth process $\{N(t); t \geqq 0\}$ with standard transition probabilities

$$p_{nm}(s,t) = P(N(t) = m \mid N(s) = n), \qquad n, m \geqq 0, \quad 0 \leqq s \leqq t$$

and $N(0) = 0$ *elementary* if

(1) $\{N(t); t \geqq 0\}$ possesses right-continuous paths;

(2) all birth rates

$$\lambda_n(t) = \lim_{h \downarrow 0} \frac{1}{h} p_{n,n+1}(t, t+h), \qquad n, t \geqq 0$$

are positive and finite;

(3) for the sequence $\{X_n; n \geqq 0\}$ of jump-times given by $X_n = \sup\{t \geqq 0; N(t) = n\}$ all finite-dimensional marginals are absolutely continuous with respect to Lebesgue measure.

It is the aim of this paper to show that $\{X_n; n \geqq 0\}$ is identically distributed with the record sequence $\{R_n; n \geqq 0\}$ from independent r.v.'s with after-record changing distributions whose c.d.f.'s are

$$F_n(t) = 1 - e^{-\Lambda_n(t)} \tag{4}$$

where $\Lambda_n(t) = \int_0^t \lambda_n(s)ds$, $t \geqq 0$, $n \geqq 0$.

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In the light of [3] we thus have obtained a complete characterization of elementary pure birth processes (EPBPS) which says that every such process essentially is a record-counting process. (This is a canonical representation of EPBPS extending the one given in [1] for ordinary Poisson processes; cf. also [4].) With respect to Monte Carlo studies this gives rise to a simple method of generating EPBP sample paths from records of independent r.v.'s with underlying c.d.f.'s of the form (4), especially in the non-homogeneous case. For example, for the Pólya–Lundberg process with birth rates

$$\lambda_n(t) = \lambda \frac{1+\alpha n}{1+\alpha\lambda t} \qquad (\alpha, \lambda > 0),$$

take

$$F_n(t) = 1 - (1+\alpha\lambda t)^{-(n+(1/\alpha))}$$

which are of Pareto type.

2. Main results

Theorem 1. $\{X_n\,; n \geqq 0\}$ is a non-decreasing Markov chain; a proper version of the transition probabilities is

$$P(X_n > t \mid X_{n-1} = s) = p_{nn}(s,t), \qquad 0 \leqq s \leqq t. \tag{5}$$

Proof. Let $0 \leqq s_0 < s_1 < \cdots < s_{n-1} < s_n = t$ and let $\varepsilon > 0$ be smaller than $\min\{s_k - s_{k-1}; 1 \leqq k \leqq n\}$. We then have

$$\bigcap_{k=0}^{n-1} \{X_k \in (s_k, s_k+\varepsilon]\} = \{N(s_0) = 0, N(s_{n-1}+\varepsilon) \geqq n\} \cap \bigcap_{k=1}^{n-1} \{N(s_{k-1}+\varepsilon) = N(s_k) = k\} \tag{6}$$

$$\{X_n > t\} \cap \bigcap_{k=0}^{n-1} \{X_k \in (s_k, s_k+\varepsilon]\} = \{N(s_0) = 0\} \cap \bigcap_{k=1}^{n} \{N(s_{k-1}+\varepsilon) = N(s_k) = k\} \tag{7}$$

(see Figure 1). Hence

$$\begin{aligned} &P\left(X_n > t \,\middle|\, \bigcap_{k=0}^{n-1} \{X_k \in (s_k, s_k+\varepsilon]\}\right) \\ &= \frac{P(N(t) = n \mid N(s_{n-1}+\varepsilon) = n)P(N(s_{n-1}+\varepsilon) = n \mid N(s_{n-1}) = n-1)}{P(N(s_{n-1}+\varepsilon) \geqq n \mid N(s_{n-1}) = n-1)} \\ &= p_{nn}(s_{n-1}+\varepsilon, t)\frac{\lambda_{n-1}(s_{n-1}) + o_1(1)}{\lambda_{n-1}(s_{n-1}) + o_2(1)} \to p_{nn}(s_{n-1}, t) \qquad \text{for } \varepsilon \downarrow 0 \end{aligned} \tag{8}$$

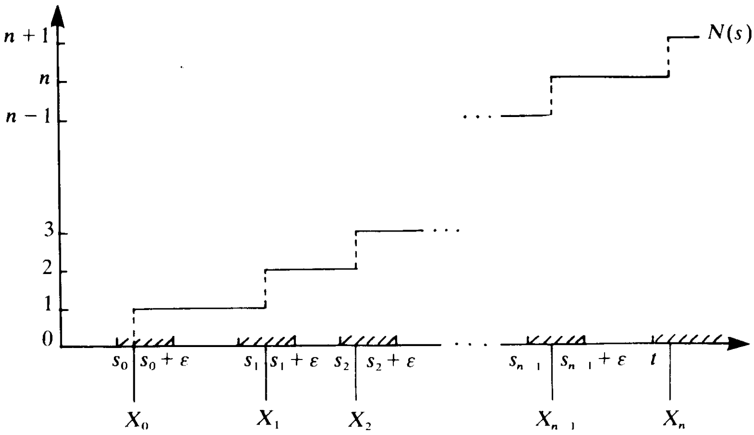

Figure 1.

by (1) and (2) since for $0 \leqq s < t$, $p_{nn}(s,t)$ is right continuous with respect to s by (1). Now (3) implies that

$$(9)\quad \begin{aligned} &P(X_n > t \mid X_0 = s_0, \cdots, X_{n-1} = s_{n-1}) \\ &= \lim_{\varepsilon \downarrow 0} P\Big(X_n > t \,\Big|\, \bigcap_{k=0}^{n-1} \{X_k \in (s_k, s_k + \varepsilon]\}\Big) = p_{nn}(s_{n-1}, t) \quad \text{a.s.} \end{aligned}$$

(cf. [2], Theorem (6.3)). Since by our assumptions $1 - p_{nn}(s_{n-1}, .)$ has all properties of a c.d.f. the theorem is proved.

Theorem 2. $\{X_n\,; n \geqq 0\}$ is identically distributed with the record sequence $\{R_n\,; n \geqq 0\}$ from a family $\{X_{00}, X_{nk}\,; n, k \geqq 1\}$ of independent r.v.'s with $F_n = 1 - e^{-\Lambda_n}$ being the c.d.f. of the X_{nk}, $n \geqq 0$. Equivalently, $\{N(t); t \geqq 0\}$ is identically distributed with the corresponding record-counting process $\#\{n\,; R_n \leqq t\}$, $t \geqq 0$.

Proof. By (3) we have conditional Lebesgue densities $f_n(t \mid s)$ of X_n given $X_{n-1} = s$, and applying Theorem 1, we obtain a forward differential equation

$$(10)\qquad f_n(t \mid s) = -\frac{\partial p_{nn}(s,t)}{\partial t} = \lambda_n(t) p_{nn}(s,t) \quad \text{a.e.,} \quad 0 \leqq s \leqq t.$$

Let $F_n(t \mid s) = 1 - p_{nn}(s,t)$. Then by (10),

$$(11)\qquad \frac{\partial}{\partial t}\{-\ln(1 - F_n(t \mid s))\} = \frac{f_n(t \mid s)}{1 - F_n(t \mid s)} = \lambda_n(t) \quad \text{a.e.}$$

or equivalently,

$$(12)\qquad 1-F_n(t\mid s)=\exp(-\{\Lambda_n(t)-\Lambda_n(s)\})=\frac{1-F_n(t)}{1-F_n(s)},\qquad 0\leqq s\leqq t.$$

Since $F_n(.\mid s)$ is a c.d.f., $\int_0^\infty \lambda_n(t)dt=\infty$, and since $\lambda_n(t)>0$ by (2), F_n is strictly increasing on $(0,\infty)$ which implies that $\xi_n=\infty$ where ξ_n is the right end of F_n, $n\geqq 1$. But in this case, $\{R_n\,; n\geqq 0\}$ also is a Markov chain with transition probabilities

$$(13)\qquad P(R_n>t\mid R_{n-1}=s)=\frac{1-F_n(t)}{1-F_n(s)},\qquad 0\leqq s\leqq t$$

([3], Corollary 2.3). Similarly, $F_0=1-e^{-\Lambda_0}$ is the c.d.f. of X_0 and R_0, respectively.

As can be seen from Theorem 2, the interarrival times of an EPBP are independent iff all $F_n, n\geqq 1$ are exponential c.d.f.'s (cf. [3], Corollary 3.2) or, equivalently, iff $\lambda_n(t)\equiv\lambda_n$, independent of t, i.e. in the homogeneous case for $n\geqq 1$.

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