4
Projective Geometry

Projective geometry is an example of mathematics that was originally created with one application in mind, and yet has unexpectedly shed light on fields that are totally unrelated to the one for which it was originally developed. Created by artists during the Renaissance for analyzing perspective, projective geometry blossomed during the eighteenth and nineteenth centuries into a complete revision of the entire field of geometry. Recently it has provided the setting for the modern study of algebraic equations, and has even played a role in physics in the mathematics of quantum field theory.

4.1 Perspective Drawing

When you look at a scene your eye does not respond directly to the objects in the scene itself. It responds instead to the light rays that it receives from points in the scene. To make a correct perspective drawing the artist first *projectivizes* the scene by extending an imaginary line to his eye from each point in the scene. He then *projects* the scene into a plane by intersecting the plane with each of the imaginary lines. Taken together these intersections form an image that looks just the same as the original scene to the artist's eye (Fig. 4.1).

Projective geometry is the study of the properties of geometric figures that are not altered by projections. Two basic projections are the following (see Fig. 4.2).

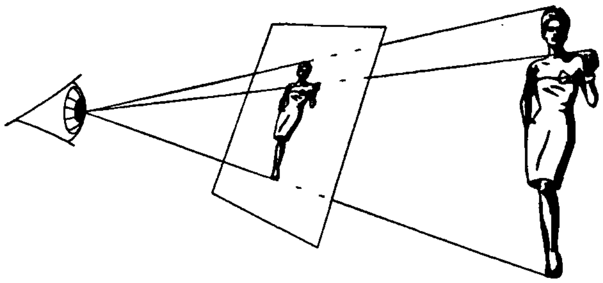


FIGURE 4.1. Projecting a scene into a plane.

1. **Central projection.** Given a point P and a plane H with $P \notin H$, define a projection function f by the formula

$$f(Q) = \overleftrightarrow{PQ} \cap H$$

for every point Q such that $\overleftrightarrow{PQ}$ is not parallel to H. f is called *projection from P into H*; P is the *center* of the projection f.

2. **Parallel projection.** Let $\vec{v}$ be a nonzero vector and H a plane that is not parallel to $\vec{v}$. For each point Q let L_Q be the line through Q that is parallel to $\vec{v}$. Define a projection function g by the formula

$$g(Q) = L_Q \cap H.$$

g is called *parallel projection into H along the direction $\vec{v}$*.

Parallel projection acts like a central projection whose center is infinitely far away.

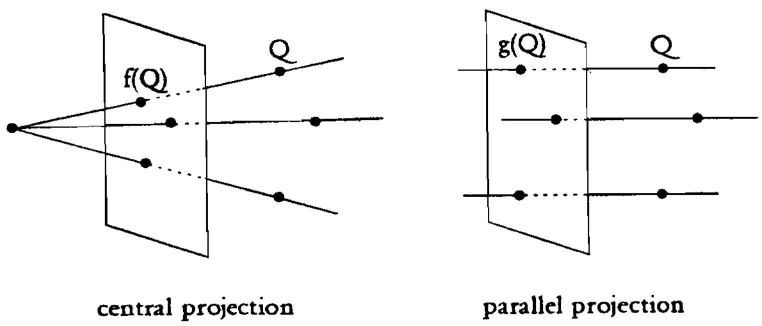


FIGURE 4.2.

Some examples of properties that are preserved by projections are: the property of being a point, the property of being a line, and the property of being a conic section. (Here and in the rest of this chapter we consider only objects that do not contain the artist's eye).

Properties that are not preserved include length, the size of angles, area, and the property of being a circle.

Projectivization. From now on we will imagine that the artist has only one eye, and that it is located at the origin, O. A *radial* line or plane is one that passes through O. The *projectivization* of a scene is the set of all radial lines that pass through points in the scene, together with all radial lines that are infinitesimally close to lines passing through points in the scene (nobody's eye is sharp enough to distinguish between lines that are infinitesimally close to each other).

As it views a scene your eye does not respond directly to the objects in the scene. Instead it responds to a projectivization of the scene, namely the projectivization that consists of all the light rays that travel along lines from points in the scene to your eye. This fact has important consequences:

1. **Radial lines look like points, and radial planes look like lines** because they are being viewed "edge on" by the eye at the origin.
2. **Radial dimensions are lost** because radial lines look like points.
3. **Non-radial lines acquire an extra "point at infinity".** The projectivization of a non-radial line L is the set of radial lines in the plane $\overline{OL}$ that connects L with the eye. Only one radial line in the plane $\overline{OL}$ does not connect a point on L to the eye. That one exception is the radial line that is parallel to L. We shall call this exceptional line P_∞, the "point at infinity" on L. To the eye P_∞ appears to be a point at infinity on L, because it is the limit of lines $\overleftrightarrow{OP}$ connecting the eye to points $P \in L$ as P approaches infinity (Fig. 4.3),

$$P_\infty = \lim_{P\to\infty} \overleftrightarrow{OP}.$$

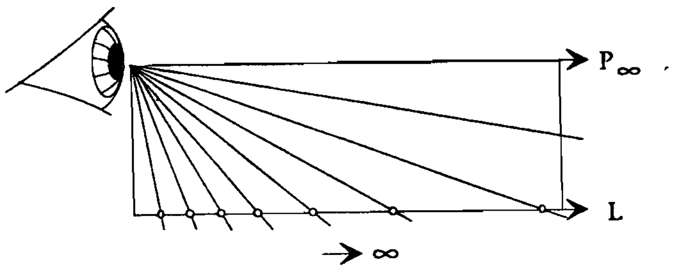


FIGURE 4.3. A point at infinity.

4. **Non-radial planes acquire an extra "line at infinity".** Let H be a non-radial plane. As $P \in H$ goes to infinity the line OP tends towards the radial plane that is parallel to H. We call this plane L_∞, the *line at infinity* of H.

$$L_\infty = \left\{ \lim_{P\to\infty} \overleftrightarrow{OP} \,\middle|\, P \in H \right\}.$$

To the eye, points in L_∞ look like they lie "at infinity" on the horizon of H (Fig. 4.4).

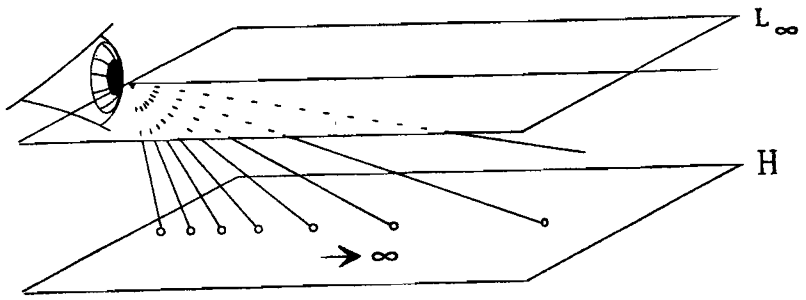
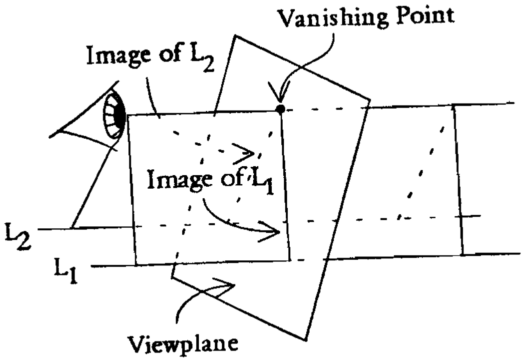


FIGURE 4.4. L_∞ is a "line at infinity".

As a general rule any figure that extends off to infinity will acquire extra "points at infinity" when it is projectivized.

Vanishing Points.

A perspective drawing is created by intersecting the projectivization of a scene with a plane. The plane, which we shall call the *viewplane*, is the artist's canvas. The *image* of the scene is the intersection of the projectivization of the scene with the viewplane. A *vanishing point* is an image of a point at infinity.



Parallel Lines have the same Vanishing Point.

FIGURE 4.5.

Parallel lines all are parallel to the same radial line, so they have the same point at infinity. Therefore the images of parallel lines all pass through the same vanishing point (Fig. 4.5). The only exception to this rule occurs when the lines are all parallel to the viewplane. In that case their point at infinity does not intersect the viewplane, so the lines have no vanishing points and their images are parallel. (Of course their common point at infinity is still visible to the eye, but it does not appear in the picture).

A plane's*horizon* is the image of its line at infinity. If the plane contains some parallel lines then their common vanishing point is on the plane's horizon.

Figure 4.6 shows four views of a rectangular box. The first is a "three point perspective"; all three vanishing points are present in the viewplane. The second view is a "two point perspective" – only two vanishing points are present and four edges have parallel images with no vanishing point.

The third view is a "one point perspective". The viewplane is parallel to a whole side of the box.

It is impossible to have a true "zero point perspective" drawing of a rectangular box since the viewplane cannot be parallel to all its edges at once. However if you move the artist's eye off to infinity then central projection through the artist's eye becomes a parallel projection, the images of parallel lines become parallel, and the image of a box has no vanishing points. The fourth view is a parallel projection.

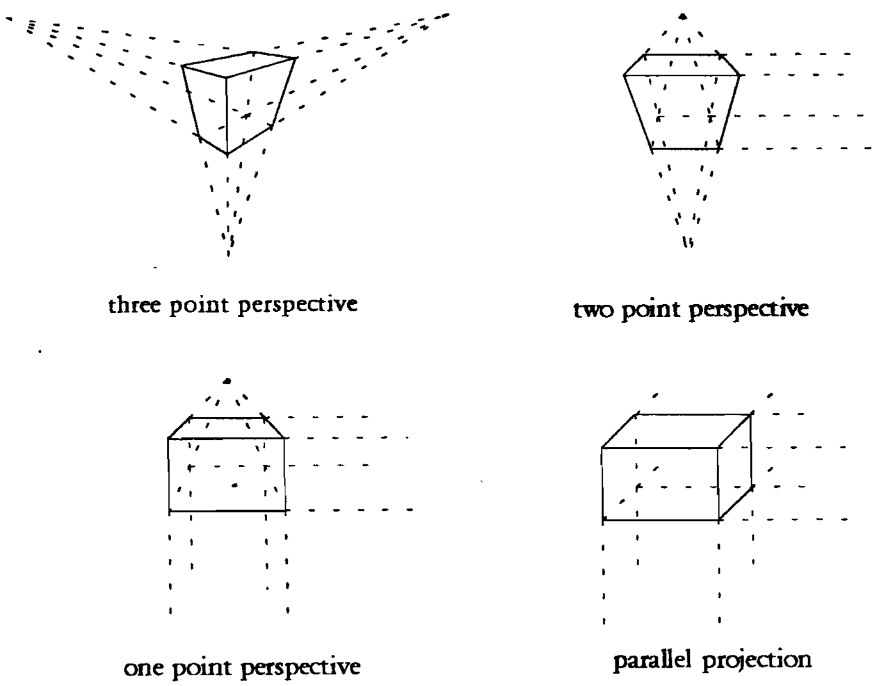


FIGURE 4.6. Four views of a box.

Parallel projections generally are preferred in technical and scientific applications, even though they look less natural than true perspective drawings, because there is a simpler relationship between distances in the drawing and distances in the original scene with parallel projections than there is with perspective drawings. This makes parallel projections easier to create, and simplifies the task of calculating the exact measurements of the original object from data that are given in drawings.

Exercise 4.1.1 If you are looking at a drawing of a rectangular box in three point perspective, where should you place your eye so that the picture will look the same to you as the original box did to the artist? Let V_1, V_2, and V_3 be the three vanishing points. Show that you should put your eye at a point P such that the lines $\overleftrightarrow{PV_1}$, $\overleftrightarrow{PV_2}$, and $\overleftrightarrow{PV_3}$ are all perpendicular to each other. Let S_1 be the sphere with diameter $\overline{V_2V_3}$, S_2 the sphere with diameter $\overline{V_1V_3}$, and S_3 the sphere with diameter $\overline{V_1V_2}$. Show that $P \in S_1 \cap S_2 \cap S_3$. There are two points in this intersection, one on each side of the viewplane.

Where should you put your eye if the box is drawn in two point perspective? In one point perspective?