On the Early History of the Singular Value Decomposition

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For Gene Golub on his fifteenth birthday

1. Introduction

One of the most fruitful ideas in the theory of matrices is that of a matrix decomposition or canonical form. The theoretical utility of matrix decompositions has long been appreciated. More recently, they have become the mainstay of numerical linear algebra, where they serve as computational platforms from which a variety of problems can be solved.

Of the many useful decompositions, the singular value decomposition — that is, the factorization of a matrix $\mathbf{A}$ into the product $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}}$ of a unitary matrix $\mathbf{U}$ a diagonal matrix $\mathbf{\Sigma}$ and another unitary matrix $\mathbf{V}^{\mathrm{H}}$ — has assumed a special role. There are several reasons. In the first place, the fact that the decomposition is achieved by unitary matrices makes it an ideal vehicle for discussing the geometry of n-space. Second, it is stable; small perturbations in $\mathbf{A}$ correspond to small perturbations in $\mathbf{\Sigma}$, and conversely. Third, the diagonality of $\mathbf{\Sigma}$ makes it easy to determine when $\mathbf{A}$ is near to a rank-degenerate matrix; and when it is, the decomposition provides optimal low rank approximations to $\mathbf{A}$. Finally, thanks to the pioneering efforts of Gene Golub, there exist efficient, stable algorithms to compute the singular value decomposition.

The purpose of this paper is to survey the contributions of five mathematicians — Eugenio Beltrami (1835–1899), Camille Jordan (1838–1921), James Joseph Sylvester (1814–1897), Erhard Schmidt (1876–1959), and Hermann Weyl (1885–1955) — who were responsible for establishing the existence of the singular value decomposition and developing its theory. Beltrami, Jordan, and Sylvester came to the decomposition through what we should now call linear algebra; Schmidt and Weyl approached it from integral equations. To give this survey context, we will begin with with a brief description of the historical background.

Is is an intriguing observation that most of the classical matrix decompositions predated the widespread use of matrices: they were cast in terms of determinants, linear systems of equations, and especially bilinear and quadratic forms. Gauss is the father of this development. Writing in 1823 [15, §31], he describes his famous