Moving Point Patterns: The Poisson Case

Dietmar Pfeifer, Hans-Peter Bäumer and Matthias Albrecht
Department of Mathematics and Computing Center, University of Oldenburg
P.O. Box 25 03, 2900 Oldenburg, Germany

Abstract: Motivated by time-dependent spatial random patterns occuring in marine or terrestrial ecosystems we investigate the limiting behaviour over time of certain Poisson point processes with possible movements of points according to a stochastic process. In particular, the possibilities of equilibrium, extinction or explosion of the system are discussed.

1 Introduction

A statistical analysis of random spatial patterns in marine or terrestrial ecosystems often requires the simultaneous consideration of time, space and migration. For instance, the spatial distribution of birds or geese in a certain observation area is varying over time due to incoming or outgoing flights and movements on the ground; similarly, the spatial distribution of sand worms in the wadden sea (like *arenicola marina*) depends on death, birth and migration of adults and larvae. Frequently the distributional patterns in space caused by such species are very much Poisson-like; hence it seems reasonable to study time-dependent point processes of this type and their long-time behaviour. Although the use of mathematical models in ecosystem theory is discussed controversely (see e.g. Wiegleb (1989) for similar problems in vegetation science) stochastic concepts in modelling of ecosystems are becoming more popular recently (see e.g. Richter and Söndgerath (1990)). A first simple model to study at least qualitative properties of systems as outlined above is a spatial birth-death process with migration (e.g., Brownian motion) whose time-marginal distributions are all Poisson point processes. Similar processes have been studied before (see e.g. Tsiatefas (1982), Volkova (1984), Madras (1989)), however with emphasis on different aspects of the model. Our approach resembles the presentation of Cox and Isham (1980), chapter 6.5 (iii); the Poisson property of the process here allows for easy calculations of possible limiting distributions over time. In particular, equilibrium, extinction, amd explosion of the system will be discussed.

2 The basic model

First we shall give a short account of the mathematical prerequisites in point process theory which are necessary to formulate our main results. For a more elementary treatment, see Pfeifer, Bäumer and Albrecht (1992) or the monographs of Daley and Vere-Jones (1988), Cressie (1991) or König and Schmidt (1992).

Although we shall for practical applications only consider Euclidean spaces such as $\mathbf{R}^1$, $\mathbf{R}^2$ or $\mathbf{R}^3$ it is easier from a theoretical point of view to start from general topological spaces such as locally compact Polish spaces $\mathcal{X}$ with corresponding σ-field $\mathcal{B}$ generated by the topology over $\mathcal{X}$ (cf. Kallenberg (1983)). Further, we may assume that the topology itself is induced by a metric ρ. Let $\mathcal{R} \subseteq \mathcal{B}$ denote the set of all relatively compact sets in $\mathcal{X}$. A measure μ on

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$\mathcal{B}$ with finite values on $\mathcal{R}$ is called a *Radon-measure*. The set $\mathcal{M}$ of Radon measures over $\mathcal{X}$ can in a natural way be equipped with a σ-field $\mathbf{M}$ generated by the so-called *evolution mappings*

$$\tau_B : (\mathcal{M}, \mathbf{M}) \longrightarrow (\mathbf{R}^1, \mathcal{B}^1) : \mu \mapsto \mu(B)$$

where $\mathcal{B}^1$ denotes the Borel σ-field over $\mathbf{R}^1$. Any random variable ξ defined on some probability space with values in $(\mathcal{M}, \mathbf{M})$ is called a *random measure*, and if, in particular, the realizations of ξ are counting measures, i.e. $\xi(B) \in \mathbb{Z}^+ \cup \{\infty\}$ for all $B \in \mathcal{B}$, then ξ is called a *point process*. The realizations of point processes can be interpreted as random point configurations in the space $\mathcal{X}$; $\xi(B)$ here is the (random) number of points which fall into the set $B \in \mathcal{B}$. Sometimes it is convenient to consider point processes of the form

$$\xi = \sum_{k=1}^{N} \varepsilon_{X_k} \tag{1}$$

where ε_x denotes the Dirac measure concentrated in the point $x \in \mathcal{X}$, N is a $\mathbb{Z}^+$-valued random variable and the $\{X_k\}$ are random variables with values in $(\mathcal{X}, \mathcal{B})$; here $\xi(B) = \sum_{k=1}^{N} \varepsilon_{X_k}(B) = \sum_{k=1}^{N} \mathbb{1}_B(X_k)$ with $\mathbb{1}_B$ denoting the indicator variable of the event $B \in \mathcal{B}$, which makes the correspondance between measures in $\mathcal{M}$ and points in $\mathcal{X}$ more transparent. If, in particular, the random variables $\{X_k\}$ have ties with probabilities zero only, then ξ is called almost surely (a.s.) *simple*; i.e. there are no multiple counts of points in the pattern with positive probability.

A basic point process is the so-called *Poisson process* which is characterized by the following two properties:

1. There exists a measure μ on $\mathcal{B}$ (not necessarily Radon) such that for all $B \in \mathcal{B}$, the random variables $\xi(B)$ are $\mathrm{P}(\lambda)$ Poisson-distributed with parameter $\lambda = \mu(B)$ with the convention that for $\mu(B) = 0$, $\xi(B) = 0$ a.s., and for $\mu(B) = \infty$, $\xi(B) = \infty$ a.s.

2. For any disjoint sequence of sets $\{B_n\}_{n \in \mathbf{N}}$, the sequence of random variables $\{\xi(B_n)\}_{n \in \mathbf{N}}$ is independent.

Especially by the latter property, point patterns realized by such a Poisson process are called *completely random*; a good illustrative example is perhaps the distributional pattern of raindrops on a walkway, or the spatial distribution of *arenicola marina* in the wadden sea.

Note that for Poisson processes, also $\mu(B) = E(\xi(B))$, $B \in \mathcal{B}$. In general,

$$E\xi(B) := E(\xi(B)), \quad B \in \mathcal{B},$$

always defines a measure on $\mathcal{B}$, called *intensity measure* of ξ; in the Poisson case, the intensity measure obviously defines the distribution of a Poisson process uniquely. Due to properties of the Poisson distribution, the superposition $\xi = \xi_1 + \xi_2$ of two independent Poisson point processes ξ_1 and ξ_2 again is a Poisson process with intensity measure $E\xi = E\xi_1 + E\xi_2$. Likewise it is possible to define a *p-thinning* of a Poisson process ξ; here a "point" of the process is retained independently of the other ones with probability $p \in [0,1]$. The resulting process ζ is again Poisson with intensity measure $E\zeta = p \cdot E\xi$. This can be proved rigorously by the fact that a Poisson process ξ with *finite* intensity measure $\nu = E\xi$ can be represented in the form (1), where N is $\mathrm{P}(\nu(\mathcal{X}))$-distributed and independent of the (also independent) random variables $\{X_k\}$ which follow the distribution $Q = \nu(\cdot)/\nu(\mathcal{X})$ provided $\nu(\mathcal{X}) > 0$

(otherwise there are a.s. no points realized). General Poisson point processes with σ-finite intensity measures can be constructed by superposition of independent Poisson processes with finite intensity measures concentrated on (at most countably many) disjoint subsets of $\mathcal{X}$. Note that, in particular, Poisson point processes are a.s. simple if the intensity measure is diffuse (i.e. all atoms have zero probabilities). For details, we refer the reader to Kallenberg (1983), Daley and Vere–Jones (1988), or König and Schmidt (1992).

Whereas for the study of Poisson point processes it is not absolutely necessary to use the full topological machinery as outlined in the beginning of this section it will become inevitable to do so when weak convergence of point processes is considered, as in the sequel of this paper. Some more notions will be needed to give simple sufficient conditions for such kinds of convergence.

A semiring $\mathcal{I} \subseteq \mathcal{R}$ is said to have the *DC-property* (i.e. being *dissecting* and *covering*), if for every set $B \in \mathcal{R}$ and every $\varepsilon > 0$ there exist finitely many sets $I_1, \dots, I_n \in \mathcal{I}$ such that $B \subseteq \bigcup_{j=1}^n I_j$ and $\sup\{\rho(x,y) \mid x, y \in I_j\} < \varepsilon$, $1 \le j \le n$. (Such a set B is sometimes also called *pre-compact* or *totally bounded.*)

The following result on weak convergence to a simple point process is due to Kallenberg (1983), Theorem 4.7.

Theorem 1. *Let $\{\xi_n\}_{n\in\mathbf{N}}$ be a sequence of (not necessarily simple) point processes and ξ an a.s. simple point process such that $\mathcal{R}_\xi = \{B \in \mathcal{R} \mid \xi(\partial B) = 0 \text{ a.s.}\}$ contains a DC-semiring $\mathcal{I}$. Then the following two conditions are sufficient for weak convergence of $\{\xi_n\}$ to ξ, i.e. $P^{\xi_n} \xrightarrow{w} P^\xi$:*

$$\lim_{n\to\infty} P\left(\xi_n\Big(\bigcup_{j=1}^k I_j\Big) = 0\right) = P\left(\xi\Big(\bigcup_{j=1}^k I_j\Big) = 0\right) \tag{2}$$

for all $k \in \mathbf{N}$, $I_1, \dots, I_k \in \mathcal{I}$;

$$\limsup_{n\to\infty} E\xi_n(I) \le E\xi(I) < \infty \tag{3}$$

for all $I \in \mathcal{I}$.

For Euclidean spaces $(\mathcal{X}, \mathcal{B}) = (\mathbf{R}^d, \mathcal{B}^d)$ with finite dimension $d \in \mathbf{N}$ (and Borel σ-field $\mathcal{B}^d$) the semiring $\mathcal{I}$ of left–open, right–closed Intervals $I = \times_{j=1}^d (a_j, b_j]$ with $a_j < b_j$ will be a suitable DC–semiring fulfilling the conditions of Theorem 1 (Kallenberg (1983), p. 11 and Lemma 4.3).

In the particular case of *Poisson point processes* $\{\xi_n\}$, $\{\xi\}$ with finite total intensities $E\xi_n(\mathcal{X}) < \infty$, $E\xi(\mathcal{X}) < \infty$, conditions (2) and (3) in Theorem 1 simplify to the following simple condition:

$$\lim_{n\to\infty} E\xi_n(I) = E\xi(I), \quad I \in \mathcal{I}. \tag{4}$$

This is obvious since Poisson processes have independent increments, and the union of sets in (2) can w.l.o.g. be taken to be pairwise disjoint; condition (2) then simplifies to the consideration of the case $k = 1$ since by independence, $P\left(\xi_n\left(\bigcup_{j=1}^k I_j\right) = 0\right) = \prod_{j=1}^k P\left(\xi_n(I_j) = 0\right)$. But by (4),

$$\lim_{n\to\infty} P\left(\xi_n(I) = 0\right) = \lim_{n\to\infty} e^{-E\xi_n(I)} = e^{-E\xi(I)} = P\left(\xi(I) = 0\right)$$

for all $I \in \mathcal{I}$, hence (2) and (3) are satisfied.

Since in actual ecosystems, only a bounded (but possibly very large) number of objects can occur, the assumptions of finite total intensities in (4) are no real restriction for modeling purposes here.

3 Time–dependent Poisson point patterns

In this section we want to investigate the long–time behaviour of spatial Poisson point patterns in which objects are allowed to be newly created (by birth) or discarded (by death), and have the possibility of movements. For this purpose, we shall consider Poisson point processes $\{\xi_t\}_{t\geq 0}$ of the form (1), depending on the time t as

$$\xi_t = \sum_{k=1}^{N(t)} \mathbb{1}_{\{T_k > t\}} \varepsilon_{X_k(t)}, \quad t \geq 0, \tag{5}$$

where $\{N(t)\}_{t\geq 0}$ is an ordinary Poisson counting process on the line with finite and positive intensity $\lambda(t) = \Lambda'(t)$, where $\Lambda(t) = E(N(t))$ with $\Lambda(0) \geq 0$, $t \geq 0$ is some weakly increasing absolutely continuous function. $\{T_k\}_{k\in\mathbf{N}}$ is a family of (also from $\{N(t)\}$) independent and identically distributed life times with absolutely continuous cdf F with $F(0) = 0$ and density $f = F'$; $\{X_k(t) \mid t \geq 0\}_{k\in\mathbf{N}}$ is a family of (also from $\{N(t)\}$ and $\{T_k\}$) independent and identically distributed stochastic processes taking values in a locally compact Polish space $(\mathcal{X}, \mathcal{B})$ such as $(\mathbf{R}^d, \mathcal{B}^d)$.

The process $\{N(t) \mid t \geq 0\}$ here governs the creation of new particles whereas the counting processes $I_k(t) = \mathbb{1}_{\{T_k > t\}}$, $t \geq 0$, describe the life lengths T_k of each individual particle in the system. The movement of (alive) particles is governed by the processes $\{X_k(t) \mid t \geq 0\}$.

Note that if the processes $\{X_k(t) \mid t \geq 0\}$ are Markov processes then the process $\{\xi_t\}_{t\geq 0}$ also is a Markov process since for any strictly increasing non–negative sequence $\{t_n\}_{n\in\mathbf{N}}$ of time points the Poisson point processes $\{\xi_{t_n}\}_{n\in\mathbf{N}}$ are obtained successively by independent thinnings, superpositions and Markovian shifts of points. This follows from the Markov chain generation theorem as in Mathar and Pfeifer (1990), Lemma 3.2.2.

Let Q_t, $t \geq 0$, denote the distribution of $X_k(t)$, $k \in \mathbf{N}$. Then by Theorem 1, we obtain the following result concerning the long–time behaviour of the pattern process $\{\xi_t\}_{t\geq 0}$.

Theorem 2. *Let $\{\xi_t\}_{t\geq 0}$ be a family of Poisson point processes of type (5) with points located in a locally compact Polish space $(\mathcal{X}, \mathcal{B})$ such that $\mathcal{B}$ contains a suitable DC–semiring $\mathcal{I}$ of relatively compact subsets fulfilling the requirements of Theorem 1. Then if there exists some Radon measure μ over $\mathcal{B}$ such that*

$$\Lambda(t)(1 - F(t))Q_t(I) \longrightarrow \mu(I), \quad t \to \infty, \quad \text{for all } I \in \mathcal{I}, \tag{6}$$

we have weak convergence of ξ_t to some Poisson point process ξ with intensity measure μ, i.e. $P^{\xi_t} \xrightarrow{w} P^{\xi}$. In particular, if $T = \inf\{t > 0 \mid \xi_s(\mathcal{X}) = 0 \text{ for all } s \geq t\}$ denotes the time of (possible) extinction of the system, then

$$\begin{aligned} P(T \leq t) &= \exp\left(-\Lambda(t)(1 - F(t)) - \int_t^\infty \lambda(s)(1 - F(s))\, ds\right) \\ &= e^{-\mu(\mathcal{X})} \exp\left(-\int_t^\infty \Lambda(s) f(s)\, ds\right), \quad t \geq 0 \end{aligned} \tag{7}$$

with $e^{-\mu(\mathcal{X})} = 0$ for $\mu(\mathcal{X}) = \infty$. In the latter case, $T = \infty$ a.s., i.e. the system will a.s. not die out.

Proof. The first part follows immediately from Theorem 1, relation (4) and the fact that for the intensity measure of ξ_t, we have

$$E\xi_t(I) = E(N(t))E\left(\mathbb{1}_{\{T_k>t\}}\varepsilon_{X_k(t)}(I)\right) = \Lambda(t)(1-F(t))Q_t(I)$$

for all $I \in \mathcal{I}$. For the second part, observe that by our assumptions,

$$\xi_t(\mathcal{X}) = \sum_{k=1}^{N(t)} \mathbb{1}_{\{T_k>t\}},\ t \geq 0,$$

is a Markov birth–death process with birth and death rates $\beta_n(t)$ and $\delta_n(t)$, resp., given by

$$\begin{aligned} \beta_n(t) &= \lim_{h\downarrow 0}\frac{1}{h}P(\xi_{t+h}(\mathcal{X}) = n+1 \mid \xi_t(\mathcal{X}) = n) = \lambda(t)(1-F(t)), \quad n \in \mathbf{Z}^+, \\ \delta_n(t) &= \lim_{h\downarrow 0}\frac{1}{h}P(\xi_{t+h}(\mathcal{X}) = n-1 \mid \xi_t(\mathcal{X}) = n) = \frac{nf(t)}{1-F(t)}, \quad n \in \mathbf{N}, \end{aligned} \tag{8}$$

for $t \geq 0$. Note that the first part of (8) is due to the fact that the counting process $\{N(t)\}$ has birth rate $\lambda(t)$ at time t, with time–depending thinning by the (joint) survival probabilitites $1-F(t)$, while for the second part, the death probability for a single individual alive particle in the time interval $[t, t+h]$ is approximately h times the hazard rate $f(t)/(1-F(t))$. To prove (7), observe that by the independent increments property ii) of Poisson processes,

$$\begin{aligned} P(T \leq t) &= P\Big(\{\xi_t(\mathcal{X}) = 0\} \cap \bigcap_{s>t}\{\xi_s(\mathcal{X}) = 0\}\Big) \\ &= P\Big(\{\xi_t(\mathcal{X}) = 0\}\Big)P\Big(\bigcap_{s>t}\{\xi_s(\mathcal{X}) = 0\} \,|\, \xi_t(\mathcal{X}) = 0\Big) \end{aligned}$$

$$= P(\{\xi_t(\mathcal{X}) = 0\}) \lim_{h\downarrow 0}\prod_{k=0}^{\infty}(1 - h\beta_0(t+kh)) = \exp\left(-E\xi_t(\mathcal{X})\right)\exp\left(-\int_t^{\infty}\beta_0(s)\,ds\right)$$

which gives the first part of (7). The second equation follows by partial integration and (6). ∎

4 Applications

We shall give an example of a possible application in $(\mathcal{X}, \mathcal{B}) = (\mathbf{R}^d, \mathcal{B}^d)$ with finite dimension $d \in \mathbf{N}$. Suppose that the lifetime distributions are exponential with mean $\frac{1}{\tau}$, $\tau > 0$, and the movements of points governed by componentwise independent Brownian motions with zero mean and variance $\sigma^2 > 0$, and initial multivariate normal distribution with independent components of zero mean and variance $\sigma_0^2 \geq 0$. Let further the cumulative intensity of the Poisson birth process be given by

$$\Lambda(t) = \sqrt{t}^{\,d} e^{c\tau t} + \Lambda(0), \quad t \geq 0, \tag{9}$$

with a parameter $c \geq 0$. Then if m_d denotes the d–dimensional Lebesgue measure, one of the following three cases will occur, depending on the choice of the parameter c:

Case 1, $c = 1$:

Here asymptotic equilibrium of the system will be achieved by weak convergence of $\{\xi_t\}$ to a Poisson point process ξ with mean measure given by

$$E\xi(B) = \frac{1}{\sqrt{2\pi\sigma^2}} m_d(B), \quad B \in \mathcal{B}^d. \tag{10}$$

This follows from Theorem 2 since for any bounded Interval I in $\mathbf{R}^d$ we have

$$Q_t(I) = \frac{1}{\sqrt{2\pi(\sigma_0^2 + \sigma^2 t)}^d} \int_I \exp\left(- \frac{1}{2(\sigma_0^2 + \sigma^2 t)} \sum_{k=1}^d x_k^2 \right) dx_1\, dx_2 \ldots dx_d, \; t > 0, \tag{11}$$

with

$$\sqrt{t}^d Q_t(I) \longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}} m_d(I), \quad t \to \infty, \tag{12}$$

hence

$$\Lambda(t)(1 - F(t))Q_t(I) = \left(e^{(c-1)\tau t} + \frac{\Lambda(0)}{\sqrt{t}^d}\right) \sqrt{t}^d Q_t(I) \longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}} m_d(I), \; t \to \infty, \tag{13}$$

which proves (10).

Case 2, $c < 1$:

Here extinction of the system will eventually happen since $\mu(\mathcal{X}) = 0$ and by (7),

$$\begin{aligned} P(T \le t) &= \exp\left(- \int_t^\infty \Lambda(s) f(s)\, ds \right) \\ &= e^{-\Lambda(0)e^{-\tau t}} \exp\left(- \int_t^\infty \sqrt{s}^d e^{-(1-c)\tau s}\, ds \right), \quad t \ge 0, \end{aligned} \tag{14}$$

which is a proper cdf. Accordingly, similar as in (10) to (13), we have weak convergence of $\{\xi_t\}$ to the void point process, i.e. the Poisson process with zero intensity measure. The following figure shows the cdf of T for the parameter choices $d = 2$, $\Lambda(0) = 0$, $\tau = 1$ and c ranging from 0.2 to 0.8 with step 0.1.

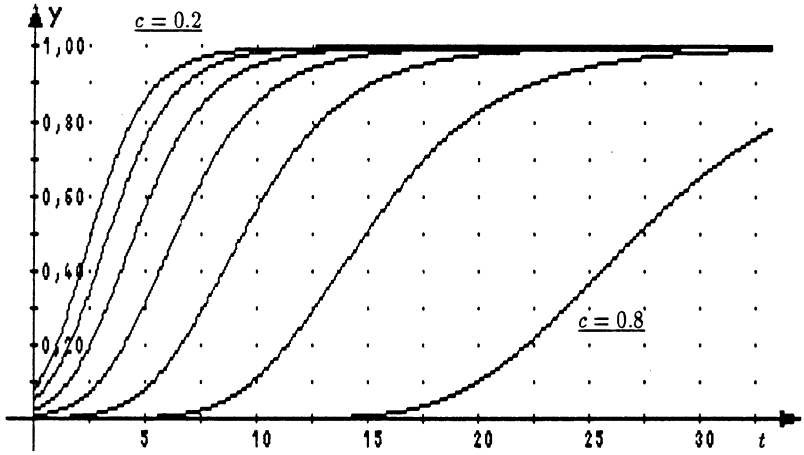

Fig. 1
Cumulative distribution function for extinction time T

Case 3, $c > 1$:

Here the system will eventually explode since the mean number of particles in any bounded set B at time $t > 0$ is $E\xi_t(B) \sim e^{(c-1)\tau t} m_d(B)/\sqrt{2\pi\sigma^2}$, which grows to ∞ for large t.

For birds, one might take $d = 2$, $c < 1$ and $\Lambda(0) = \sigma_0^2 = 0$; then $E\xi_t(\mathcal{X}) = t \cdot \exp(-(1-c)\tau t)$, $t \geq 0$, hence the system will start from zero individuals, growing up to a maximal average flock size of $1/[e(1-c)\tau]$ (achieved at time $t = 1/[(1-c)\tau]$), and then gradually decrease to zero again. For $c = 1$, the system would again start with zero individuals, but increase gradually to a stable average flock size within every bounded region of $\mathbf{R}^2$. In the case of worms, one might take $d = 2$, $c \leq 1$ and $\Lambda(0) > 0$, $\sigma_0^2 > 0$; then the system starts with a random configuration of an average positive number of individuals, and is either asymptotically stable over bounded regions (for $c = 1$) or dies out (for $c < 1$). Of course, similar results would hold true with other initial distributions $P^{X_k(0)}$ than normal distributions. For instance, in revitalization experiments in the wadden sea rectangular areas are covered with impermeable sheets such that after some time, no more individuals will be inside this area. This corresponds to a deterministic thinning procedure of a Poisson point process at the beginning ($t = 0$), hence one starts with an initial distribution for the location of individuals which is non–normal. However, the same analysis as before shows that in case $c = 1$, a random spatial pattern will develop over time which approaches a homogeneous Poisson process as before, which is in coincidence with observations made in field experiments.

Fig. 2 shows a simulation study of such a system for a starting homogeneous Poisson point process with total deletion of points within a rectangular area (taken from Pfeifer, Bäumer and Albrecht (1992)).

Software for visualizing moving point patterns is presently being developed by the authors and will be available upon request.

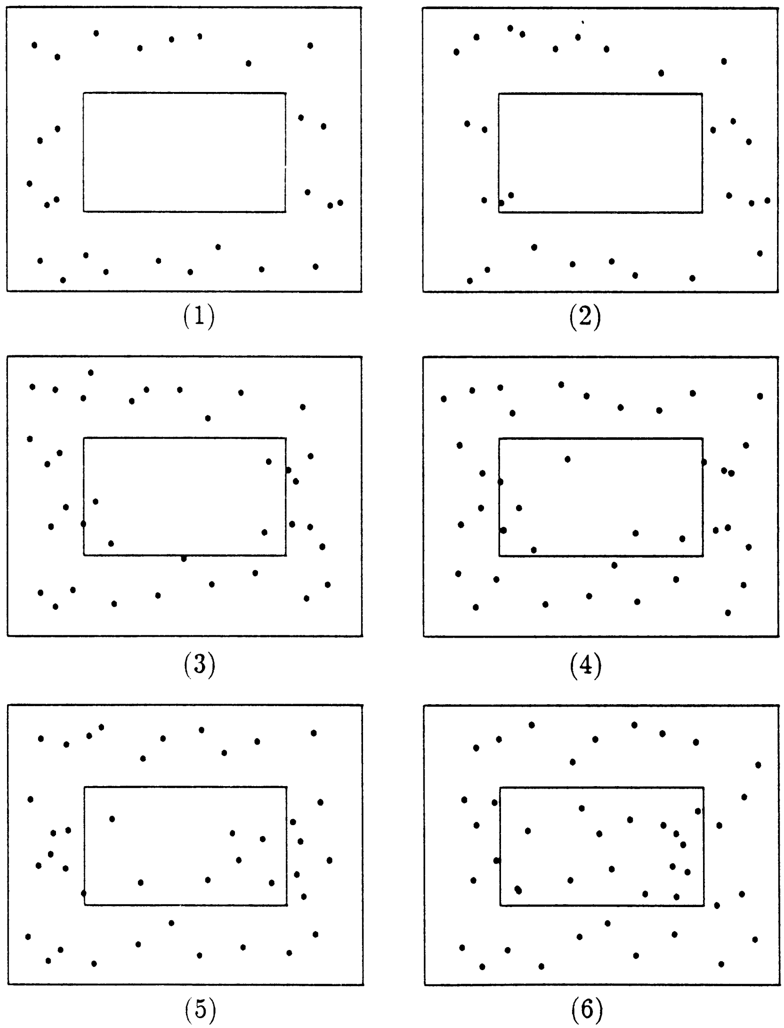

Fig. 2
Simulation of a revitalization experiment by Poisson point patterns

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